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INTEGRABLE CONNECTIONS RELATED TO MANIN AND SCHECHTMAN'S HIGHER BRAID GROUPS

BY

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To the memory of K. T. Chen

Introduction

In [11], Manin and Schechtman introduced a series of hyperplane arrangements called the discriminantal arrangements, which generalize the braid arrangements. For positive integers $k \leq n$, the discriminantal arrangement $\mathscr{B}(n, k)$ consists of hyperplanes D_J in $\mathbf{C}^n$ labelled with $J \subset \{1, \ldots, n\}$ such that $|J| = k$. The complementary space $U(n, k) = \mathbf{C}^n - \bigcup_J D_J$ parametrizes affine hyperplanes $(H_1, \ldots, H_n)$ in $\mathbf{C}^k$ which are in general position and each H_i is parallel to H_i^0 for some fixed affine hyperplanes $(H_1^0, \ldots, H_n^0)$ in general position. The fundamental group of $U(n, k)$ is called a higher braid group and coincides with the pure braid group with n strings if $k = 1$. The combinatorial aspects of such arrangements were studied by Manin and Schechtman [11].

Combining with the theory of K. T. Chen [2], [3], we see that the completion of the group ring of the higher braid group over $\mathbf{C}$ with respect to the powers of the augmentation ideal is generated by X_J, which are in one-to-one correspondence with the hyperplanes D_J, with relations

(i) $[X_J, \Sigma_{I \subset K} X_I] = 0$ for any $K \subset \{1, \ldots, n\}$ with $|K| = k + 2$,

(ii) $[X_{J_1}, X_{J_2}] = 0$ if $|J_1 \cup J_2| \geq k + 3$.

These relations generalize the infinitesimal pure braid relations in the sense of [9] and [10]. They give the integrability condition of the connection of the form $\Sigma_J X_J \, d \log \varphi_J$ where X_J is a constant matrix and φ_J is a linear form with $\operatorname{Ker} \varphi_J = D_J$. In the case $k = 1$ the above infinitesimal pure braid relations were studied in relation with the classical Yang-Baxter equation and we obtain one parameter family of linear representations of the pure braid groups as the holonomy of this connection for any simple Lie algebra and its representations (see [10]). The purpose of this note is to give a generalization of this construction to the higher braid groups. Namely, given a finite dimensional

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complex simple Lie algebra $\mathfrak{g}$ and its irreducible representations

$$\rho_i\colon \mathfrak{g} \to \mathrm{End}(V_i), \quad 1 \le i \le n,$$

we construct a one parameter family of integrable connections over $U(n, k)$. The holonomy of these connections give linear representations of the higher braid groups. The author would like to express his sincere gratitude to Prof. Manin who kindly communicated to him a program to study the hierarchy related to the higher braid groups depending on the parameter $1 \le k \le n$.

1. Arrangements and monodromy representations

Let V be a finite dimensional complex vector space and let $\mathscr{A}$ be a finite set of complex hyperplanes in V containing the origin. We call $\mathscr{A}$ an arrangement and we put $\mathscr{A} = \{H_1, \ldots, H_n\}$. Let $M(\mathscr{A})$ be the complementary space $V - \bigcup_{1 \le i \le n} H_i$. We are mainly interested in linear representations of the fundamental group $\pi_1(M(\mathscr{A}))$. First, we recall the description of the cohomology ring of $M(\mathscr{A})$ due to Orlik and Solomon [13]. Associated with the arrangement $\mathscr{A}$, we consider the geometric lattice $L(\mathscr{A})$ defined in the following way. As a set, $L(\mathscr{A})$ consists of the subspaces of V of the form

$$X = H_{i_1} \cap \cdots \cap H_{i_p} \quad \text{where } \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$$

and we partially order $L(\mathscr{A})$ by the reverse inclusion, which means $X' \le X$ if and only if $X \supset X'$ as subspaces of V. For $H_i \in \mathscr{A}$ let φ_i be a linear form with $\operatorname{Ker} \varphi_i = H_i$. We define holomorphic differential forms ω_i, $1 \le i \le n$, on $M(\mathscr{A})$ by $\omega_i = d \log \varphi_i$. Let $\mathscr{E}$ be the exterior algebra of the complex vector space with basis consisting of the elements e_i, $1 \le i \le n$, which are in one-to-one correspondence with the hyperplanes $H_i \in \mathscr{A}$. For a subset $S = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ we put

$$e_S = e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

We define a $\mathbf{C}$-linear differential ∂ in $\mathscr{E}$ by

$$\partial(1) = 0, \quad \partial(e_i) = 1, 1 \le i \le n,$$

and

$$\partial(e_S e_T) = \partial e_S \cdot e_T + (-1)^{|S|} e_S \cdot \partial e_T$$

We say that S is dependent if

$$\operatorname{codim}_{\mathbf{C}} \bigcap_{j \in S} H_j > |S|.$$

Let us denote by $\mathscr{I}$ the ideal of $\mathscr{E}$ generated by ∂e_S for any dependent $S \subset \{1, \ldots, n\}$.

PROPOSITION 1.1 [13]. *We have an isomorphism of graded rings*

$$\varphi\colon H^*(M(\mathscr{A}); \mathbf{C}) \simeq \mathscr{E}/\mathscr{I}$$

such that $\varphi([\omega_i]) = e_i \bmod \mathscr{I}$.

Let us consider a 1-form ω on $M(\mathscr{A})$ written as

$$\omega = \sum_{i=1}^{n} Z_i \, d \log \varphi_i$$

where Z_i, $1 \le i \le n$, are $m \times m$ constant matrices with some fixed m. Now we are going to discuss the integrability of ω. For this purpose we introduce the following notations. Let K be a codimension two element in $L(\mathscr{A})$. We define a set of hyperplanes $\mathscr{A}_K$ by

$$\mathscr{A}_K = \{ H_i \in \mathscr{A};\ H_i \supset K \}.$$

We put $\mathscr{A}_K = \{H_{i_1}, \ldots, H_{i_p}\}$, $i_1 < \cdots < i_p$.

LEMMA 1.2. *We have* $\omega \wedge \omega = 0$ *if and only if*

$$\left[Z_{i_\nu}, Z_{i_1} + \cdots + Z_{i_p} \right] = 0, \quad 1 \le \nu \le p, \tag{1.3}$$

for any codimension two element $K \in L(\mathscr{A})$, *where* $\mathscr{A}_K = \{H_{i_1}, \ldots, H_{i_p}\}$.

Proof. First, we describe a basis of $H^2(M(\mathscr{A}); \mathbf{C})$. To any codimension two element K in the associated geometric lattice $L(\mathscr{A})$ we associate the set $\mathscr{B}_K = \{\varphi_{i_1} \wedge \varphi_{i_\nu};\ 2 \le \nu \le p\}$ where $\mathscr{A}_K = \{H_{i_1}, \ldots, H_{i_p}\}$. Then it turns out that the union $\bigcup_K \mathscr{B}_K$ gives a basis of $H^2(M(\mathscr{A}); \mathbf{C})$. Let us now express $\omega \wedge \omega$ as a linear combination of the above 2-forms. The coefficient of $\varphi_{i_1} \wedge \varphi_{i_\nu}$ is equal to $[Z_{i_1} + \cdots + Z_{i_p}, Z_{i_\nu}]$. Our lemma follows immediately.

Let us suppose that $\omega = \sum_i Z_i \, d \log \varphi_i$ satisfies the condition $\omega \wedge \omega = 0$. Since ω is a closed form this implies that ω defines an integrable connection on the trivial vector bundle over $M(\mathscr{A})$ with a fiber W isomorphic to $\mathbf{C}^m$. As the holonomy of this connection we obtain a linear representation

$$\phi\colon \pi_1(M(\mathscr{A}), x_0) \to \mathrm{End}(W).$$

In other words, we consider the total differential equation $d\Phi = \omega\Phi$ with values in $\mathrm{End}(W)$, and the solution Φ satisfying $\Phi(x_0) = \mathrm{id}$ gives a linear representation determined by

$$\gamma^*\Phi = \Phi \cdot \phi(\gamma) \quad \text{for } \gamma \in \pi_1(M(\mathscr{A}), x_0).$$

This monodromy representation can be expressed by the following infinite sum using Chen's iterated integrals (see [3])

$$(1.4) \qquad 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots + \int_\gamma \omega \ldots \omega + \cdots$$

Let us recall the group theoretical meaning of the relations in (1.3). Let $\mathbf{C}\langle\langle X_1, \ldots, X_n \rangle\rangle$ denote the ring of non-commutative formal power series with indeterminates $X_1, \ldots, X_n$ which are in one-to-one correspondence with the hyperplanes $H_1, \ldots, H_n$. Let $\mathscr{I}$ be the two-sided ideal generated by $[X_{i_\nu}, X_{i_1} + \cdots + X_{i_p}]$, $1 \leq \nu \leq p$, for any codimension two element $K \in L(\mathscr{A})$ corresponding to (1.3). As a universal expression of the monodromy representation (1.4) we obtain a homomorphism

$$\tilde{\phi}^{\sim} : \pi_1(M(\mathscr{A}), x_0) \to \mathbf{C}\langle\langle X_1, \ldots, X_n \rangle\rangle / \mathscr{I}$$

which is given by the iterated integrals of the Chen's formal connection

$$\tilde{\omega} = \sum_i X_i \otimes d \log \varphi_i.$$

Let $\mathbf{C}\pi_1(M(\mathscr{A}), x_0)^\wedge$ denote the completion of the group ring of the fundamental group of $M(\mathscr{A})$ with respect to the powers of the augmentation ideal J, by which we mean the projective limit

$$\varprojlim_r \mathbf{C}\pi_1(M(\mathscr{A}), x_0)/J^r.$$

By using the result of Chen [2] and the mixed Hodge structure on the above completion of the fundamental group due to Hain [5] and Morgan [12] we have an isomorphism

$$(1.5) \qquad \mathbf{C}\pi_1(M(\mathscr{A}), x_0)^\wedge \simeq \mathbf{C}\langle\langle X_1, \ldots, X_n \rangle\rangle / \mathscr{I}$$

as complete Hopf algebras (see [1], [2], [5] and [8]).

(1.6) *Example.* Let $z_1, \ldots, z_N$ be the coordinate functions of the vector space V and let H_{ij}, $1 \leq i < j \leq N$, be the hyperplanes defined by $z_i = z_j$. The corresponding arrangement $\mathscr{A} = \{H_{ij}\}$ is called the *braid arrangement*. The relations (1.3) are expressed as

$$\left[Z_{ij}, Z_{jk} + Z_{ik}\right] = \left[Z_{ij} + Z_{jk}, Z_{ik}\right] = 0 \quad \text{for any } i < j < k,$$
$$\left[Z_{ij}, Z_{kl}\right] = 0 \quad \text{for any distinct } i, j, k, l$$

which we call the infinitesimal pure braid relations (see [9] and [10]). In the next section we discuss a generalization of the braid arrangement due to Manin and Schechtman [11].

2. Discriminantal arrangement

In this section we recall a construction of Manin and Schechtman [11], which gives a family of arrangements generalizing the braid arrangements. Let $V_0 = \mathbf{C}^k$ and we fix a family of affine hyperplanes in general position $\mathscr{A}^0 = \{H_1^0, \dots, H_n^0\}$ with $k \le n$. Let $U(n, k)$ be the set of affine hyperplanes $\mathscr{A} = \{H_1, \dots, H_n\}$ in V_0 satisfying the following two conditions:

(i) H_i is parallel to H_i^0 for $1 \le i \le n$;

(ii) $H_1, \dots, H_n$ are in general position.

Let $V = \mathbf{C}^n$ be the space of parallel translations of the affine hyperplanes of $\mathscr{A}^0$. Manin and Schechtman showed that $U(n, k)$ is the complementary space of an arrangement in V, which is described in the following manner. We denote by $C(n, a)$ the set of subsets of $\{1, \dots, n\}$ of cardinality a. For $K \in C(n, a)$ let D_K be the set of parallel translations $(H_1, \dots, H_n)$ considered as an element of V such that $\bigcap_{i \in K} H_i \neq \phi$. If $|K| \ge k + 1$, then we have

$$\operatorname{codim}_{\mathbf{C}} D_K = |K| - k.$$

In particular, for $J \in C(n, k+1)$, the set D_J is a hyperplane in V. The arrangement

$$\mathscr{B}(n, k) = \{ D_J;\ J \in C(n, k+1)\}$$

is called a *discriminantal arrangement* and it turns out that $U(n, k)$ is its complementary space

$$M(\mathscr{B}(n, k)) = V - \bigcup_{J \in C(n, k+1)} D_J.$$

The fundamental group $\pi_1(U(n, k))$ is called a *higher braid group* by Manin and Schechtman. Let us note that $\pi_1(U(n, 1))$ is the pure braid group with n strings.

For our purpose it is important to describe the codimension two elements of the associated geometric lattice $L(\mathscr{B}(n, k))$.

LEMMA 2.1 [11]. *The codimension two elements in $L(\mathscr{B}(n, k))$ are of one of the following two types*:

(i) D_K *for* $K \in C(n, k+2)$

(ii) $D_{J_1} \cap D_{J_2}$ *for* $J_1, J_2 \in C(n, k+1)$ *with* $|J_1 \cup J_2| \geq k+3$

Moreover, for $J \in C(n, k+1)$, *we have* $D_K \subset D_J$ *if and only if* $J \supset K$, *and we have* $D_{J_1} \cap D_{J_2} \subset D_J$ *if and only if* $J = J_1$ *or* $J = J_2$.

Combining (1.5) and (2.1), we obtain the following description of the completion of the group algebra of the higher braid groups, which is described in [11] by means of the nilpotent completion.

PROPOSITION 2.2. *Let* $\mathbf{C}\pi_1(U(n, k))^\wedge$ *denote the completion of the group algebra of the higher braid group with respect to the powers of the augmentation ideal. Then* $\mathbf{C}\pi_1(U(n, k))^\wedge$ *is isomorphic to* $\mathbf{C}\langle\langle X_J \rangle\rangle / \mathscr{I}$, *where* $\mathbf{C}\langle\langle X_J \rangle\rangle$ *denotes the ring of the non-commutative formal power series with indeterminates* X_J, $J \in C(n, k+1)$ *and* $\mathscr{I}$ *it its two-sided ideal generated by the following elements*:

(i) $[X_J, \Sigma_{I \subset K} X_I]$ *for any* $I \in C(n, k+1)$ *with* $K \supset J$;

(ii) $[X_{J_1}, X_{J_2}]$ *for any* J_1, J_2 *with* $|J_1 \cup J_2| \geq k+3$.

3. Construction of integrable connections

In this section we introduce a series of integrable connections defined over $U(n, k)$. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra and let ρ_i: $\mathfrak{g} \to \mathrm{End}(V)$, $1 \leq i \leq n$, be a family of finite dimensional irreducible representations of $\mathfrak{g}$. We denote by $\{I_\mu\}$ an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We consider $t = \Sigma_\mu I_\mu \otimes I_\mu$ in $\mathfrak{g} \otimes \mathfrak{g}$, which is also expressed as

$$t = \tfrac{1}{2}\{\Delta(c) - c \otimes 1 - 1 \otimes c\},$$

where Δ: $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the comultiplication of the universal enveloping algebra and c denotes the Casimir element in $U(\mathfrak{g})$. We define

$$\Omega_{ij} \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n), \quad 1 \leq i < j \leq n,$$

by

$$\Omega_{ij} = \sum_\mu 1 \otimes \cdots \otimes 1 \otimes \rho_i(I_\mu) \otimes 1 \otimes \cdots \otimes 1 \otimes \rho_j(I_\mu) \otimes 1 \otimes \cdots \otimes 1$$

where $\rho_i(I_\mu)$ and $\rho_j(I_\mu)$ act on the i-th and j-th components and all the other components are the identity transformations. For a subset $I \subset \{1, \ldots, n\}$ with

$|I| \geq 2$ we define $\Omega_I \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ by

$$\Omega_I = \sum_{\{i,j\} \subset I} \Omega_{ij}$$

where the RHS is the sum with respect to any $i < j$ such that $\{i, j\}$ is contained in I. As in the previous section, we denote by φ_J a linear form with $\mathrm{Ker}\,\varphi_J = D_J$ for $J \in C(n, k+1)$.

THEOREM 3.1. *With the above notations the* 1*-form*

$$\omega = \sum_{J \in C(n, k+1)} \lambda \Omega_J \, d \log \varphi_J$$

defines an integrable connection on the trivial vector bundle over $U(n, k)$ *with fiber* $V_1 \otimes \cdots \otimes V_n$ *for any complex parameter* λ, *and as the holonomy we obtain a one parameter family of linear representations of higher braid groups in the sense of Manin and Schechtman.*

Proof. In view of (1.2) and (2.1) it suffices to verify the following relations among Ω_J, $J \in C(n, k+1)$.

(3.2) For any $K \in C(n, k+2)$ with $K \supset J$, we have $[\Omega_J, \sum_{I \subset K} \Omega_I] = 0$
(3.3) If $|J_1 \cup J_2| \geq k+3$, then we have $[\Omega_{J_1}, \Omega_{J_2}] = 0$

We introduce the following notation. For any subset $S \subset \{1, \ldots, n\}$ we define the multi-diagonal map $\Delta_S \colon U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}$ to be the homomorphism of algebras determined by

$$\Delta_S(X) = \sum_{i \in S} 1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1$$

for $X \in \mathfrak{g}$, where X is situated on the i-th component and all the other components are the identities. First, we show the relation (3.2). We have

$$\sum_{I \subset K} \Omega_I = k \sum_{\{i,j\} \subset K} \Omega_{ij}$$

The expression $\sum_{\{i,j\} \subset K} \Omega_{ij}$ is equal to the action of

$$\frac{1}{2}\left\{\Delta_K(c) - \sum_{i \in K} 1 \otimes \cdots \otimes 1 \otimes c \otimes 1 \otimes \cdots \otimes 1\right\}$$

on $V_1 \otimes \cdots \otimes V_n$ via ρ_i, $1 \leq i \leq n$, where $1 \otimes \cdots \otimes 1 \otimes c \otimes 1 \otimes \cdots \otimes 1$ signifies that c is situated on the i-th component and the other components are the identity. By using the fact that the Casimir element c lies in the center of $U(\mathfrak{g})$, we obtain the relation

$$\left[\Omega_{kl}, \sum_{\{i,j\}\subset K} \Omega_{ij}\right] = 0 \quad \text{for any } \{k, l\} \subset K,$$

which implies (3.2). Now we show the relation (3.3). We put $J_\alpha = J \cup K_\alpha$ (disjoint union), $\alpha = 1, 2$. By the hypothesis we have $|K_\alpha| \geq 2$. We see that Ω_{J_α} is written as

$$\Omega_{J_\alpha} = \Omega_J + \Omega_{K_\alpha} + \Omega_{JK_\alpha}$$

Here Ω_{JK_α} stands for the sum $\sum_{j\in J, k\in K_\alpha}\Omega_{jk}$ and is equal to the action of

$$\sum_\mu \Delta_J(I_\mu) \cdot \sum_\nu \Delta_{K_\alpha}(I_\nu)$$

on $V_1 \otimes \cdots \otimes V_n$ via ρ_i, $1 \leq i \leq n$. Let us observe that for any subset $S \subset \{1, \ldots, n\}$ we have $[\Omega_S, \Delta_S(X)] = 0$ for any $X \in U(\mathfrak{g})$. By using this we obtain $[\Omega_J, \Omega_{JK_\alpha}] = 0$, $\alpha = 1, 2$ and $[\Omega_{JK_1}, \Omega_{JK_2}] = 0$. Combining these we get the relation (3.3). This completes the proof.

(3.4) *Final Remark.* The above construction shows that we can associate a one parameter family of linear representations of the higher braid group $\pi_1(U(n, k))$ to any simple Lie algebra $\mathfrak{g}$ and its finite dimensional irreducible representations ρ_i, $1 \leq i \leq n$. In the case $k = 1$, $\mathfrak{g}$ is non-exceptional and ρ_i is the vector representation, the representations of the braid group obtained as the holonomy of the above connection are expressed by means of solutions of Yang-Baxter equation (see [10]) and it turns out that these representations commute with the action of the quantized universal enveloping algebra due to Drinfel'd [4] and Jimbo [6]. This type of connection also appears in the conformal field theory on the sphere with gauge symmetry due to Knizhnik and Zamolodchikov [7], where the parameter λ is a special rational number. Its monodromy property was studied by Tsuchiya and Kanie [14] from the field theoretical viewpoint. In [11], Manin and Schechtman considers the hierarchy depending on the parameter k in relation with the higher braid groups. It would be an interesting problem to study the algebraic structure of the monodromy representations of the higher braid groups obtained as the holonomy of our connection.

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