ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 1990

HIGHER LOGARITHMS

BY

RICHARD M. HAIN[1] AND ROBERT MACPHERSON[2]

Dedicated to the memory of Professor Kuo-Tsai Chen

"And finally, in an attempt to unify the entire subject into a coherent whole, difficulties of a different order are encountered, and some central unifying principle has still to be discovered." (Lewin, Polylogarithms and Associated Functions [L] p. xv.)

1. Introduction

Few mathematicians would disagree with the assertion that the logarithm is one of the most important functions in mathematics. During the nineteenth century an analogous function, the *dilogarithm*, was the subject of much research. First defined by Leibnitz in 1696, the dilogarithm was subsequently studied by Euler, Spence, Abel, Hill, Jonquière Kummer, Lindelöf, Lobachevsky, and many others [L]. Recently there has been a resurgence of interest in this remarkable function, due in large part to the

Received March 15, 1989.

[1]Supported in part by grants from the National Science Foundation and an American Mathematical Society Fellowship.

[2]Supported in part by grants from the National Science Foundation.

work of Bloch [B1], [B2], [B3] in number theory and K-theory; Gabrielov, Gelfand and Losik [GGL] on the combinational formula for the first Pontrjagin class; Wigner (see [B2] and [Dp1]) on group cohomology; not to mention the scholarly work of L. Lewin [L]. Other recent work includes [A1], [A2], [A3], [Be1], [BGSV], [D4], [Dp1], [Dp2], [DS], [GM], [Li], [Lo], [M1], [M2], [R1], [R2], [R3], [Z1], [Z2].

The dilogarithm has properties analogous to those of the logarithm. It has been widely believed, both in the nineteenth century and more recently, that these two functions should be the first two elements of an infinite sequence of higher logarithms which share analogous properties. To date, several sequences of such functions have been proposed, but no function beyond the dilogarithm in any of these sequences is known to possess all the desired properties.

In this paper, we propose a new approach to constructing higher logarithms $\{L_p\}$ which will produce what we believe should be the true generalizations of the logarithm and the dilogarithm. The difficulty in this approach lies in constructing the functions L_p; once existence is established, the function will automatically possesses the desired properties. This is to be contrasted with the classical approach where the difficulty lies in establishing that given functions possess the sought after properties. As evidence for our program, we have constructed the first four functions; the first three being constructed in this paper.

In the epilogue we explain how a p-logarithm L_p appears as a component of an interesting cocycle whose class in the Deligne-Beilinson cohomology

$$H^{2p}_{\mathscr{D}}(G^p_\cdot, \mathbf{Q}(p))$$

of a certain simplicial space $G^p_\cdot$, of Zariski open subsets of various Grassmann manifolds, is a kind of universal pth Chern class. This provides an explanation for the importance of the dilogarithm in K-theory.

The fascinating and important number theoretic aspects of polylogarithms have been completely neglected in this paper. Nonetheless, we believe that our new trilogarithm function will possess interesting number theoretic properties analogous to those of the dilogarithm. An excellent survey of these remarkable properties of the dilogarithm can be found in [Z1].

The problem of generalizing the logarithm and the dilogarithm

The *logarithm* $\log x$, may be defined as the analytic continuation of the power series

$$(1.1) \qquad -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1$$

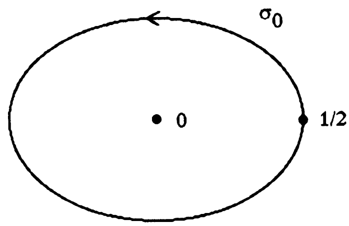


FIG. 1

to $\mathbf{C}^*$. It possesses three fundamental properties: one analytic, one topological, and one algebraic in nature.

Analytic property. The logarithm may be written as a line integral

$$\log x = \int_1^x \frac{dz}{z}$$

of a logarithmic 1-form on $\mathbf{C}^*$.

Topological property. The logarithm is a multivalued function on $\mathbf{C}^*$. Let σ_0 be homotopy class of loops based at (say) $1/2$ in $\mathbf{C}$ with winding number 1 about 0 (Fig. 1). Let $M(\sigma_0)$ be the monodromy operator whose value on a function is its analytic continuation along σ_0. Then

$$M(\sigma_0)\log x = \log x + 2\pi i.$$

In other words, $M(\sigma_0)$ acts on the two dimensional vector space of germs of functions at $z = 1/2$ with basis $\log x, 1$ through the matrix

$$M(\sigma_0) = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}.$$

The monodromy group is the discrete, 1-step unipotent group

$$\begin{pmatrix} 1 & \mathbf{Z}(1) \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{Z}(p)$ denotes the subgroup $(2\pi i)^p\mathbf{Z}$ of $\mathbf{C}$.

Algebraic property. The logarithm satisfies the three term functional equation

$$\log x - \log xy + \log y = 0$$

provided that appropriate branches have been chosen. The logarithm owes much of its utility to this functional equation which can be thought of as a 2-cocycle condition. When suitably interpreted, the logarithm represents the universal first Chern class.

The classical *dilogarithm function* $\ln_2(x)$, which we review in Section 4, may be defined as the analytic continuation of the power series

$$\ln_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \tag{1.2}$$

It has three properties analogous to those of the logarithm.

Analytic property. The dilogarithm may be written as an iterated line integral of logarithmic 1-forms of length two

$$\ln_2(x) = \int_0^x \frac{dz}{1-z}\frac{dz}{z} := \int_0^x \int_0^z \frac{dw}{1-w}\frac{dz}{z}$$

of the type studied systematically by K.-T. Chen (see [C2], for example). This expression shows that the dilogarithm can be analytically continued to any point of $\mathbf{C} - \{0, 1\}$.

Topological property. The dilogarithm is a multivalued function on $\mathbf{C} - \{0, 1\}$. Let σ_0 be the homotopy class of loops in $\mathbf{C} - \{0, 1\}$ based at $1/2$ that encircle 0, and σ_1 the homotopy class of loops based at $1/2$ that encircle 1 (Fig. 2). Denote the corresponding monodromy operators by $M(\sigma_0)$ and $M(\sigma_1)$. Then

$$M(\sigma_0)\ln_2(x) = \ln_2(x), \quad M(\sigma_1)\ln_2(x) = \ln_2(x) - 2\pi i \log x.$$

In other words, $M(\sigma_0)$ and $M(\sigma_1)$ act on the three dimensional vector space of germs of functions at $z = 1/2$ with basis $\ln_2(x)$, $\log x$, and 1 via the matrices

$$M(\sigma_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\pi i \\ 0 & 0 & 1 \end{pmatrix}, \quad M(\sigma_1) = \begin{pmatrix} 1 & -2\pi i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The monodromy group associated to $\ln_2(x)$ is the discrete, 2-step unipotent

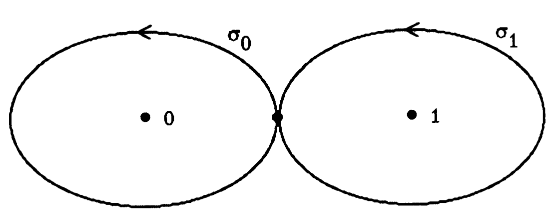


FIG. 2

group

$$\begin{pmatrix} 1 & \mathbf{Z}(1) & \mathbf{Z}(2) \\ 0 & 1 & \mathbf{Z}(1) \\ 0 & 0 & 1 \end{pmatrix}.$$

Algebraic property. If we define $\phi(x)$, a form of the dilogarithm due to Rogers [Ro], by

$$\phi(x) = \tfrac{1}{2}(\ln_2(x) - \ln_2(1-x)),$$

then $\phi(x)$ satisfies the 5 term functional equation

$$\phi(x) - \phi(y) + \phi(y/x) - \phi\left(\frac{y-1}{x-1}\right) + \phi\left(\frac{x(y-1)}{y(x-1)}\right) = \frac{\pi^2}{6}$$

provided that we choose the branches of each of the five terms carefully. The five functions

$$x, y, y/x, (y-1)/(x-1), \quad x(y-1)/y(1-x)$$

arise naturally as the cross ratios of the four element subsets of the configuration $(y, x, 1, 0, \infty)$ of five points on the projective line. A functional equation, equivalent to the one above, was discovered by Spence in 1809, rediscovered by Abel in 1828, and then again by many others. (The form above is due to Rogers [Ro].) When suitably interpreted, this five term equation is a 4-cocycle condition, and the cocycle associated to $\phi(x)$ represents the second Chern class in certain cases ([GGL], [B2], [GM], [Dp1], [DS], [Be1]).

The quest for higher logarithms

No red blooded mathematician could compare the properties of the logarithm and dilogarithm above without wondering if they were the first two terms of an infinite sequence of higher logarithms possessing the following properties:

Analytic property. The pth logarithm should be defined by integrating a closed iterated integral of logarithmic 1-forms of length $\leq p$.

Topological property. The pth logarithm should be a multivalued function whose associated monodromy group is discrete, and unipotent of length exactly p. (The last condition will imply, in particular, that the pth logarithm cannot be expressed as a polynomial of functions obtained by integrating iterated integrals of length $\leq (p-1)$.)

Algebraic property. The pth logarithm should satisfy a natural functional equation. Since it is expected that the pth logarithm will be naturally associated with the pth Chern class, this equation should be a $2p$-cocycle condition, and therefore be of the form

$$\sum_{j=0}^{2p} (-1)^j \mathscr{L}_p(A_j(x)) = 0,$$

where the $A_j(x)$ are algebraic functions from a variety into the domain of $\mathscr{L}_p$.

The classical higher logarithms, or *polylogarithms* as we shall call them, are the naive generalizations of the logarithm and dilogarithm obtained by extrapolating from (1.1) and (1.2):

$$\ln_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \quad |x| < 1.$$

These first appeared in the literature in the late eighteenth century (see [L]). The integral formula

$$\ln_p(x) = \int_0^x \ln_{p-1}(z)\frac{dz}{z} = \int_0^x \frac{dz}{1-z}\overbrace{\frac{dz}{z}\cdots\frac{dz}{z}}^{p-1}$$

shows that $\ln_p(x)$ can be analytically continued to a multivalued function on $\mathbf{C} - \{0, 1\}$ and that it is an iterated integral of length p. Its monodromy group is a discrete, unipotent group of length exactly p [R2]. The polylogarithms therefore possess the desired analytic and topological properties. As for the algebraic property, the lower polylogarithms satisfy many functional equations, but the number of terms in the functional equation of each considered most natural by Lewin [L; p. 239] is given in the following table.

p	1	2	3	4	5
No. of terms	3	5	9	20	33

No pattern in these equations is discernable and no generalization has been found for the higher polylogarithms.

Another definition of real higher logarithms was proposed in [GM] for even p. They are real valued functions defined on a real algebraic variety. These satisfy the algebraic property; however, it is not clear that these extend to complex valued functions defined on the complex points of their domains, nor that these functions would possess the analytic and topological properties. We hope that these functions will turn out to be the analogues of the

Bloch-Wigner-Ramakrishnan functions [R3], [Z2] of the higher logarithms proposed later in this paper.

Lewin, in his discussion of functional equations satisfied by the higher polylogarithms [L; pp. 238–240], comments that

"*It is difficult to believe that no formulas exist for the higher orders, but a radically new structure is necessary for further progress.*" (p. 238),

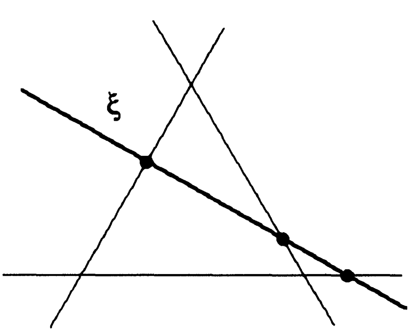
and

"*... the complexity of the present results make a completely new approach imperative if much progress is to be made.*" (p. 240).

Higher logarithms

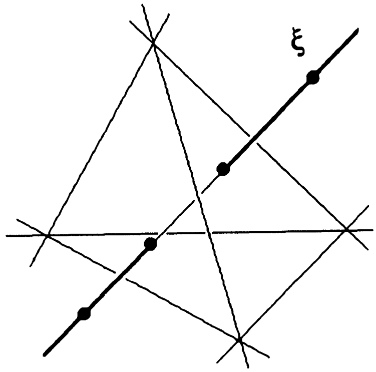
The higher logarithms that we propose will be functions, not of a single complex variable, but of a point in a complex algebraic manifold G^p_{p-1} of dimension p^2 which is an open subset of the self dual Grassmann manifold of $p-1$ dimensional linear subspaces of $\mathbf{P}^{2p-1}$. More precisely, G^p_q is the open subset of the Grassmann manifold $G(q, \mathbf{P}^{p+q})$ of q dimensional linear subspaces ξ of $\mathbf{P}^{p+q}$ that are transverse to the configuration of coordinate hyperplanes. Figures 3–5 depict elements of the real G^1_1, G^2_1 and G^2_0. In Fig. 3, the line ξ is required to avoid the vertices of the coordinate simplices, so $G^1_1 = \mathbf{C}^* \times \mathbf{C}^*$ with coordinates (a, b). In Fig. 4, ξ is required to avoid the edges of the coordinate simplices. In Fig. 5, ξ is required to avoid the coordinate axes, so that $G^2_0 = \mathbf{C}^* \times \mathbf{C}^*$.

Since the transversality requirement defining G^p_q is a generic condition, we call G^p_q the *generic part* of the grassmannian. The manifold G^p_{p-1} on which we expect the p-logarithm to be defined is "self dual".



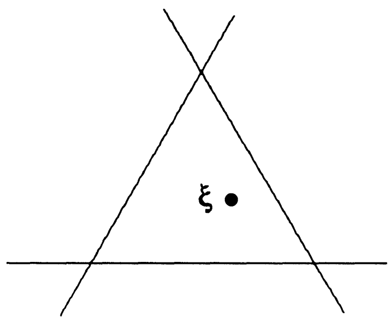
G^1_1

FIG. 3



G_1^2

FIG. 4



G_0^2

FIG. 5

The idea that higher logarithms should be functions on the G_q^p is due to Gelfand and MacPherson [GM]; their higher logarithms are defined on the real points of G_q^{2p}. Damiano [Da] showed that the only possibly non-zero Gelfand-MacPherson higher logarithms occur when $q = 2p - 1$.

When $q > 0$, there are $p + q + 1$ face maps A_i: $G_q^p \to G_{q-1}^p$; A_i takes the element $\xi \subseteq \mathbf{P}^{p+q}$ of G_q^p to its intersection with the ith coordinate hyperplane $\approx \mathbf{P}^{p+q-1}$ of $\mathbf{P}^{p+q}$. The face maps A_i: $G_1^1 \to G_0^1$ and A_i: $G_1^2 \to G_0^2$ are illustrated Figs. 6 and 7.

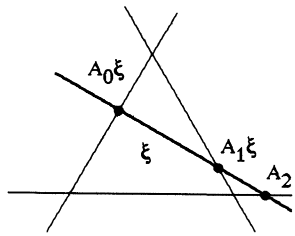


FIG. 6

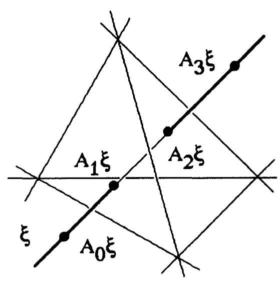


FIG. 7

In our approach, the existence of a p-logarithm function L_p will guarantee that it satisfies the functional equation

$$\sum_{j=0}^{2p}(-1)^j A_j^* L_p = 0, \tag{1.3}$$

where the A_j are the $2p+1$ face maps $G_p^p \to G_{p-1}^p$. Because the function L_p will be multivalued, one has to be very careful with branches. (These issues are dealt with in Section 2, where we lay down a categorical framework for dealing with multivalued functions and forms.)

The higher logarithm bicomplex

The p-logarithm function is a component of a cochain in a certain double complex which we now describe briefly. A detailed description of it is given in Sections 3 and 5.

For fixed p, the face maps $A_i\colon G_q^p \to G_{q-1}^p$ satisfy the usual identities dual to those that hold between the faces of a simplex. This means that $\{G_q^p\}_{q=0,\ldots,p}$ is a truncated simplicial variety $G_\bullet^p$. It is natural to put G_q^p in dimension $p+q$ as there are $p+q+1$ face maps emanating from it. If we apply a contravariant, abelian group valued functor to $G_{\cdot}^{p}$, we will obtain a cochain complex with differential

$$A^* = \sum_{i=0}^{p} (-1)^i A_i^*.$$

More generally, if we apply a contravariant, cochain complex valued functor, we will obtain a double complex. We will apply the multivalued de Rham complex functor $\tilde{\Omega}^{\cdot}$ which is constructed in detail in Section 3.

Briefly, the complex of multivalued differential forms on a complex algebraic manifold X is

$$\tilde{\Omega}^{\cdot}(X) = \tilde{\mathscr{O}}(X) \otimes \Omega^{\cdot}(X),$$

where $\Omega^{\cdot}(X)$ denotes the holomorphic forms on X with logarithmic singularities at infinity, and $\tilde{\mathscr{O}}(X)$ consists of all multivalued functions on X obtained by integrating a relatively closed iterated integral[3] of elements of $\Omega^1(X)$. Both $\tilde{\mathscr{O}}(X)$ and $\tilde{\Omega}^{\cdot}(X)$ come equipped with a canonical filtration, called the *weight filtration*[4]; if $X = G_q^p$, then $W_{2l}\tilde{\mathscr{O}}(X)$ consists of those functions obtained by integrating iterated integrals of length not exceeding l. Thus, for example,

$$\log x = \int_1^x \frac{dz}{z} \in W_2\tilde{\mathscr{O}}(\mathbf{C}^*),$$

$$\ln_p(x) = \int_0^x \frac{dz}{1-z} \overbrace{\frac{dz}{z} \cdots \frac{dz}{z}}^{p-1} \in W_{2p}\tilde{\mathscr{O}}(\mathbf{C} - \{0,1\}).$$

Combining these filtrations defines a weight filtration on $\tilde{\Omega}^{\cdot}(X)$.

Neglecting the problem of choosing branches, which is dealt with in Section 5, we obtain a double complex

$$\left(W_{2p}\tilde{\Omega}^{\cdot}(G_{\cdot}^{p}), D\right)$$

by applying $W_{2p}\tilde{\Omega}^{\cdot}$ to the simplicial space $G_{\cdot}^{p}$. The differential D is the total

[3]An iterated line integral I is *relatively closed* if its value on a path γ depends only on the homotopy class of γ relative to its endpoints.

[4]This is the weight filtrations in the sense of Deligne [D1].

differential $d \pm A^*$. For example, the double complex for $p = 3$ is

$$\begin{array}{ccccccc}
W_6\tilde{\mathcal{O}}(G_3^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_3^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_3^3) & \longrightarrow & \Omega^3(G_3^3) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
W_6\tilde{\mathcal{O}}(G_2^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_2^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_2^3) & \longrightarrow & \Omega^3(G_2^3) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
W_6\tilde{\mathcal{O}}(G_1^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_1^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_1^3) & \longrightarrow & \Omega^3(G_1^3) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
W_6\tilde{\mathcal{O}}(G_0^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_0^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_0^3) & \longrightarrow & \Omega^3(G_0^3).
\end{array}$$

The manifold G_0^p is just $\mathbf{P}^p$ minus the union of the coordinate hyperplanes, and is isomorphic to $(\mathbf{C}^*)^p$. On this there is a canonical p-form, the "volume" form:

$$\mathrm{vol}_p = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_p}{x_p} \in \Omega^p(G_0^p).$$

Definition. A *p-logarithm* is a $2p-1$ cochain Z_p in the Grassmann bicomplex $W_{2p}\tilde{\Omega}^{\cdot}(G_{\cdot}^p)$ that satisfies the equation $DZ_p = \mathrm{vol}_p$.

Note that it is not at all clear that such a cochain Z_p exists. The first obstruction is that vol_p be closed in the double complex as

$$A^*\,\mathrm{vol}_p = D\,\mathrm{vol}_p = D^2(Z_p) = 0.$$

This can be verified by direct computation for small values of p.

Theorem (9.7). *For all p, $A^*\,\mathrm{vol}_p = 0$.*

This we prove by taking residues and exploiting the action of the symmetric group on G_1^p. P. Cartier [Ca] has given a very elegant proof of the vanishing of $A^*\,\mathrm{vol}_p$ using Cartan's theory of basic forms.

Note that the component L_p of a p-logarithm Z_p in $W_{2p}\tilde{\mathcal{O}}(G_{p-1}^p)$ satisfies

(i) $A^*L_p = \sum_{i=0}^{2p}(-1)^i A_i^* L_p = 0$,

(ii) L_p is obtained by integrating a relatively closed iterated integral of logarithmic 1-forms of length not exceeding p.

We shall call L_p a *p-logarithm function*. It clearly possesses the desired analytic and algebraic properties. As for the topological property, the analytical property implies that L_p has unipotent monodromy group of length $\leq p$.

To say that the group has length exactly p is equivalent to showing that L_p is indecomposable; that is, L_p cannot be expressed as a polynomial of elements of $W_{2p-2}\tilde{\mathscr{O}}(G_{p-1}^p)$.

Neither the cochain Z_p nor the function L_p is unique. The cochain can be adjusted by coboundaries and the function by functions of the form A^*G, where $G \in W_{2p}\tilde{\mathscr{O}}(G_{p-2}^p)$, which satisfy the functional equation trivially. (One can find more canonical representatives with the aid of the symmetric group. This is discussed in Section 9.)

Here is our main result.

THEOREM. *For $p = 1, 2, 3$, there is a p-logarithm which is unique modulo coboundaries. In each case, the associated p-logarithm function is indecomposable and non-trivial in*

$$W_{2p}\tilde{\mathscr{O}}(G_{p-1}^p)/A^*W_{2p}\tilde{\mathscr{O}}(G_{p-2}^p).$$

When $p = 1$, the cochain $\log x \in W_2\tilde{\mathscr{O}}(\mathbf{C}^*)$ represents Z_1 and the functional equation $A^* \log x = 0$ is the usual one. (See (6.3).)

At first glance it seems that the 2-logarithm function cannot be the classical dilogarithm or Rogers' function $\phi(x)$ as L_2 is defined on G_1^2 while $\phi(x)$ is defined on $\mathbf{C} - \{0, 1\}$. However, an element of G_1^2 is a line in $\mathbf{P}^3$ that intersects the four coordinate hyperplanes in four distinct points. Taking ξ to the cross ratio of these four points defines a function

$$\pi\colon G_1^2 \to \mathbf{C} - \{0, 1\}.$$

In (6.4) we show that there is a representative of Z_2 where

$$L_2 = \pi^*\phi - \pi^2/6.$$

Similarly, there is a projection of G_2^2 onto the domain of the functional equation of $\phi(x)$, and the functional equation $A^*L_2 = 0$ is just the pullback of Rogers' functional equation.

Both the logarithm and the dilogarithm have single, real valued cousins which also satisfy natural functional equations. In the case of the logarithm, this function is the logarithm of the absolute value $D_1\colon \mathbf{C}^* \to \mathbf{R}$ defined by $z \mapsto \log|x|$. It satisfies the functional equation

$$D_1(x) - D_1(xy) + D_1(y) = 0$$

and possesses the *symmetry* property

$$\sigma^*D_1 = \operatorname{sgn}(\sigma) D_1$$

for all σ in the symmetric group on 2 letters, Σ_2, which acts on $\mathbf{C}^*$ by letting the generator take z to z^{-1}. The functional equation implies that D_1 represents a cohomology class in $H^1(GL(\mathbf{C}), \mathbf{R})$. This class is the first Cheeger-Simons Chern class $\hat{c}_1$ of the universal flat bundle over $BGL(\mathbf{C})^\delta$, the classifying space of stable, flat complex vector bundles. It also defines the first regulator mapping r_1: $K_1(\mathbf{C}) = \mathbf{C}^* \to \mathbf{R}$.

The single valued cousin of the dilogarithm is the *Bloch-Wigner function* [B1] (see also [Z1])

$$D_2 \colon \mathbf{C} - \{0,1\} \to \mathbf{R}$$

which is defined by

$$D_2(z) = \operatorname{Im} \ln_2(z) + \arg(1-z)\log|z|.$$

It satisfies the 5-term functional equation

$$D_2(x) - D_2(y) + D_2(y/x) - D_2\left(\frac{y-1}{x-1}\right) + D_2\left(\frac{x(y-1)}{y(x-1)}\right) = 0.$$

Viewing $\mathbf{C} - \{0, 1\}$ as the space of ordered 4-tuples of distinct points on the projective line modulo projective equivalence, we see that Σ_4, the symmetric group on 4 letters, acts naturally on $\mathbf{C} - \{0, 1\}$. The Bloch-Wigner function satisfies the symmetry condition

$$\sigma^* D_2 = \operatorname{sgn}(\sigma) D_2$$

for all $\sigma \in \Sigma_4$. Just as D_1 represents $\hat{c}_1$, D_2 represents the second Cheeger-Simons Chern class

$$\hat{c}_2 \in H^3(GL(\mathbf{C}), \mathbf{R})$$

of the universal flat bundle over $BGL(\mathbf{C})^\delta$ [Dp1], and, by composition with the Hurewicz homomorphism $K_3(\mathbf{C}) \to H_3(GL(\mathbf{C}))$, it defines the regulator mapping r_2: $K_3(\mathbf{C}) \to \mathbf{R}$ [B2].

In Section 11 we construct the single valued cousin of the 3-logarithm function constructed in Section 8. It is a single valued function D_3: $Y_2^3 \to \mathbf{R}$, where

$$Y_q^3 = \left\{ \begin{matrix} \text{ordered } (4+q)\text{-tuples of} \\ \text{points in } \mathbf{P}^2, \text{ no 3 on a line} \end{matrix} \right\} \Big/ \text{projective equivalence}.$$

The seven ways of forgetting a point give 7 maps A_j: $Y_3^3 \to Y_2^3$. We show that D_3 satisfies the functional equation

$$\sum_{j=0}^{6} (-1)^j A_j^* D_3 = 0$$

and that it possesses the symmetry property

$$\sigma^* D_3 = \operatorname{sgn}(\sigma) D_3$$

for all $\sigma \in \Sigma_6$, where Σ_6 acts on Y_2^3 by permuting the points. We expect that D_3 represents the third universal Cheeger-Simons Chern class $\hat{c}_3 \in H^5(GL(\mathbf{C}), \mathbf{R})$ and that the composition of the Hurewicz homomorphism

$$K_5(\mathbf{C}) \to H_5(GL(\mathbf{C}))$$

with D_3 is the regulator mapping r_3: $K_5(\mathbf{C}) \to \mathbf{R}$. As a step towards establishing these properties, J. Yang has proved that D_3 represents an element of $H^5_{\mathrm{cts}}(GL_3(\mathbf{C}), \mathbf{R})$ and thus defines a map $K_5(\mathbf{C}) \to \mathbf{R}$ which is necessarily a multiple of Borel's regulator.

The definition of higher logarithms given above is only part of a more complicated definition of higher logarithms as cocycles in the multivalued Deligne-Beilinson complex of the simplicial variety $G^p_\cdot$. We give this definition in the Epilogue and sketch a proof of the result that, when $p = 1, 2, 3$, the Deligne-Beilinson cohomology group

$$H^{2p}_{\mathscr{D}}(G^p_\cdot, \mathbf{Q}(p))$$

is isomorphic to $\mathbf{Q}$ and spanned by the class of the p-logarithm. The expected role of these classes as universal Chern classes for algebraic K-theory is explained in [BMS].

This paper is the result of rumination over work done at the Institute for Advanced Study in 1985–86 and at the Institut des Hautes Études Scientifiques in the spring of 1986. We thank both these institutions for their hospitality and support. The first author would also like to gratefully acknowledge the support of an American Mathematical Society Fellowship.

It is great pleasure to thank A. Beilinson, P. Cartier, P. Deligne, M. Goresky, M. Levine, J.-L. Loday, M. Nori, A. Nicas, D. Ramakrishnan, V. Schechtman, and the many others with whom we have had fruitful conversations related to this work. Special thanks go to Pierre Cartier and Jun Yang, who both read the manuscript carefully and saved us from many an embarrassment. Needless to say, all remaining errors and misprints are our responsibility. We also thank Frances Chen for her patient and careful typing of this manuscript.

2. Multivalued differential forms

The complex of holomorphic differential forms on a complex manifold M will be denoted by $E^{\cdot}M$ and the category of connected complex manifolds and holomorphic maps by $\mathscr{A}n$. A *multivalued differential form* on M, a complex manifold, is a holomorphic differential form on its universal covering manifold $\tilde{M}$. To ensure that multivalued forms and their pullbacks under holomorphic maps are well defined, we have to enrich the category $\mathscr{A}n$. Define $\widetilde{\mathscr{A}n}$ to be the category whose objects are universal covering projections π: $\tilde{M} \to M$ and whose morphisms f: $\pi \to \pi'$ are commutative squares

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M}' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M'. \end{array}$$

We will usually abuse notation and represent objects of $\widetilde{\mathscr{A}n}$ by the underlying manifold M. The complex $\tilde{E}^\cdot(\pi)$ of multivalued differential forms on π: $\tilde{M} \to M$ is defined to be $E^\cdot\tilde{M}$ and will frequently be denoted by $\tilde{E}^\cdot M$. The multivalued de Rham complex functor $M \to \tilde{E}^\cdot M$ defines a contravariant functor

$$\tilde{E}^\cdot : \widetilde{\mathscr{A}n} \to \text{d.g. algebras}, \tag{2.1}$$

from $\widetilde{\mathscr{A}n}$ into the category of differential graded algebras. We can define the fundamental group $\pi_1(M)$ of an object $\tilde{M} \to M$ of $\widetilde{\mathscr{A}n}$ to be the set of deck transformations of $\tilde{M} \to M$. Thus we have a short exact sequence of groups

$$1 \to \pi_1(M) \to \mathrm{Aut}_{\widetilde{\mathscr{A}n}} M \to \mathrm{Aut}_{\mathscr{A}n} M \to 1$$

and the action of $\pi_1(M)$ on $\tilde{M}$ gives $\tilde{E}^\cdot M$ the structure of a right $\pi_1(M)$ module. The assignment of $\pi_1(M)$ to $\tilde{M} \to M$ defines a functor

$$\widetilde{\mathscr{A}n} \to \text{Groups}. \tag{2.2}$$

We now give an alternative and more concrete description of the category $\widetilde{\mathscr{A}n}$ and the functor $\tilde{E}^\cdot$. Define the category $\mathscr{A}n_*$ to be the category whose objects are pairs (M, x), where M is a connected complex manifold and $x \in M$. The morphisms $(M, x) \to (N, y)$ of $\mathscr{A}n_*$ are pairs (f, γ), where f: $M \to N$ is holomorphic and γ is a homotopy class of paths in N from y to $f(x)$.

The standard construction of a pointed universal covering $(\tilde{M}, \tilde{x})$ of a pointed manifold (M, x) as the set of homotopy classes of paths in M

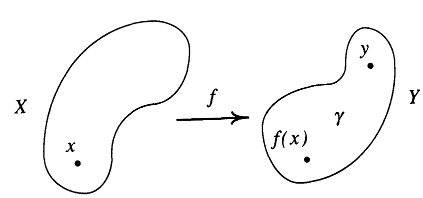


FIG. 8

emanating from x defines a functor $\mathscr{A}n_* \to \widetilde{\mathscr{A}n}$. This is an equivalence of categories. The basepoint $\tilde{x}$ of $\tilde{M}$ fixes an action of $\pi_1(M, x)$ on $\tilde{M}$. Consequently, the composite of $\mathscr{A}n_* \to \widetilde{\mathscr{A}n}$ with the fundamental group functor (2.2) defines a functor $\mathscr{A}n_* \to$ Groups that takes (M, x) to $\pi_1(M, x)$.

Define the multivalued de Rham complex functor

$$\tilde{E}^{\cdot}: \mathscr{A}n_* \to \text{d.g. algebras}$$

to be the composite of the functor $\mathscr{A}n_* \to \widetilde{\mathscr{A}n}$ with the multivalued de Rham complex functor (2.1). This functor has an alternative description. Let (M, x) be an object of $\mathscr{A}n_*$ and $(\tilde{M}, \tilde{x}) \to (M, x)$ the corresponding object of $\widetilde{\mathscr{A}n}$. By identifying a neighbourhood of $\tilde{x}$ in $\tilde{M}$ with a neighbourhood of x in M, we see that a multivalued differential form on (M, x) can be viewed as the germ of a holomorphic differential form on M at x that admits analytic continuation along all paths in M. The pullback of a multivalued form ω on (N, y) along a morphism (f, γ): $(M, x) \to (N, y)$ can be computed as follows: First analytically continue ω along γ from y to $f(x)$ and then pull back the resulting germ to obtain a germ of a form at x. The right action of $\pi_1(M, x)$ on $\tilde{E}^{\cdot}(M, x)$ also admits a simple description. If $g \in \pi_1(M, x)$ and $\omega \in \tilde{E}^{\cdot}(M, x)$, then $\omega \cdot g$ is obtained by analytically continuing ω around a loop in M that represents g.

Finally we remark that the fundamental group functor $\mathscr{A}n_* \to$ Groups takes the morphism (f, γ): $(M, x) \to (N, y)$ to the composite

$$\pi_1(M, x) \xrightarrow{f_\#} \pi_1(N, f(x)) \xrightarrow{\phi_\gamma} \pi_1(N, y),$$

where ϕ_γ is the natural isomorphism defined by γ.

(2.3) Denote the category of smooth irreducible complex algebraic varieties and morphisms by $\mathscr{A}$. This category can be enriched in the same way as $\mathscr{A}n$ to obtain equivalent categories $\tilde{\mathscr{A}}$ and $\mathscr{A}_*$.

(2.4) Examples of multivalued functions can be constructed using iterated integrals. Suppose that M is a complex manifold. Denote the space of piecewise smooth paths γ: $[0, 1] \to M$ by PM. Let π: $PM \to M \times M$ be the function that takes a path γ to its endpoints $(\gamma(0), \gamma(1))$. Let $P_{x,y}M$ be the fiber $\pi^{-1}(x, y)$. Let $\{\omega_i$: $i \in A\}$ be a finite set of holomorphic 1-forms on M and

$$I = \sum_{s=0}^{n} \sum_{K \in A^s} a_K \int \omega_{k_1} \ldots \omega_{k_s}, \quad a_K \in \mathbf{C}, \tag{2.5}$$

be an iterated line integral on PM. We say that I is *relatively closed* if its restriction to each $P_{x,y}M$ is closed. That is, the value of I on a path γ depends only on its homotopy class relative to its endpoints. For each choice of basepoint $x \in M$, I defines a multivalued holomorphic function $f_{I,x}$ on M. Its value at a point p of a simply connected neighbourhood U of x is

$$f_{I,x}(p) = I(\gamma)$$

where $\gamma \in P_{x,p}U$.

Chen [C2; (1.5.2)] has proved that I is relatively closed if and only if $dI = 0$, where

$$\begin{aligned} d\int \omega_1 \dots \omega_r = &- \sum_{i=1}^{r} \int \omega_1 \dots \omega_{i-1}\, d\omega_i \omega_{i+1} \dots \omega_r \\ &- \sum_{i=1}^{r-1} \int \omega_1 \dots \omega_{i-1} (\omega_i \wedge \omega_{i+1}) \omega_{i+2} \dots \omega_r. \end{aligned}$$

(2.6) The formula

$$\int_{\alpha\beta} \omega_1 \dots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \dots \omega_i \int_\beta \omega_{i+1} \dots \omega_r$$

([C1], [C2; (2.2.2)], see also [H1; (2.9)]) implies that the analytic continuation of $f_{I,x}$ along the path γ from x to y is $f_{J,y}$, where I is given by (2.5) and

$$J = \sum_{s=0}^{n} \sum_{|K|=s} \sum_{i=0}^{s} \left(a_K \int_\gamma \omega_{k_1} \dots \omega_{k_i} \right) \int \omega_{k_{i+1}} \dots \omega_{k_2}.$$

3. A class of multivalued forms appropriate for algebraic geometry

Suppose that X is a smooth algebraic variety. As is well known [Hk], there is a compact algebraic manifold $\overline{X}$ that contains X as a Zariski open subset and such that $D = \overline{X} - X$ is a divisor in $\overline{X}$ with normal crossings. (That is, locally D has equation $z_1 z_2 \dots z_r = 0$ with respect to local holomorphic coordinates $(z_1, \dots, z_n)$ in X.)

A holomorphic p-form ω on X is said to have *logarithmic singularities at infinity* (with respect $X \hookrightarrow \overline{X}$) if it is meromorphic on $\overline{X}$ and locally ω is of the form

$$\sum_{k=0}^{p} \sum_J \varphi_J \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_k}}{z_{j_k}}, \tag{3.1}$$

where φ_J is holomorphic on U, a coordinate neighbourhood in $\overline{X}$ in which D has equation

$$z_1 z_2 \dots z_r = 0$$

with respect to the holomorphic coordinates $(z_1, \dots, z_r, w_1, \dots, w_s)$. Following Deligne [D1], we say that ω has *weight* $\leq l$ if each term of (3.1) has fewer than $l - p$ logarithmic terms (i.e., $k \leq l - p$). In particular, ω has weight p if and only if it extends to a holomorphic p-form on $\overline{X}$.

Denote the vector space of holomorphic p-forms on X with logarithmic singularities at infinity by $\Omega^p(\overline{X} \log D)$ and those of weight $\leq l$ by $W_l\Omega^{\cdot}(\overline{X} \log D)$. The former comprises a complex $\Omega^{\cdot}(\overline{X} \log D)$ which has the following properties due to Deligne [D1; (3.2.14)].

(3.2) PROPOSITION. (i) *Every element of* $\Omega^{\cdot}(\overline{X} \log D)$ *is closed.*
(ii) *The natural map* $\Omega^{\cdot}(\overline{X} \log D) \to H^{\cdot}(X, \mathbf{C})$ *is injective. In particular*

$$\dim \Omega^{\cdot}(\overline{X} \log D) < \infty.$$

(iii) *The image of* $W_l\Omega^p(\overline{X} \log D) \to H^p(X, \mathbf{C})$ *is* $F^pW_lH^p(X, \mathbf{C})$, *where* $F^{\cdot}$ *and* $W_{.}$ *denote the Hodge and weight filtrations of the canonical mixed Hodge structure on* $H^{\cdot}(X)$. ∎

(3.3) COROLLARY. *The image of the natural inclusion* $W_l\Omega^{\cdot}(\overline{X} \log D) \to E^{\cdot}X$ *is independent of the compactification* $X \to \overline{X}$.

Proof. If Y, Z are compact algebraic manifolds with normal crossing divisors $E \subseteq Y$, $F \subseteq Z$ such that

$$X = Y - E = Z - F,$$

then there is a third compact algebraic manifold $\overline{X}$ containing a normal crossings divisor D and morphisms $f\colon \overline{X} \to Y$, $g\colon \overline{X} \to Z$ which induce isomorphisms

$$\overline{X} - D \to Y - E \quad \text{and} \quad \overline{X} - D \to Z - F.$$

Consequently, we have a commutative diagram

$$\begin{tikzcd}
W_l\Omega^{\cdot}(\overline{X} \log D) \arrow[drr] & & W_l\Omega^{\cdot}(Z \log F) \arrow[ll, "g^*"'] \arrow[d] \\
W_l\Omega^{\cdot}(Y \log E) \arrow[u, "f^*"] \arrow[rr] & & E^{\cdot}X.
\end{tikzcd}$$

Since each map in this diagram is injective, the result follows from (3.2)(iii). ∎

In other words, the notion of a holomorphic form on a smooth variety X having logarithmic singularities at infinity is independent of the compactification $X \to \overline{X}$ chosen, and we denote the algebra of such forms by $\Omega^{\cdot}(X)$.

Likewise, the weight filtration $W_{\cdot}$ of $\Omega^{\cdot}(X)$ is independent of the compactification and we obtain a functor $(\Omega^{\cdot}(X), W_{\cdot})$ from the category $\mathscr{A}$ of smooth varieties into the category of filtered, graded commutative algebras.

Our complex of multivalued forms will be defined as $\tilde{\mathscr{O}}$ linear combinations of elements of $\Omega^{\cdot}(X)$, where $\tilde{\mathscr{O}}$ is a ring of multivalued functions that we now define.

Denote the vector space of all iterated integrals of the form

$$\sum \int \omega_{i_1}\omega_{i_2}\ldots\omega_{i_r}, \tag{3.4}$$

where each $\omega_{i_j} \in \Omega^1(X)$, by $A(X)$. View $A(X)$ as a subalgebra of the smooth functions $PX \to \mathbf{C}$. Pointwise multiplication of functions induces the shuffle product

$$\int \omega_1 \ldots \omega_r \otimes \int \omega_{r+1} \ldots \omega_{r+s} \mapsto \sum_{\sigma} \int \omega_{\sigma(1)} \ldots \omega_{\sigma(r+s)} \tag{3.5}$$

where σ ranges over the shuffles of type (r, s) ([C1], see also [H1; (2.11)]).

The homomorphism

$$\Phi\colon T(\Omega^1(X)) \to A(X), \qquad \omega_1 \otimes \ldots \otimes \omega_r \mapsto \int \omega_1 \ldots \omega_r$$

from the tensor algebra on $\Omega^1(X)$ into $A(X)$ is clearly surjective. Applying Chen's algebraic description of iterated integrals [C2; (4.1.1)] we obtain:

(3.6) PROPOSITION. (a) *Φ is an isomorphism.*

(b) *A homogeneous iterated integral of length s*

$$I_s = \sum_{|J|=s} \int \omega_{j_1} \ldots \omega_{j_s} \in A(X)$$

is relatively closed if and only if, for each integer $r \in [1, s-1]$,

$$\sum_{|J|=s} \omega_{j_1} \otimes \ldots \otimes \omega_{j_{r-1}} \wedge \omega_{j_r} \otimes \ldots \otimes \omega_{j_s} = 0$$

in

$$\left[\bigotimes^{r-1} H^1(X)\right] \otimes H^2(X) \otimes \left[\bigotimes^{s-r-1} H^1(X)\right]. \qquad \blacksquare$$

The weight filtration

$$0 = W_0 \subseteq W_1 \subseteq W_2 = \Omega^1(X)$$

of $\Omega^1(X)$ extends to a weight filtration $W_\cdot$ of the tensor algebra $T(\Omega^1(X))$ and hence to $A(X)$. The shuffle product (3.5) preserves the weight filtration. That is, the image of $W_iA(X) \otimes W_jA(X)$ is contained in $W_{i+j}A(X)$. An algebra with a filtration $W_\cdot$ satisfying this condition will be called a *filtered* algebra.

Denote the space of relatively closed elements of $A(X)$ by $H^0(A(X))$. This is a subalgebra of $A(X)$. The weight filtration of $A(X)$ restricts to a weight filtration of $H^0(A(X))$.

As in (2.5), for each choice of basepoint $x \in X$, the linear map

$$(H^0)(A(X)) \to \tilde{E}^0(X, x)$$

that takes I to $f_{I,x}$ is an injective algebra homomorphism. Denote its image by $\tilde{\mathscr{O}}(X, x)$. It is a filtered algebra.

The following result follows directly from (2.6).

(3.7) PROPOSITION. *If γ is a pth in X from x to y, then the natural map*

$$\tilde{E}^0(X, x) \to \tilde{E}^0(X, y)$$

defined by analytically continuing germs along γ restricts to a filtration preserving algebra homomorphism

$$\tilde{\mathscr{O}}(X, x) \to \tilde{\mathscr{O}}(X, y).$$

Consequently, the assignment of $\tilde{\mathscr{O}}(X, x)$ to (X, x) defines functors from the categories $\mathscr{A}_*$ and $\tilde{\mathscr{A}}$ (see (2.3)) into the category of filtered commutative algebras. ∎

The *multivalued algebraic de Rham complex* of the object X of $\tilde{\mathscr{A}}$ is defined to be

$$\tilde{\Omega}^\cdot(X) = \tilde{\mathscr{O}}(X) \otimes_{\mathbf{C}} \Omega^\cdot(X). \tag{3.8}$$

This has a natural weight filtration defined by

$$W_l\Omega^\cdot(X) = \sum_{i+j=l} W_i\tilde{\mathscr{O}}(X) \otimes W_j\Omega^\cdot(X).$$

To justify calling $\tilde{\Omega}^\cdot(X)$ a complex of multivalued forms, one needs to show that the natural homomorphism

$$\tilde{\Omega}^\cdot(X) \to \tilde{E}^\cdot(X)$$

is injective. This follows from another application of the algebraic characterization of iterated integrals [C2; (4.2.1)]. The differential of $\tilde{E}^{\cdot}(X)$ thus induces a differential on $\tilde{\Omega}^{\cdot}(X)$. The following proposition can be proved, either directly using the definition of iterated integrals or by using the general formula [C2; (1.5.2)] for the differential of an iterated integral.

(3.9) PROPOSITION. *The differential of $\tilde{\Omega}^{\cdot}(X, x)$ is induced by the linear map*

$$\nabla\colon A(X) \otimes \Omega^{\cdot}(X) \to A(X) \otimes \Omega^{\cdot}(X),$$
$$(\omega_1 \otimes \dots \otimes \omega_r) \otimes \omega \mapsto (\omega_1 \otimes \dots \otimes \omega_{r-1}) \otimes \omega_r \wedge \omega. \qquad \blacksquare$$

(3.10) COROLLARY. *The association of $\tilde{\Omega}^{\cdot}(X)$ to an object X of $\tilde{\mathscr{A}}$ defines a functor from $\tilde{\mathscr{A}}$ into the category of filtered, commutative differential graded algebras. In particular, each term of the weight filtration of $\tilde{\Omega}^{\cdot}(X)$ is a subcomplex.* $\blacksquare$

Since $\Omega^1(X)$ is finite dimensional, each $W_l\tilde{\mathscr{O}}(X)$ is a finite dimensional complex vector space. Analytically continuing elements of $W_l\tilde{\mathscr{O}}(X)$ about loops based at $x \in X$ defines a monodromy representation

$$\pi_1(X, x) \to \operatorname{Aut} W_l\tilde{\mathscr{O}}(X).$$

It follows from (2.6) that this action is trivial mod $W_{l-1}\tilde{\mathscr{O}}(X)$ which proves:

(3.11) PROPOSITION. *Each $W_l\tilde{\mathscr{O}}(X)$ is a unipotent $\pi_1(X, x)$ module.* $\blacksquare$

(3.12) *Examples.* (i) If X is a smooth variety and $f \in \mathscr{O}^*(X) := H^0(X, \mathscr{O}_X^*)$ is an invertible function, then

$$\log f(x) = \log f(x_0) + \int_{x_0}^{x} \frac{df}{f} \in W_2\tilde{\mathscr{O}}(X).$$

(ii) The dilogarithm is given by the formula

$$\ln_2(x) = \int_0^x \frac{dz}{1-z}\frac{dz}{z} \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\}),$$

the kth polylogarithm by

$$\ln_k(x) = \int_0^x \frac{dz}{1-z}\overbrace{\frac{dz}{z}\cdots\frac{dz}{z}}^{k-1} \in W_{2k}\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\}).$$

(3.13) *Remark.* Define the *irregularity* $q(X)$ of a smooth variety X to be

$$\dim H^0(\overline{X}, \Omega^1_{\overline{X}}) = \dim \Omega^1(\overline{X}),$$

where $\overline{X}$ is any smooth completion of X. According to Deligne (3.2)(iii), $q(X) = \dim W_1\Omega^1(X)$. Thus, if $q(X) = 0$, then $\Omega^1(X)$ is purely of weight 2. In this case the weight filtration of $\tilde{\mathscr{O}}(X, x)$ is essentially its filtration by length:

$$W_l\tilde{\mathscr{O}}(X, x) \cong \left\{ \begin{array}{l} \text{closed iterated integrals in } A(X) \text{ of} \\ \text{length} \le s \text{ when } l = 2s, 2s+1. \end{array} \right\}$$

The irregularity of every Zariski open subset of a simply connected variety is zero. In particular, every Zariski open subset of a grassmannian has irregularity zero.

The tensor algebra $T(\Omega^1(X))$ is graded by tensor length. The isomorphism Φ induces a grading

$$A(X) = \bigoplus_{l=0}^{\infty} A_l(X)$$

of $A(X)$. By (3.6)(b), this grading passes to a grading

$$H^0(A(X)) = \bigoplus_{l=0}^{\infty} H^0(A_l(X))$$

of the relatively closed elements of $A(X)$. When $q(X) = 0$,

$$Gr^W_{2l-1}\tilde{\mathscr{O}}(X) = 0 \quad \text{and} \quad Gr^W_{2l}\tilde{\mathscr{O}}(X) = H^0(A_l(X)).$$

For example, if $q(X) = 0$, then

$$Gr^W_2\tilde{\mathscr{O}}(X) \cong H^1(X),$$
$$Gr^W_4\tilde{\mathscr{O}}(X) \cong \ker\{\otimes^2 H^1(X) \xrightarrow{\text{cup}} H^2(X)\}.$$

4. The dilogarithm

In this section we give a more or less classical approach to the functional equation of the dilogarithm or, more accurately, its variant, Rogers' function.

Throughout this section, we identify $\mathbf{C} - \{0, 1\}$ with the space of equivalence classes of ordered 4-tuples of distinct points in $\mathbf{P}^1$ under the action of

$PGL(2)$; the orbit of (z_0, z_1, z_2, z_3) goes to its cross ratio

$$[z_0 : z_1 : z_2 : z_3] = \frac{(z_0 - z_2)/(z_1 - z_2)}{(z_0 - z_3)/(z_1 - z_3)} \in \mathbf{C} - \{0, 1\}.$$

Let Y be the space of equivalence classes of ordered 5-tuples of distinct points in $\mathbf{P}^1$ modulo the action of $PGL(2)$. Since $PGL(2)$ acts 3 transitively on $\mathbf{P}^1$, each orbit of $PGL(2)$ on the set of ordered 5-tuples of distinct points contains a unique point of the form $(y, x, 1, 0, \infty)$. It follows that Y is isomorphic to

$$(\mathbf{C} - \{0, 1\})^2 - \text{diagonal}$$

with coordinates (x, y).

There are 5 maps

$$A_j \colon Y \to \mathbf{C} - \{0, 1\}; \tag{4.1}$$

the jth map takes the orbit of $(z_0, z_1, \ldots, z_4)$ to the orbit of $(z_0, \ldots, \hat{z}_j, \ldots, z_4)$. An elementary calculation yields the formulas:

$$A_j(x, y) = \begin{cases} x, & j = 0, \\ y, & j = 1, \\ y/x, & j = 2, \\ (y - 1)/(x - 1), & j = 3, \\ \dfrac{x(y - 1)}{y(x - 1)}, & j = 4. \end{cases} \tag{4.2}$$

To lift the diagram

$$Y \; \overset{A_0}{\underset{A_4}{\stackrel{\longrightarrow}{\underset{\longrightarrow}{\vdots}}}} \; \mathbf{C} - \{0, 1\}$$

to the category $\tilde{\mathscr{A}}$ (see (2.3)), choose a basepoint (x_0, y_0) of Y, a basepoint a of $\mathbf{C} - \{0, 1\}$, and paths γ_j in $\mathbf{C} - \{0, 1\}$ from a to $A_j(x_0, y_0)$, $j = 0, \ldots, 4$. We would like to find all

$$F \in W_4 \tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\})$$

that satisfy the functional equation

$$A^* F := \sum_{j=0}^{4} (-1)^j A_j^* F = c,$$

where c is a constant. Adding a constant λ to F changes the constant c in the functional equation by λ. So we might as well assume that $c = 0$ and seek only those F that satisfy $A^*F = 0$.

There is a second natural normalization. Set

$$\omega_0 = \frac{dz}{z}, \quad \omega_1 = \frac{dz}{z-1} \in \Omega^1(\mathbf{C} - \{0, 1\}).$$

Choose loops σ_0, σ_1 in $\mathbf{C} - \{0, 1\}$ based at a, that satisfy

$$\int_{\sigma_j} \omega_k = 2\pi i \delta_{jk}.$$

Analytically continuing any $F \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\})$ along the commutator

$$[\sigma_0, \sigma_1] = \sigma_0\sigma_1\sigma_0^{-1}\sigma_1^{-1}$$

alters F by a constant. This can be seen using (2.6). When the classical dilogarithm $\ln_2$ is continued along $[\sigma_0, \sigma_1]$, its value increases by $(2\pi i)^2$. We seek all

$$F \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\})$$

which satisfy

$$[\sigma_0, \sigma_1]^*F = F + (2\pi i)^2, \quad A^*F = 0.$$

If $F \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\})$ satisfies $A^*F = 0$, then so does its equivalence class $\bar{F}$ in

$$Gr_4^W\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\}) \cong H^1(\mathbf{C} - \{0, 1\}, \mathbf{C})^{\otimes 2}.$$

(This last isomorphism is a special case of (3.13).) The equivalence class $\bar{F}$ does not change when F is analytically continued (2.6). We shall call $\bar{F}$ the *symbol* of F. For example, the symbol of $\ln_2$ is $-\omega_1 \otimes \omega_0$.

(4.3) PROPOSITION. *There is no $F \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\})$ which is congruent to $\ln_2 \bmod W_2\tilde{\mathscr{O}}$ and which satisfies the equation $A^*F = 0$.*

This follows by applying the homomorphism

$$d\log \otimes d\log\colon \mathscr{O}^*(Y) \otimes_{\mathbf{Z}} \mathscr{O}^*(Y) \to H^1(Y)^{\otimes 2}$$

to the following formula. Note that the previous result does not contradict the classical Abel-Spence functional equation for $\ln_2$ [L; §1.5] as that equation has a term which is a quadratic polynomial in $\log x$ and $\log(1 - x)$.

(4.4) PROPOSITION. *In* $\mathscr{O}^*(Y) \otimes_{\mathbf{Z}} \mathscr{O}^*(Y)$,

$$\sum_{j=0}^{4}(-1)^j(A_j - 1) \otimes A_j = x \otimes x + y \otimes y - (x \otimes y + y \otimes x)$$
$$+ (y \otimes (x-1) + (x-1) \otimes y)$$
$$-((y-1) \otimes y + y \otimes (y-1)). \qquad \blacksquare$$

A straightforward calculation shows that

$$\sum_{j=0}^{4}(-1)^j \frac{dA_j}{A_j} \otimes \frac{dA_j}{A_j}, \quad \sum_{j=0}^{4}(-1)^j \frac{dA_j}{A_j - 1} \otimes \frac{dA_j}{A_j - 1},$$
$$\sum_{j=0}^{4}(-1)^j\left(\frac{dA_j}{A_j - 1} \otimes \frac{dA_j}{A_j} + \frac{dA_j}{A_j} \otimes \frac{dA_j}{A_j - 1}\right)$$

are linearly independent in $H^1(Y)^{\otimes 2}$. Since the image of the right hand side of (4.4) in $H^1(Y)^{\otimes 2}$ is symmetric, we have:

(4.5) PROPOSITION. *If* $F \in W_4(\mathbf{C} - \{0,1\})$ *satisfies* $A^*F = 0$, *then* F *has symbol*

$$\bar{F} = \lambda(\omega_0 \otimes \omega_1 - \omega_1 \otimes \omega_0). \qquad \blacksquare$$

By (2.6), we have

$$[\sigma_0, \sigma_1]^*F = F + 2\lambda \begin{vmatrix} \int_{\sigma_0} \omega_0 & \int_{\sigma_1} \omega_0 \\ \int_{\sigma_0} \omega_1 & \int_{\sigma_1} \omega_1 \end{vmatrix}$$
$$= F + 2\lambda(2\pi i)^2.$$

So, if we normalize F so that $[\sigma_0, \sigma_1]^*F = F + (2\pi i)^2$, then F must have symbol

$$\tfrac{1}{2}(\omega_0 \otimes \omega_1 - \omega_1 \otimes \omega_0).$$

(4.6) To guarantee the existence of a function F with symbol $1/2(\omega_0 \otimes \omega_1 - \omega_1 \otimes \omega_0)$ satisfying $A^*F = 0$, it is necessary to choose the lift to $\tilde{\mathscr{A}}$ of the diagram

$$Y \begin{array}{c} \xrightarrow{A_0} \\ \vdots \\ \xrightarrow[A_4]{} \end{array} \mathbf{C} - \{0,1\}$$

more carefully. Choose a basepoint (x_0, y_0) of Y and a basepoint a of $\mathbf{C} - \{0, 1\}$ satisfying:

(i) a, x_0, y_0 are real;

(ii) $a > 1$, $x_0 > 1$ and $y_0 - 1 < x_0 < y_0$.

Then each $A_j(x_0, y_0)$ is real and > 1. Choose the path γ_j from a to $A_j(x_0, y_0)$ to be the unique linear path from a to $A_j(x_0, y_0)$ in $\mathbf{R} - \{0, 1\}$.

(4.7) THEOREM. *There exists a unique function* $F \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\})$ *which satisfies*

$$[\sigma_0, \sigma_1]^*F = F + (2\pi i)^2, \quad A^*F = 0 \tag{4.8}$$

where the functional equation is computed using the lift of the A_j *to* $\tilde{\mathscr{A}}$ *described in* (4.6). *In fact,* $F = \phi - \pi^2/6$, *where* ϕ *is Rogers' function.*

Proof. We first prove uniqueness. The uniqueness of F modulo $W_2\tilde{\mathscr{O}}$ was established in the discussion following (4.5). So if $F + \Delta$ is another function satisfying (4.8), then

$$\Delta \in W_2\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\}).$$

The equivalence class $\bar{\Delta}$ of Δ modulo $W_0\tilde{\mathscr{O}} \cong \mathbf{C}$ is then an element of

$$Gr_2^W\tilde{\mathscr{O}}(\mathbf{C} - \{0, 1\}) \cong H^1(\mathbf{C} - \{0, 1\})$$

which satisfies the equation $A^*\bar{\Delta} = 0$ in $H^1(Y)$. Since $A^*\colon H^1(\mathbf{C} - \{0, 1\}) \to H^1(Y)$ is injective, it follows that $\bar{\Delta} = 0$. That is, Δ is a constant c. But then

$$c = A^*F + A^*C = A^*(F + \Delta) = 0,$$

which establishes the uniqueness of F.

To prove existence, first cut $\mathbf{C}$ by a curve that does not intersect the upper half plane $\operatorname{Im} z > 0$, and intersects the real axis only at 0 and 1 (Fig. 9). Choose the branch of $\log z$ on this cut plane that agrees with the usual logarithm on the positive real axis.

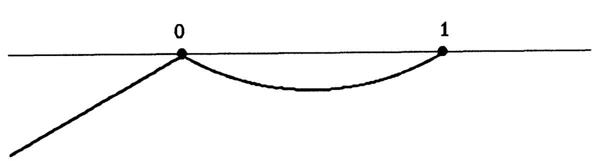


FIG. 9

The choice of basepoint in (4.6) ensures that the numberator a and denominator b of each of the 10 functions

$$A_0 = x,\quad A_1 = y,\quad A_2 = y/x,\quad A_3 = \frac{y-1}{x-1},\quad A_4 = \frac{x(y-1)}{y(x-1)},$$
$$A_0 - 1 = x - 1,\quad A_1 - 1 = y - 1,\quad A_2 - 1 = \frac{y-x}{x},$$
$$A_3 - 1 = \frac{y-x}{x-1},\quad A_4 - 1 = \frac{y-x}{y(x-1)}$$

is real and positive in a neighbourhood of (x_0, y_0) in

$$Y_{\mathbf{R}} = \left(\mathbf{R}^2 - \{0,1\}\right)^2 - \text{diagonal}.$$

Consequently, $\log(a/b) = \log a - \log b$ for each of these 10 functions in a neighbourhood of (x_0, y_0) in Y.

Define F by

$$\begin{aligned} F(z) &= \frac{1}{2}\int_a^z\left(\log z\,\frac{dz}{z-1} - \log(z-1)\frac{dz}{z}\right) \\ &= \frac{1}{2}\int_a^z(\omega_0\omega_1 - \omega_1\omega_0) + \frac{1}{2}\left(\log a\int_a^z\omega_1 - \log(a-1)\int_a^z\omega_0\right). \end{aligned}$$

This has symbol $1/2(\omega_0 \otimes \omega_1 - \omega_1 \otimes \omega_0)$.

Now

$$\begin{aligned} d(A^*F) &= A^*\,dF \\ &= A^*\left(\log z\,\frac{dz}{z-1} - \log(z-1)\frac{dz}{z}\right). \end{aligned}$$

Since we have chosen our basepoints and branch of $\log z$ carefully, we may apply $\log \otimes d\log$ to (4.4) to conclude that $dA^*F = 0$ in a neighbourhood of (x_0, y_0) in Y. Since F is holomorphic, this implies that A^*F equals a constant c. Replacing F by $F - c$ we obtain the desired function. ■

5. The Grassmannian complex

Denote the ordered set $\{0, 1, \ldots, n\}$ by $[n]$. View $\mathbf{C}^{n+1}$ as the complex vector space with basis $[n]$ and canonical coordinates $(z_0, \ldots, z_n)$. View $\mathbf{P}^n$ as the corresponding projective space. Each strictly increasing function f: $[n] \to [m]$ induces a linear inclusion $\mathbf{P}(f)$: $\mathbf{P}^n \to \mathbf{P}^m$. In particular, the jth face map d_j is the unique order preserving injection $[n-1] \to [n]$ that omits the value j. The image of the induced map $\mathbf{P}(d_j)$: $\mathbf{P}^{n-1} \to \mathbf{P}^n$ is the jth coordinate hyperplane H_j, canonically coordinatized. Define the *coordinate simplex* $\Delta(\mathbf{P}^n)$ of $\mathbf{P}^n$ to be the union of the coordinate hyperplanes H_j. Define the *k-skeleton* of $\Delta(\mathbf{P}^n)$ by

$$\begin{aligned}\Delta_k(\mathbf{P}^n) &= \bigcup f(\mathbf{P}^k)\\ &= \{\text{Union of the } k\text{-dimensional coordinate planes of } \mathbf{P}^n\}.\end{aligned}$$

where f ranges over the strictly increasing functions $[k] \to [n]$.

Denote the Grassmann manifold of n dimensional linear subspaces of $\mathbf{P}^m$ by $G(n, \mathbf{P}^m)$.

5.1 Definition. For positive integers p, q, we define

$$G_q^p = \left\{\xi \in G(q, \mathbf{P}^{p+q}) : \xi \cap \Delta_{p-1}(\mathbf{P}^{p+q}) = \varnothing\right\}.$$

This is the top stratum of the "pieceification" of $G(q, \mathbf{P}^{p+q})$, introduced in [GGMS].

The condition that $\xi \cap \Delta_{p-1}$ be empty is easily seen to be equivalent to the condition that ξ be transverse to each stratum of $\Delta(\mathbf{P}^{p+q})$. Consequently, each strictly increasing map

$$f : [p + r] \to [p + q]$$

induces a morphism $G_q^p \to G_r^p$ that takes $\xi \in G_q^p$ to its intersection $f^{-1}(\xi \cap f(\mathbf{P}^{p+r}))$ with $\mathbf{P}^{p+r}$. In particular, we have the face maps

$$A_j : G_q^p \to G_{q-1}^p, \quad 0 \le j \le p + q, \tag{5.2}$$

obtained by intersecting with the coordinate hyperplanes. These satisfy the usual identities satisfied by face maps of simplicial spaces.

Suppose that r and s are positive integers with $r \le s$. Let $\mathbf{\Delta}[r, s]$ be the full sub-category of the category of all ordinals and strictly increasing maps whose objects are the ordinals $[k]$ with $r \le k \le s$. A contravariant functor from $\mathbf{\Delta}[r, s]$ into a category $\mathscr{C}$ will be called an *(r, s)-truncated simplicial object of $\mathscr{C}$*.

(5.3) *Notation*. The $(p, 2p)$-truncated simplicial variety whose k simplices are G_{k-p}^p and whose face maps are as in (5.2) will be denoted by $G_\bullet^p$.

(5.4) Proposition. *The $(p, 2p)$-truncated simplicial variety $G_\bullet^p$ has a canonical lift to a $(p, 2p)$-truncated simplicial object $\tilde{G}_\bullet^p$ of $\tilde{\mathscr{A}}$.*

To construct the lift we need an alternative description of G_q^p. The general linear group acts on the generalized Stiefel variety

$$S_m^n = \{(v_1, \ldots, v_{n+m}) : v_j \in \mathbf{C}^n \text{ and } v_1, \ldots, v_{n+m} \text{ span } \mathbf{C}^n\}.$$

via the diagonal action: $g(v_1, \ldots, v_{n+m}) = (gv_1, \ldots, gv_{n+m})$. There is a natural bijection

$$(5.5) \qquad S_m^n / GL(n) \to G(m, \mathbf{C}^{n+m}),$$

where $G(m, \mathbf{C}^{n+m})$ denotes the Grassmannian of m planes in $\mathbf{C}^{n+m}$: To $(v_1, \ldots, v_{n+m})$ associate the kernel of the linear map $\mathbf{C}^{n+m} \to \mathbf{C}^n$ that takes the jth standard basis vector e_j to v_j. Conversely, if V is an m plane in $\mathbf{C}^{n+m}$, choose an isomorphism $\mathbf{C}^{n+m}/V \cong \mathbf{C}^n$ and let v_j be $e_j + V$.

(5.6) PROPOSITION. *Under the isomorphism* (5.5), G_q^p *corresponds to*

$$\{(v_0, \ldots, v_{p+q}) : v_j \in \mathbf{C}^p, \textit{ each } p \textit{ of the vectors } v_j \textit{ span } \mathbf{C}^p\}$$

and the face map A_j: $G_q^p \to G_{q-1}^p$ *is induced by the map*

$$(v_0, \ldots, v_{p+1}) \mapsto (v_0, \ldots, \hat{v}_j, \ldots, v_{p+1}).$$

Proof of (5.4). Consider the curve v: $\mathbf{C} \to \mathbf{C}^p$ defined by

$$v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{p-1} \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ t_0 & t_1 & \ldots & t_n \\ \vdots & \vdots & & \vdots \\ t_0^n & t_1^n & \ldots & t_n^n \end{pmatrix} = \prod_{i>j} (t_i - t_j),$$

it follows that

$$(v(0), v(1), \ldots, v(p+q))$$

corresponds, via (5.6), to a point x_q of G_q^p. For each strictly increasing function

$$f: [p+r] \to [q+r],$$

define a morphism

$$(f, \gamma_f)\colon (G_q^p, x_q) \to (G_r^p, x_r)$$

in $\tilde{\mathscr{A}}$ by $\gamma_f(t) = (v_0(t), \ldots, v_{p+r}(t))$, where

$$v_j(t) = v((1-t)j + tf(j)), \quad 0 \le t \le 1.$$

(5.7) To better understand the topology of the G_q^p, we introduce the space

$$Y_q^p = \left\{ \begin{matrix} \text{ordered } (p+q+1)\text{-tuples of points in} \\ \mathbf{P}^{p-1}, \text{ no } p \text{ of which lie in a hyperplane} \end{matrix} \right\} \Big/ PGL(p).$$

This is an affine variety of dimension $(p-1)q$. There is a principal $(\mathbf{C}^*)^{p+q}$ fibration $G_q^p \to Y_q^p$. This can be seen using the description (5.6) of G_q^p: each $(p+q+1)$-tuple of vectors in $\mathbf{C}^p$ determines $p+q+1$ points in $\mathbf{P}^{p-1}$.

Since $PGL(p)$ acts transitively on the generic ordered $(p+1)$-tuples of points in $\mathbf{P}^{p-1}$, we can choose homogeneous coordinates in $\mathbf{P}^{p-1}$ such that each point in Y_q^p is given by the columns of the $p \times (p+q+1)$ matrix

$$\left(\begin{array}{c|c|c} & 1 & \\ I_p & \vdots & A \\ & 1 & \\ \hline & 1 & 1 \ldots 1 \end{array} \right) \tag{5.8}$$

all of whose $p \times p$ minors are non-zero. This matrix also defines a point in G_q^p which shows that the principal bundle $G_q^p \to Y_q^p$ has a section. This proves:

(5.9) PROPOSITION. *As algebraic varieties* $G_q^p = Y_q^p \times (\mathbf{C}^*)^{p+q}$.

One should note that this product decomposition is not canonical as it depends, for example, upon a choice of $p+1$ of the $p+q+1$ points in $\mathbf{P}^{p-1}$.

For example, since Y_0^p is a point, $G_0^p \cong (\mathbf{C}^*)^p$. Since $Y_1^2 \cong \mathbf{C} - \{0,1\}$ and Y_2^2 is the space Y defined in Section 4, $G_1^2 \cong (\mathbf{C} - \{0,1\}) \times (\mathbf{C}^*)^3$ and

$$G_2^2 = \left((\mathbf{C} - \{0,1\})^2 - \text{diagonal}\right) \times (\mathbf{C}^*)^4.$$

(5.10) Applying the multivalued de Rham complex functor (3.8) to the canonical lift of $G_\bullet^p$ to $\tilde{\mathscr{A}}$, described in the proof of (5.4), we obtain a filtered double complex

$$\left(\tilde{\Omega}^\cdot(G_\bullet^p), W_\cdot\right).$$

The differential

$$A^*: \tilde{\Omega}^r(G^p_{q-1}) \to \tilde{\Omega}^r(G^p_q)$$

is the alternating sum

$$\sum_{j=0}^{q} (-1)^j A_j^*$$

of the maps induced by the face maps. Since the A_j satisfy the usual simplicial identities, $(A^*)^2 = 0$.

It is natural to put $\tilde{\Omega}^s(G^p_t)$ in total degree $p + s + t$. The differential

$$D: \tilde{\Omega}^k(G^p_q) \to \tilde{\Omega}^{k+1}(G^p_q) \oplus \tilde{\Omega}^k(G^p_{q+1})$$

is defined to be $d + (-1)^k A^*$. It satisfies $D^2 = 0$.

6. Higher logarithms

Fix an integer $p \geq 1$. Denote the coordinates of $\mathbf{P}^p$ by $[x_0, \ldots, x_p]$. Denote the hyperplane $x_j = 0$ by H_j. Then

$$G^p_0 = \mathbf{P}^p - \bigcup_{j=0}^{p} H_j.$$

Identify G^p_0 with $(\mathbf{C}^*)^p$ by taking $(x_1, \ldots, x_p) \in (\mathbf{C}^*)^p$ to $[1, x_1, \ldots, x_p] \in G^p_0$.

Since

$$\Omega^p(G^p_0) \subseteq W_{2p}\tilde{\Omega}^p(G^p_\cdot).$$

the p-form

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_p}{x_p} \in \Omega^p(G^p_0)$$

defines an element of $W_{2p}\tilde{\Omega}^\cdot(G^p_\cdot)$ of degree $2p$ that we shall denote by vol_p.

(6.1) Definition. A *p-logarithm* is an element Z_p of degree $2p - 1$ in the double complex $W_{2p}\tilde{\Omega}^\cdot(G^p_\cdot)$ that satisfies the equation $DZ_p = \mathrm{vol}_p$.

Denote the component of Z_p in $W_{2p}\tilde{\Omega}^0(G^p_{p-1})$ by L_p. This is a multivalued function defined by integrating an iterated integral of rational 1-forms on G^p_{p-1} of length $\leq p$. The condition $DZ_p = \mathrm{vol}_p$ implies that L_p satisfies the

$2p + 1$ term functional equation

$$A^*L_p = \sum_{j=0}^{2p} (-1)^j A_j^* L_p = 0.$$

Since $D^2 = 0$, a necessary condition for the existence of a p-log is that $A^* \operatorname{vol}_p = 0$. This can be verified by direct calculation for $p \le 3$. We give a general proof of this in Section 9.

The following elementary observation will be useful for constructing p-logarithms when p is small.

(6.2) PROPOSITION. *If X is a smooth variety, then, for each l,*

$$H^1\big(W_l \tilde{\Omega}^{\cdot}(X)\big) = 0.$$

Proof. If ω is a closed element of $W_l\tilde{\Omega}^1(X)$, then the multivalued function

$$F\colon x \mapsto \int_{x_0}^{x} \omega$$

is well defined and satisfies $dF = \omega$. If

$$\omega = \sum_r \sum_{|I|=r} \left(\int \omega_{i_1} \dots \omega_{i_r} \right) \otimes \omega_I,$$

then $F = f_{I, x_0}$, where

$$I = \sum_r \sum_{|I|=r} \int \omega_{i_1} \dots \omega_{i_r} \omega_I,$$

which shows that $F \in W_l \tilde{\mathscr{O}}(X)$. ■

We now show how the classical logarithm and dilogarithm fit into this picture of higher logarithms.

(6.3) *The logarithm.* In the case $p = 1$, the double complex is

$$\begin{array}{ccc} W_2\tilde{\mathscr{O}}(G_1^1) & \xrightarrow{d} & \Omega^1(G_1^1) \\ {\scriptstyle A^*}\uparrow & & \uparrow{\scriptstyle A^*} \\ W_2\tilde{\mathscr{O}}(G_0^1) & \xrightarrow{d} & \Omega^1(G_0^1). \end{array}$$

Identify G_0^1 ($\subseteq \mathbf{P}^1$) with $\mathbf{C}^*$ by identifying $\lambda \in \mathbf{C}^*$ with $[-\lambda, 1] \in G_0^1$. In these coordinates,

$$\mathrm{vol}_1 = -d\lambda/\lambda.$$

Identify G_1^1 with $\mathbf{C}^* \times \mathbf{C}^*$ by taking the line

$$x_0 + mx_1 + mbx_2 = 0, \quad m \neq 0, b \neq 0,$$

in $\mathbf{P}^2$ to (b, m). With respect to these coordinates, the face maps A_j: $G_1^1 \to G_0^1$ are given by the formulas

$$A_j(b,m) = \begin{cases} b, & j = 0, \\ bm, & j = 1, \\ m, & j = 2. \end{cases}$$

Thus

$$-A^* \mathrm{vol}_1 = \frac{db}{b} - \frac{d(bm)}{bm} + \frac{dm}{m} = 0.$$

Lift the diagram

$$G_1^1 \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} G_0^1$$

to $\tilde{\mathscr{A}}$ by letting $(1,1)$ be the basepoint of G_1^1 and 1 be the basepoint of G_0^1. Since $A_j(1,1) = 1$, this defines a lift of the diagram to $\tilde{\mathscr{A}}$.

Let $L \in W_2\tilde{\mathscr{O}}(G_0^1)$ be the multivalued function

$$L(x) = -\int_1^x \frac{d\lambda}{\lambda}.$$

Then $dL = \mathrm{vol}_1 \in \Omega^1(G_0^1)$ and $dA^*L = A^*dL = A^*\mathrm{vol}_1 = 0$. Consequently, $A^*L = c$, a constant, which is easily seen to be zero by evaluating at $(1,1)$. This function is clearly the unique element of $W_2\tilde{\mathscr{O}}(G_0^1)$ satisfying $dL = \mathrm{vol}_1$ and $A^*L = 0$.

This shows that the classical logarithm is a 1-logarithm.

(6.4) *The dilogarithm.* When $p = 2$, the double complex is

$$\begin{array}{ccccc}
W_4\tilde{\mathscr{O}}(G_2^2) & \longrightarrow & W_4\tilde{\Omega}^1(G_2^2) & \longrightarrow & \Omega^2(G_2^2) \\
\uparrow & & \uparrow & & \uparrow \\
W_4\tilde{\mathscr{O}}(G_1^2) & \longrightarrow & W_4\tilde{\Omega}^1(G_1^2) & \longrightarrow & \Omega^2(G_1^2) \\
\uparrow & & \uparrow & & \uparrow \\
W_4\tilde{\mathscr{O}}(G_0^2) & \longrightarrow & W_4\tilde{\Omega}^1(G_0^2) & \longrightarrow & \Omega^2(G_0^2).
\end{array}$$

Verifying that $A^* \operatorname{vol}_2 = 0$ is left as an exercise. Identify G_0^2 with $(\mathbf{C}^*)^2$ as in the beginning of this section. With respect to these coordinates,

$$\operatorname{vol}_2 = \frac{dx}{x} \wedge \frac{dy}{y}.$$

Let

$$\omega = \frac{1}{2}\left(\log x \frac{dy}{y} - \log y \frac{dx}{x}\right) \in W_4\tilde{\Omega}^1(G_0^2),$$

Then $d\omega = \operatorname{vol}_2$. Now

$$dA^*\omega = A^* d\omega = A^* \operatorname{vol}_2 = 0$$

so that $A^*\omega$ is a closed element of $W_4\tilde{\Omega}^1(G_1^2)$. By (6.2), $\omega = df$, where

$$f(x) = \int_{x_0}^{x} A^*\omega \in W_4\tilde{\mathscr{O}}(G_1^2).$$

Since

$$d(A^*f) = A^*(df) = (A^*)^2\omega = 0,$$

$A^*f = c$, a constant. Set $L_2 = f - c$, then $A^*L_2 = 0$. Thus (ω, L_2) is a 2-log.

Next we show that the 2-logarithm is unique up to a coboundary. The difference between two 2-logs is a 3-cocycle in the complex. Suppose that

$$(\xi, G) \in W_4\tilde{\Omega}^1(G_0^2) \oplus W_4\tilde{\mathscr{O}}(G_1^2)$$

is a 3-cocycle. (That is, $A^*G = 0$, $d\xi = 0$ and $dG = A^*\xi$.) By (6.2), there exists a function $H \in W_4\tilde{\mathscr{O}}(G_0^2)$ such that $dH = \xi$. Now

$$d(G - A^*H) = dG - A^*\xi = 0,$$

which implies that $G - A^*H = c$, a constant. But, as usual,

$$c = A^*(G - A^*H) = A^*G = 0.$$

Consequently, $G = A^*H$ and $DH = (\xi, G)$, so that the 2-log is unique up to a 2-coboundary.

It remains to relate L_2 and its functional equation to the Rogers' function and its functional equation.

We identify the space Y_1^2, the space of ordered 4 tuples of points in $\mathbf{P}^1$ modulo $PGL(2)$, with $\mathbf{C} - \{0, 1\}$ via the cross ratio. The space Y_2^2 is just the space Y defined in Section 4. Let

$$\pi_1 \colon G_1^2 \to \mathbf{C} - \{0,1\} \quad \text{and} \quad \pi_2 \colon G_2^2 \to Y$$

be the standard projections defined in (5.7). For each j, the diagram

$$\begin{tikzcd} G_2^2 \arrow[r, "A_j"] \arrow[d, "\pi_2"'] & G_1^2 \arrow[d, "\pi_1"] \\ Y \arrow[r, "A_j"'] & \mathbf{C} - \{0,1\} \end{tikzcd} \tag{6.5}$$

commutes. In fact it commutes in $\tilde{\mathscr{A}}$: The canonical basepoint

$$x_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \cdot GL(2)$$

of G_1^2 defined in (5.6) projects to $a = 4/3 \in \mathbf{C} - \{0,1\}$, while the basepoint

$$x_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \cdot GL(2)$$

of G_2^2 projects to $(x_0, y_0) = (4/3, 3/2) \in Y$. These basepoints are consistent with the choices made in (4.6). Moreover, the path from x_1 to $A_j(x_2)$ in G_1^2 projects to a path homotopic to the path γ_j defined in (4.6). Thus, if we lift the basepoint preserving projections π_1, π_2 to $\tilde{\mathscr{A}}$ in the obvious way, then the diagram (6.5) commutes in $\tilde{\mathscr{A}}$.

Let $F \in W_4\tilde{\mathscr{O}}(\mathbf{C} - \{0,1\})$ be the function given by (4.7). Recall that $F = \phi - \pi^2/6$, where ϕ is Rogers' function. Then, as (6.5) commutes in $\tilde{\mathscr{A}}$, $\pi_1^*F \in W_4\tilde{\mathscr{O}}(G_1^2)$ satisfies the functional equation $A^*(\pi_1^*F) = 0$. We do not know whether $L_2 = -\pi_1^*F$, but at least we have:

(6.6) Proposition. $L_2 + \pi_1^*F \in A^*W_2\tilde{\mathscr{O}}(G_0^2)$. *Consequently there is a representative of the* 2*-logarithm of the form*

$$(-\pi_1^*F, \eta) \in W_4\tilde{\mathscr{O}}(G_1^2) \oplus W_4\tilde{\Omega}^1(G_0^2).$$

Proof. We begin by showing that $L_2 \equiv -\pi_1^*F \bmod W_2\tilde{\mathscr{O}}(G_1^2)$. As in Section 4, we shall call the residue $\overline{G}$ of a function $G \in W_4\tilde{\mathscr{O}}(G_1^2)$ modulo $W_2\tilde{\mathscr{O}}(G_1^2)$ the *symbol* of G. By (2.6), $\overline{G}$ does not change under analytic continuation. From (3.13) it follows that the set of symbols is

$$Gr_4^W\tilde{\mathscr{O}}(X) \cong \operatorname{Ker}\{H^1(X)^{\otimes 2} \xrightarrow{\text{cup}} H^2(X)\},$$

when $X = \mathbf{C} - \{0,1\}, G_0^2, G_1^2$, (more generally, for any variety with $q(X) = 0$).

The assertion that $L_2 \equiv -\pi_1^* F \bmod W_2\tilde{\mathscr{O}}$ thus follows from the assertion that

$$\pi_1^*\left(\frac{dz}{z} \otimes \frac{dz}{z-1} - \frac{dz}{z-1} \otimes \frac{dz}{z}\right) + A^*\left(\frac{dx}{x} \otimes \frac{dy}{y} - \frac{dy}{y} \otimes \frac{dx}{x}\right)$$
$$= 0 \quad \text{in } \Omega^1\left(G_1^2\right)^{\otimes 2}.$$

This can be verified by a straightforward calculation. (Note that this formula also implies that $A^* \operatorname{vol}_2 = 0$.)

To show that $L_2 + \pi_1^* F \in A^* W_2 \tilde{\mathscr{O}}(G_0^2)$, first note that, since $L_2 + \pi_1^* F \in W_2\tilde{\mathscr{O}}(G_1^2)$,

$$d(L_2 + \pi_1^* F) \in \Omega^1\left(G_1^2\right)$$

and moreover that $A^*(d(L_2 + \pi_1^* F)) = 0$.

The following fact may be verified by a direct calculation.

(6.7) Proposition. *The sequence*

$$H^1\left(G_0^2\right) \xrightarrow{A^*} H^1\left(G_1^2\right) \xrightarrow{A^*} H^1\left(G_2^2\right)$$

is exact.

This result thus implies that $d(L_2 + \pi_1^* F) = A^*\omega$ for some $\omega \in \Omega^1(G_0^2)$. Setting

$$G(x) = \int_{x_0}^{x} \omega \in W_2\tilde{\mathscr{O}}\left(G_0^2\right)$$

we have $L_2 + \pi_1^* F - A^* G = c$, a constant, but

$$c = A^* c = A^* L_2 - A^* \pi_1^* F - (A^*)^2 G = 0.$$

That is, $L_2 \equiv -\pi_1^* F \bmod A^* W_2 \tilde{\mathscr{O}}(G_0^2)$ as claimed. ∎

Finally, we show that the function L_2 does not trivially satisfy the functional equation $A^* L_2 = 0$. That is, $L_2 \notin A^* W_4 \tilde{\mathscr{O}}(G_0^2)$. To see this first recall from (5.9) that

$$G_1^2 = (\mathbf{C}^*)^3 \times (\mathbf{C} - \{0, 1\})$$

which implies that

$$\pi_1\left(G_1^2\right) \cong \mathbf{Z}^3 \times \langle \sigma_0, \sigma_1 \rangle,$$

where $\langle \sigma_0, \sigma_1 \rangle$ is the free group generated by loops σ_0, σ_1 in $\mathbf{C} - \{0,1\}$ that encircle 0 and 1, respectively. It follows from (4.7) and (6.6) that the analytic continuation $[\sigma_0, \sigma_1]^* L_2$ around the commutator $[\sigma_0, \sigma_1]$ satisfies $[\sigma_0, \sigma_1]^* L_2 = L_2 - (2\pi i)^2$. On the other hand, the fact that $\pi_1(G_0^2)$ is abelian implies that $A_{j*}[\sigma_0, \sigma_1] = 1$ in $\pi_1(G_0^2)$ for each j. From this it follows that elements of $A^*\tilde{\mathscr{O}}(G_0^2)$ are invariant under analytic continuation around $[\sigma_0, \sigma_1]$ which shows that $L_2 \notin A^*\tilde{\mathscr{O}}(G_0^2)$.

(6.8) *The trilogarithm-first steps.* In this case the double complex looks like

$$\begin{array}{ccccccc}
W_6\tilde{\mathscr{O}}(G_3^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_3^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_3^3) & \longrightarrow & \Omega^3(G_3^3) \\
\uparrow & & \uparrow & & \uparrow & & \downarrow \\
W_6\tilde{\mathscr{O}}(G_2^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_2^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_2^3) & \longrightarrow & \Omega^3(G_2^3) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
W_6\tilde{\mathscr{O}}(G_1^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_1^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_1^3) & \longrightarrow & \Omega^3(G_1^3) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
W_6\tilde{\mathscr{O}}(G_0^3) & \longrightarrow & W_6\tilde{\Omega}^1(G_0^3) & \longrightarrow & W_6\tilde{\Omega}^2(G_0^3) & \longrightarrow & \Omega^3(G_0^3).
\end{array}$$

Once again, $A^* \operatorname{vol}_3 = 0$ in $\Omega^3(G_1^3)$, which can be verified by a direct calculation using (3.2)(iii).

We begin the process of finding a 3-logarithm by setting

$$\eta = \frac{1}{3}\left(\log x \frac{dy}{y} \wedge \frac{dz}{z} - \log y \frac{dx}{x} \wedge \frac{dz}{z} + \log z \frac{dx}{x} \wedge \frac{dy}{y}\right) \in W_6\tilde{\Omega}^2(G_0^3).$$

This satisfies $d\eta = \operatorname{vol}_3$. Now

$$d(A^*\eta) = A^*(d\eta) = A^* \operatorname{vol}_3 = 0.$$

To proceed, we have to be able to solve the equation

$$d\xi = -A^*\eta, \quad \xi \in W_6\tilde{\Omega}^1(G_1^3).$$

Thus the first (and, as we shall see shortly, the only) obstruction to finding a 3-logarithm is the cohomology class

$$[A^*\eta] \in H^2(W_6\tilde{\Omega}^{\cdot}(G_1^3)).$$

(Unfortunately we cannot simply integrate $A^*\eta$ to find ξ as we did in the cases of the logarithm and dilogarithm.) The vanishing of this cohomology group will be established in the Section 8, Corollary (8.7). Given this vanishing result, the existence and uniqueness of the 3-log follow:

If $[A^*\eta] = 0$, then there exists $\xi \in W_6\tilde{\Omega}^1(G_1^3)$ which satisfies

$$d\xi + A^*\eta = 0.$$

Now $A^*\xi \in W_6\tilde{\Omega}^1(G_2^3)$ is a closed 1-form as

$$d(A^*\xi) = A^*(d\xi) = -A^*(A^*\eta) = 0.$$

We may therefore apply (6.2) to obtain a multivalued function $F \in W_6\tilde{\mathscr{O}}(G_2^3)$ satisfying $dF = A^*\xi$. Now

$$d(A^*F) = A^*(dF) = (A^*)^2\xi = 0,$$

so that $A^*F = c$, a constant. Setting $L_3 = F - c$, we obtain a function $L_3 \in W_6\tilde{\mathscr{O}}(G_2^3)$ that satisfies the 7-term functional equation

$$\sum_{j=0}^{6} (-1)^j A_j L_3 = A^*L_3 = 0;$$

and $Z_3 = (\eta, \xi, L_3)$ is a 3-logarithm.

To establish the uniqueness of Z_3 modulo coboundaries, we have to show that

$$H^5(W_6\Omega^\cdot(G_\cdot^3)) = 0.$$

Suppose that

$$C = (\eta, \xi, G) \in W_6(\tilde{\Omega}^2(G_0^3) \oplus \tilde{\Omega}^1(G_1^3) \oplus \tilde{\mathscr{O}}(G_2^3))$$

is a 5-cocycle. The condition $DC = 0$ is equivalent to the conditions

$$d\eta = 0, \quad d\xi + A^*\eta = 0, \quad dG = A^*\xi.$$

Provided that we can solve the equation

$$(6.9) \qquad d\omega = \eta, \quad \omega \in W_6\tilde{\Omega}^1(G_0^3),$$

we can solve the equations

$$dH - A^*\omega = \xi, \quad A^*H = G, \quad H \in W_6\tilde{\mathscr{O}}(G_1^3).$$

by what should now be a familiar argument using (6.2). The existence of a solution to (6.9) follows from the following result.

(6.10) PROPOSITION. *For all $p, k \geq 1$ and $l \geq 0$ $H^k(W_l\tilde{\Omega}^\cdot(G_0^p)) = 0$.*

Proof. In this case $\tilde{\mathscr{O}}(G_0^p)$ is the polynomial ring

$$\mathbf{C}[\log x_1, \dots, \log x_p]$$

generated by the logarithms of the coordinate functions. Consequently,

$$\tilde{\Omega}(G_0^p) = \mathbf{C}[\log x_1, \dots, \log x_p] \otimes \Lambda\left(\frac{dx_1}{x_1}, \dots, \frac{dx_p}{x_p}\right)$$

which is the free graded commutative algebra generated by the vector space spanned by

$$\log x_j, \quad \frac{dx_j}{x_j}, \quad j = 1, \dots, p.$$

Since $d \log x_j = dx_j/x_j$, this vector space is acyclic. Thus each

$$W_l\tilde{\Omega}(G_0^p) = \left\{\text{polynomials in } \log x_j, \frac{dx_j}{x_j} \text{ of degree} \leq \frac{1}{2}l\right\}$$

is acyclic as claimed. ∎

Modulo the existence of the first lift η, we have proved the existence and uniqueness of the 3-logarithm. The non-triviality of L_3 modulo $A^*W_6\tilde{\mathscr{O}}(G_1^3)$ will be established in Section 10.

7. Higher albanese manifolds

Consider a complex algebraic manifold X satisfying $\Omega^1(X) = H^1(X, \mathbf{C})$. An equivalent condition is that $q(X) := h^{1,0}(\overline{X})$ be zero, where $\overline{X}$ is any smooth completion of X. In particular, each G_q^p and Y_q^p is such a variety. We use the methods of Chen [C3] to construct an inverse system of complex nilmanifolds and maps of X into this inverse system:

$$\begin{tikzcd} & & X \arrow[dl, "\theta^2"'] \arrow[d, "\theta^3"] \arrow[dr, "\theta^1"] & \\ \cdots \arrow[r] & \mathrm{Alb}^3(X) \arrow[r] & \mathrm{Alb}^2(X) \arrow[r] & \mathrm{Alb}^1(X). \end{tikzcd}$$

There is a more general construction of higher albanese manifolds, due to Deligne [D3], that applies to all smooth varieties. For details, see [HZ] and [H2].

The strategy behind proving the vanishing of the obstruction to the existence of a 3-logarithm is to replace G_1^3 by its albanese $\mathrm{Alb}^2(G_1^3)$. In second half of this section we construct a multivalued de Rham complex for $\mathrm{Alb}^s(X,x)$ and establish the vanishing

$$H^k\big(W_l\tilde{\Omega}^{\cdot}(\mathrm{Alb}^s(X,x))\big) = 0$$

when $k > 0$. In the next section we give a condition under which the natural homomorphism

$$W_l\tilde{\Omega}^{\cdot}(\mathrm{Alb}^s(X,x)) \to W_l\tilde{\Omega}^{\cdot}(X,x)$$

is a quasi-isomorphism and show that this condition is satisfied by each G_1^p.

The free Lie algebra $\mathbf{L}(H_1(X))$ generated by $H_1(X,\mathbf{C})$ is naturally graded by bracket length:

$$\mathbf{L}(H_1(X)) = \bigoplus_{s=1}^{\infty} \mathbf{L}^s(H_1(X))$$

where $\mathbf{L}^s$ denotes the elements of $\mathbf{L}$ which are homogeneous of degree s. The lower central series

$$\mathbf{L}(H_1(X)) = I\mathbf{L} \geq I^2\mathbf{L} \geq \dots$$

of $\mathbf{L}$ satisfies

$$I^s\mathbf{L} = \bigoplus_{r \geq s} \mathbf{L}^r.$$

Observe that there is a natural isomorphism

$$\mathbf{L}^2(H_1(X)) \cong \Lambda^2 H_1(X).$$

Let R be the ideal of $\mathbf{L}(H_1(X))$ generated by the image of the dual of the cup product

$$H_2(X) \to \Lambda^2 H_1(X).$$

Set

$$\mathfrak{g}_s = \mathbf{L}(H_1(X))/(R + I^{s+1}\mathbf{L}).$$

The grading of $\mathbf{L}$ induces a grading

$$\mathfrak{g}_s = \bigoplus_{p=1}^{s} \mathfrak{g}_s^p$$

of each $\mathfrak{g}_s$. Since $H_1(X)$ is pure of weight -2, it is natural to define a weight filtration on $\mathfrak{g}_s$ by

$$W_l \mathfrak{g}_s = \bigoplus_{2p \geq -l} \mathfrak{g}_s^p,$$

The $\mathfrak{g}_s$ form an inverse system

$$\mathfrak{g}_1 \leftarrow \mathfrak{g}_2 \leftarrow \mathfrak{g}_3 \leftarrow \cdots$$

of filtered Lie algebras. Let

$$G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$$

be the corresponding inverse system of simply connected nilpotent Lie groups.

Consider the element ω_s of

$$\Omega^1(X) \otimes H_1(X) \subseteq \Omega^1(X) \otimes \mathfrak{g}_s$$

that, using the identification $H^1(X) \cong \Omega^1(X)$, corresponds to the identity homomorphism

$$\mathrm{id} \in \mathrm{Hom}(H_1(X), H_1(X)) \cong H^1(X) \otimes H_1(X) \cong \Omega^1(X) \otimes H_1(X).$$

By (3.2), $d\omega_s = 0$. Since $[\omega_s, \omega_s] = 2\omega_s \wedge \omega_s = 0$ by the choice of the ideal R, ω_s defines an integrable connection ∇_s on the trivial principal G_s bundle $G_s \times X \to X$: if u: $X \to G_s$ is a locally defined section, then

$$\nabla_s u = du - u\omega_s. \tag{7.1}$$

Parallel transport of $1 \in G_s$ along a path γ in X defines an element $T(\gamma)$ of G_s. The resulting function on the path space

$$T_s\colon PX \to G_s$$

is given by the formula

$$T_s(\gamma) = 1 + \int_\gamma \omega_s + \int_\gamma \omega_s \omega_s + \cdots .$$

(The formula should be interpreted inside a faithful, unipotent matrix representation of G_s.)

For each choice of basepoint $x \in X$, we obtain a sequence of compatible monodromy representations

$$\rho_s\colon \pi_1(X, x) \to G_s, \quad \gamma \mapsto T_s(\gamma).$$

The following result follows directly from Chen's method of power series connections [C2], [C3].

(7.2) Proposition. *The homomorphism*

$$\rho\colon \pi_1(X, x) \to \varprojlim G_s$$

induced by the ρ_s is the complex form of the Malcev completion of $\pi_1(X, x)$ [Mv]. *In particular, the homomorphisms ρ_s induce isomorphisms*

$$[Gr_\Gamma^s \pi_1(X, x)] \otimes \mathbf{C} \xrightarrow{\approx} Gr_\Gamma^s G_t, \quad t \geq s$$

where $G = \Gamma^1 \geq \Gamma^2 \geq \Gamma^3 \geq \cdots$ denotes the lower central series of the group H.

Denote the image of ρ_s by Γ_s. Proposition (7.2) implies that Γ_s is a discrete subgroup of G_s. Define the sth *albanese manifold* of (X, x) by

$$\mathrm{Alb}^s(X, x) = \Gamma_s \backslash G_s.$$

These form an inverse system of complex nilmanifolds

$$\mathrm{Alb}^1(X, x) \leftarrow \mathrm{Alb}^2(X, x) \leftarrow \cdots$$

with

$$\mathrm{Alb}^1(X, x) = H_1(X, \mathbf{C}^*)$$

and with the fiber of the map $\mathrm{Alb}^s \to \mathrm{Alb}^{s-1}$ being $[Gr_\Gamma^s \pi_1(X, x)] \otimes \mathbf{C}^*$.

(7.3) *Example.* When $X = \mathbf{C} - \{0, 1\}$,

$$\mathfrak{g}_2 = \begin{pmatrix} 0 & \mathbf{C} & \mathbf{C} \\ 0 & 0 & \mathbf{C} \\ 0 & 0 & 0 \end{pmatrix},$$

the Lie algebra of the Heisenberg group

$$G_2 = H = \begin{pmatrix} 1 & \mathbf{C} & \mathbf{C} \\ 0 & 1 & \mathbf{C} \\ 0 & 0 & 1 \end{pmatrix}.$$

The connection form ω_2 is

$$\begin{pmatrix} 0 & \dfrac{dz}{z-1} & 0 \\ 0 & 0 & \dfrac{dz}{z} \\ 0 & 0 & 0 \end{pmatrix}$$

and the homomorphism ρ_2 takes γ to

$$\begin{pmatrix} 1 & \displaystyle\int_\gamma \frac{dz}{z-1} & \displaystyle\int_\gamma \frac{dz}{z-1}\frac{dz}{z} \\ 0 & 1 & \displaystyle\int_\gamma \frac{dz}{z} \\ 0 & 0 & 1 \end{pmatrix}.$$

The group Γ_2 is conjugate to the subgroup

$$H_{\mathbf{Z}} = \begin{pmatrix} 1 & \mathbf{Z}(1) & \mathbf{Z}(2) \\ 0 & 1 & \mathbf{Z}(1) \\ 0 & 0 & 1 \end{pmatrix}.$$

of G_2. Here $\mathbf{Z}(p)$ denotes the subgroup $(2\pi i)^p\mathbf{Z}$ of $\mathbf{C}$. ■

Define the sth *albanese mapping*

$$\theta_x^s \colon X \to \mathrm{Alb}^s(X, x)$$

by $\theta_x^s(y) = T_s(\gamma)$, where γ is a path in X from x to y and T_s is the transport map associated to the connection ω_s. Since ω_s is integrable, $T_s(\gamma)$ depends only on the relative homotopy class of γ.

(7.4) *Example.* (a) When $s = 1$, then

$$\theta_x^1 \colon X \to \mathrm{Alb}^1(X, x) \cong H_1(X, \mathbf{C}^*)$$

is given by $\theta_x^1(y) = (f_1(y), \dots, f_m(y))$ where $f_1, \dots, f_m \in \mathscr{O}^*(X)$ are rational functions satisfying $f_j(x) = 1$ and whose logarithmic derivatives form a basis of $H^1(X, \mathbf{Z}(1))$.

(b) When $X = \mathbf{C} - \{0,1\}$ and $s = 2$,

$$\theta_x^2 \colon \mathbf{C} - \{0,1\} \to \mathrm{Alb}^2(\mathbf{C} - \{0,1\}, x)$$

takes $y \in \mathbf{C} - \{0,1\}$ to the Γ_2 coset of

$$\begin{pmatrix} 1 & \int_x^y \frac{dz}{z-1} & \int_x^y \frac{dz}{z-1}\frac{dz}{z} \\ 0 & 1 & \int_x^y \frac{dz}{z} \\ 0 & 0 & 1 \end{pmatrix}.$$

The composite of θ_x^2 with the biholomorphism

$$\mathrm{Alb}^2(\mathbf{C} - \{0,1\}, x) \to H_{\mathbf{Z}} \backslash H_{\mathbf{C}}, \quad X \mapsto A^{-1}X,$$

where

$$A = \begin{pmatrix} 1 & \log(1-x) & -\ln_2(x) \\ 0 & 1 & \log x \\ 0 & 0 & 1 \end{pmatrix},$$

takes y to the $H_{\mathbf{Z}}$ coset of

$$\begin{pmatrix} 1 & \log(1-y) & -\ln_2(y) \\ 0 & 1 & \log y \\ 0 & 0 & 1 \end{pmatrix}. \qquad \blacksquare$$

Consider $\mathrm{Alb}^s(X,x)$ as an element of the category $\widetilde{\mathscr{A}n}$ defined in Section 2 by choosing the coset $\bar{1}$ of the identity as a basepoint. Since the albanese map θ_x^s takes x to $\bar{1}$, it lifts to a morphism of $\widetilde{\mathscr{A}n}$.

(7.5) PROPOSITION. *The association of* $\mathrm{Alb}^s(X,x)$ *to* (X,x) *defines a functor from* $\tilde{\mathscr{A}}$ *to* $\widetilde{\mathscr{A}n}$. *The albanese mapping* θ^s *is natural with respect to the morphisms of these categories.*

Identify the dual $\mathfrak{g}_s^*$ of $\mathfrak{g}_s$ with the left invariant 1-forms of G_s. These descend to 1-forms on $\mathrm{Alb}^s(X,x)$. The inclusion of $\mathfrak{g}_s^*$ into $E^1(\mathrm{Alb}^s(X,x))$ induces a d.g. algebra homomorphism

$$\tau_s\colon \mathscr{C}(\mathfrak{g}_s) \to E^{\cdot}(\mathrm{Alb}^s(X,x)).$$

Here $\mathscr{C}(\mathfrak{g}_s)$ denotes the Chevalley-Eilenberg complex associated to $\mathfrak{g}_s$. (As a graded vector space, it is isomorphic to the exterior algebra on $\mathfrak{g}_s^*[-1]$. The differential is induced by the dual of the bracket.)

The following result is due to Nomizu [N]. It can be proved by induction on s using the Leray-Serre spectral sequence.

(7.6) PROPOSITION. *τ_s is a quasi-isomorphism, so that*

$$H^{\cdot}(\mathrm{Alb}^s(X,x)) \cong H^{\cdot}(\mathscr{E}(\mathfrak{g}_s)) \cong H^{\cdot}(\mathfrak{g}_s, \mathbf{C}).$$

Although the higher albanese manifolds are rarely algebraic, the forms $\mathscr{E}(\mathfrak{g}_s)$ behave like logarithmic forms:

(7.7) PROPOSITION. *The image of the homomorphism*

$$\theta_x^*: \mathscr{E}(\mathfrak{g}_s) \to E^{\cdot}(X)$$

is contained in $\Omega^{\cdot}(X)$. Consequently θ_x^s induces a d.g. algebra homomorphism

$$\theta_x^*: \mathscr{E}(\mathfrak{g}_s) \to \Omega^{\cdot}(X).$$

Proof. It suffices to show that the image of $\mathfrak{g}_s^*$ in $E^1(X)$ is contained in $\Omega^1(X)$. It follows from (7.1) that, locally, θ_x^s satisfies the equation

$$d\theta_x^s = \theta_x^s \omega_s.$$

So if $Y \in T_yX$ is a tangent vector at $y \in X$ and $\varphi \in \mathfrak{g}_s^* \subseteq E^1(\mathrm{Alb}^s(X))$, then

$$\langle \theta_x^*(\varphi), Y\rangle = \langle \varphi, d\theta_x(Y)\rangle = \langle \varphi, Y \lrcorner \omega_s\rangle = \langle \varphi \circ \omega_s, Y\rangle.$$

It follows that $\theta_x^*(\varphi) = \varphi \circ \omega_s$, which is an element of $\Omega^1(X)$ as claimed. ■

We can associate a multivalued de Rham complex to $\mathrm{Alb}^s(X,x)$ in much the same way as we did for X. Consider the space

$$H^0(B(\mathscr{E}(\mathfrak{g}_s)))$$

of relatively closed iterated integrals of the form

$$\sum \int \omega_{i_1} \dots \omega_{i_r}$$

where each $\omega_{i_j} \in \mathscr{E}^1(\mathfrak{g}_s)$. As in (2.4), the distinguished basepoint $\bar{1} \in \mathrm{Alb}^s(X,x)$ determines an injective ring homomorphism

$$H^0(B(\mathscr{E}(\mathfrak{g}_s))) \to \tilde{E}^0(\mathrm{Alb}^s(X,x), \bar{1}).$$

Denote its image by $\tilde{\mathscr{O}}(\mathrm{Alb}^s(X,x))$.

Alternatively, we may describe $\tilde{\mathscr{O}}(\mathrm{Alb}^s(X,x))$ as follows: Since G_s is a simply connected nilpotent Lie group, the exponential map $\mathfrak{g}_s \to G_s$ is a biholomorphism so that the composite

$$\mathfrak{g}_s \xrightarrow{\exp} G_s \to \mathrm{Alb}^s(X, x)$$

is a universal covering of $\mathrm{Alb}^s(X, x)$. With this choice of universal covering, $\tilde{\mathscr{O}}(\mathrm{Alb}^s(X, x))$ may be identified with the subring of $E^0(\mathfrak{g}_s)$ consisting of polynomials on $\mathfrak{g}_s$. The equivalence of these two definitions can be proved using the tautological connection on the principal G_s bundle $G_s \times G_s \to G_s$ whose connection form is the Maurer-Cartan form

$$\mathrm{id} \in \mathfrak{g}_s^* \otimes \mathfrak{g}_s \subseteq E^1(G_s) \otimes \mathfrak{g}_s.$$

The weight filtration on $\mathfrak{g}_s$ induces a weight filtration on $\mathscr{C}(\mathfrak{g}_s)$. This, in turn, induces a weight filtration on the relatively closed iterated integrals. This transfers to a weight filtration on $\tilde{\mathscr{O}}(\mathrm{Alb}^s(X, x))$ via this isomorphism

$$H^0(B(\mathscr{C}(\mathfrak{g}_s))) \cong \tilde{\mathscr{O}}(\mathrm{Alb}^s(X, x)).$$

The weight filtration may also be constructed from the isomorphism

$$\tilde{\mathscr{O}}(\mathrm{Alb}^s(X, x)) \cong \mathrm{Sym}(\mathfrak{g}_s^*)$$

as the filtration corresponding to the weight filtration on $\mathrm{Sym}(\mathfrak{g}_s^*)$ obtained by extending the natural weight filtration $W_l(\mathfrak{g}_s^*) = (\mathfrak{g}_s/W_{-l-1})^*$ to $\mathrm{Sym}(\mathfrak{g}_s^*)$.

Define the *multivalued de Rham complex* $\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s(X, x))$ of $\mathrm{Alb}^s(X, x)$ to be

$$\tilde{\mathscr{O}}(\mathrm{Alb}^s(X, x)) \otimes \mathscr{C}(\mathfrak{g}_s).$$

It is closed under exterior differentiation. The natural weight filtrations on $\tilde{\mathscr{O}}(\mathrm{Alb}^s)$ and $\mathscr{C}(\mathfrak{g}_s)$ induce a weight filtration on $\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s)$.

The following result generalizes (6.10).

(7.8) Proposition. *For each $l \geq 0$ the complex $W_l\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s(X, x))$ is acyclic.*

Proof. Since $\mathfrak{g}_s$ is graded, the weight filtration of $\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s)$ is naturally split and $\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s)$ is quasi-isomorphic to its associated graded

$$Gr_{\bullet}^W\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s(X, x)). \tag{7.9}$$

Therefore it suffices to prove that $\tilde{\Omega}^{\bullet}(\mathrm{Alb}^s)$ is acyclic.

To see this, first observe that, as graded algebras,

$$\tilde{\Omega}(\mathrm{Alb}^s) \cong \mathrm{Sym}(\mathfrak{g}_s^*) \otimes \Lambda(\mathfrak{g}_s^*[-1]),$$

the free graded algebra generated by $Q = \mathfrak{g}_s^* \oplus \mathfrak{g}_s^*[-1]$. Filtering $\tilde{\Omega}$ by the powers of its augmentation ideal I, we obtain a spectral sequence whose E_1 term is the free graded commutative algebra generated by $I/I^2 \cong Q$. Since the matrix of the induced differential on I/I^2 is

$$\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$$

with respect to the direct sum decomposition above, I/I^2 and hence the E_1 term of this spectral sequence is acyclic. The acyclicity of $\tilde{\Omega}(\mathrm{Alb}^s)$ follows.

Alternatively, the acyclicity of $\tilde{\Omega}(\mathrm{Alb}^s)$ follows from the observation that it is isomorphic to the one sided bar construction $B(\mathbf{C}, \mathscr{C}(\mathfrak{g}_s), \mathscr{C}(\mathfrak{g}_s))$ which is well known to be acyclic. ■

Since θ_x^s induces a d.g. algebra homomorphism $\mathscr{C}(\mathfrak{g}_s) \to \Omega^{\cdot}(X)$, it induces a homomorphism on iterated integrals, and hence on $\tilde{\mathscr{O}}$.

(7.9) PROPOSITION. *The albanese maps θ_x^s: $X \to \mathrm{Alb}^s(X, x)$ induces a homomorphism of filtered d.g. algebras*

$$\varinjlim \tilde{\Omega}^{\cdot}(\mathrm{Alb}^s(X, x)) \to \tilde{\Omega}^{\cdot}(X, x)$$

which is an isomorphism in degree 0.

8. Rational $K(\pi, 1)$'s and the existence of the 3-logarithm

In view of (7.8), the existence of the 3-log will follow if we can prove that

$$W_6\tilde{\Omega}^{\cdot}(\mathrm{Alb}^s(G_1^3)) \to W_6\tilde{\Omega}^{\cdot}(G_1^3)$$

is a quasi-isomorphism for s sufficiently large. Suppose that X is a variety satisfying $q(X) = 0$. Set

$$\tilde{\Omega}^{\cdot}(\mathrm{Alb}(X)) = \varinjlim_s \tilde{\Omega}^{\cdot}(\mathrm{Alb}^s(X)).$$

In this section we give a criterion for the map

$$\theta_x^*: W_l\tilde{\Omega}^{\cdot}(\mathrm{Alb}(X)) \to W_l\tilde{\Omega}^{\cdot}(X)$$

to be a quasi-isomorphism for all l. (Such a map is commonly called a $W_{\cdot}$-filtered quasi-isomorphism.) Since

$$\tilde{\mathscr{O}}(\mathrm{Alb}(X)) \to \tilde{\mathscr{O}}(X)$$

is an isomorphism of filtered algebras (7.9), the problem reduces to giving a criterion for the homomorphism $C(\mathfrak{g}) \to \Omega^{\cdot}(X)$ to be a quasi-isomorphism, where

$$\mathscr{C}(\mathfrak{g}) = \varinjlim_{s} \mathscr{C}(\mathfrak{g}_s).$$

If we think of $\mathfrak{g}$ as a topological Lie algebra whose neighbourhoods of 0 are the kernels of the natural homomorphisms $\mathfrak{g} \to \mathfrak{g}_s$, then $\mathscr{C}(\mathfrak{g})$ is the complex of *continuous* cochains on $\mathfrak{g}$ (see [H4]).

A variety X with $q(X) = 0$ is a *rational* $K(\pi, 1)$ if the composite

$$(8.1) \qquad \mathscr{C}(\mathfrak{g}) \to \Omega^{\cdot}(X) \hookrightarrow A^{\cdot}(X)$$

is a quasi-isomorphism. (Here $A^{\cdot}(X)$ denotes the complex of C^{∞} forms on X.) For example, since $\mathrm{Alb}^s((\mathbf{C}^*)^n) = (\mathbf{C}^*)^n$ for all s, $(\mathbf{C}^*)^n$ is a rational $K(\pi, 1)$.

(8.2) Proposition. *If X is a smooth, rational $K(\pi, 1)$ variety satisfying $q(X) = 0$ then*:

(i) θ_x^*: $\mathscr{C}(\mathfrak{g}) \to \Omega^{\cdot}(X)$ *is a filtered quasi-isomorphism.*

(ii) θ_x^*: $\tilde{\Omega}^{\cdot}(\mathrm{Alb}(X)) \to \tilde{\Omega}^{\cdot}(X)$ *is a filtered quasi-isomorphism.*

(iii) (Falk [F]) $H^{\cdot}(X, \mathbf{C}) = \Omega^{\cdot}(X) \cong \Lambda H^1(X)/(R)$, *where* $R \subseteq \Lambda^2 H^1(X)$. (*Such an algebra is called a quadratic algebra in* [BG].)

(iv) (Kohno [K]) $\prod_{n \geq 1}(1 - t^n)^{\varphi_n} = p_X(-t)$, *where $p_X(t)$ is the Poincaré series of X and $\varphi_n = \dim \mathfrak{g}^n$, the rank of the n^{th} term of the lower central series of $\pi_1(X)$* (cf. (7.2)).

Proof. The first assertion follows from (3.2). The second assertion follows from the first using a spectral sequence argument. The third assertion follows from (i), while the fourth follows from the fact that the complex

$$Gr_{2n}^W \mathscr{C}(\mathfrak{g}) \to H^n(X) \to 0$$

is acyclic and therefore has Euler characteristic zero. ∎

The terminology "rational $K(\pi, 1)$" deserves further explanation. Suppose for the moment that X is a manifold with finitely generated fundamental group. To $\pi_1(X)$ one can associate a complete pronilpotent Lie algebra $\mathfrak{g}$ which, in the case when X is an algebraic manifold with $q(X) = 0$, is the Lie algebra $\mathfrak{g} = \varprojlim \mathfrak{g}_s$ defined above. (An elegant construction of $\mathfrak{g}$ is given in Appendix A of [Q].) There are various ways of constructing a d.g. algebra homomorphism $\mathscr{C}(\mathfrak{g}) \to A^{\cdot}X$ which is an isorophism on H^1 and injective on H^2. One method, analogous to our construction of (8.1), is to use the inverse system of real albanese manifolds constructed by Chen in [C3]. In Sullivan's terminology, $\mathscr{C}(\mathfrak{g}) \to A^{\cdot}X$ is the 1-minimal model of X [S1]. One defines X

to be a rational $K(\pi, 1)$ if $\mathscr{E}(\mathfrak{g}) \to A^{\cdot}X$ is a quasi-isomorphism, or equivalently, if $\mathscr{E}(\mathfrak{g})$ is the minimal model of X. This terminology is motivated by the fact that, for simply connected spaces, the generators of the minimal model are dual to the homotopy groups of X. A space whose minimal model has generators only in dimension 1 is, in some sense, a "rational $K(\pi, 1)$". As we shall see later, there are $K(\pi, 1)$'s that are not rational $K(\pi, 1)$'s, and there are rational $K(\pi, 1)$'s that are not $K(\pi, 1)$'s.

It is convenient to generalize this notion:

(8.3) Definition. Let n be a positive integer. A manifold X with finitely presented fundamental group is a rational n-$K(\pi, 1)$ if the natural map $\mathscr{E}(\mathfrak{g}) \to A^{\cdot}X$ induces an isomorphism on H^k when $k \le n$ and an injection on H^{n+1}.

Every space is a rational 1-$K(\pi, 1)$. Standard results in rational homotopy theory yield the next result.

(8.4) Theorem. (i) *The property of being a rational n-$K(\pi, 1)$ depends only on the homotopy type of X.*

(ii) *If X and Y are rational n-$K(\pi, 1)$'s, then so are $X \times Y$ and $X \vee Y$.*

(iii) (Falk [F]) *Suppose that $F \to E \to B$ is a fibration and that $\pi_1(B)$ acts unipotently on $H^{\cdot}(F, \mathbf{Q})$. If F and B are rational n-$K(\pi, 1)$'s, then so is E.*

Ideas behind the proof. (ii) Since $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, we have $\mathfrak{g}_{X \times Y} = \mathfrak{g}_X \times \mathfrak{g}_Y$ so that $\mathscr{E}(\mathfrak{g}_{X\times Y}) = \mathscr{E}(\mathfrak{g}_X) \otimes \mathscr{E}(\mathfrak{g}_Y)$ from which the assertion about $X \times Y$ follows. To prove the dual assertion, we give another characterization of rational n-$K(\pi, 1)$'s. Let (A_X, δ) denote the complete Hopf algebra associated to X by Chen's method of formal power series connections [C2; §3], [C3; §1]. For X to be a rational n-$K(\pi, 1)$, it is necessary and sufficient for $H_k(A_X, \delta)$ to vanish when $0 < k < n$. The assertion for $X \vee Y$ follows from the fact that $(A_{X \vee Y}, \delta)$ is the completed free product of (A_X, δ) and (A_Y, δ) and the fact that homology commutes with (completed) free products.

Assertion (iii) follows from the main result of [Ha] or one can consult Falk's proof. ■

Since $S^1 \simeq \mathbf{C}^*$, and since every affine curve is homotopy equivalent to a bouquet of circles, we have the next result.

(8.5) Corollary. *Every affine curve is a rational $K(\pi, 1)$.*

The existence of the 3-log will follow from the next result.

(8.6) Proposition. *Each G_1^p is a rational $K(\pi, 1)$.*

Proof. If suffices to show that Y_1^p is a rational $K(\pi, 1)$. Using the coordinates (5.8), we see that

$$Y_1^p = (\mathbf{C} - \{0,1\})^{p-1} - \Delta,$$

where Δ denotes the fat diagonal of points $(z_1, \ldots, z_{p-1})$ whose coordinates z_j are not distinct. Forgetting the first coordinate defines a fibration $Y_1^p \to Y_1^{p-1}$ whose fiber is the complement of p distinct points in $\mathbf{C}$. Since $Y_1^2 = \mathbf{C} - \{0,1\}$ and each of these bundles has trivial monodromy, the result follows from (8.4) by induction on p. ■

Combining this with (7.8) yields:

(8.7) COROLLARY. *For each l, $W_l\tilde{\Omega}^{\cdot}(G_1^p)$ is acyclic.*

Combining this with (6.8) establishes the existence of a 3-log.

(8.8) THEOREM. *There exists a unique* 3-log $Z_3 \in W_6\tilde{\Omega}^{\cdot}(G_\cdot^3)$*/coboundaries.*

It is worth pursuing these ideas a little further as they may establish the existence of p-logarithms for $p > 3$. (At worst they give a unified and conceptual construction of the p-logarithms for $p = 1, 2, 3$.)*

(8.9) THEOREM. *Fix an integer $p > 0$.*
(1) *If G_q^p is a rational $(p-q-1)$-$K(\pi,1)$ for $1 \le q \le p$, then the p-logarithm, if it exists, is unique mod coboundaries.*
(2) *If each G_q^p is a rational $(p-q)$-$K(\pi,1)$ for $1 \le q \le p$, then there exists a (necessarily unique) p-logarithm*

$$Z_p \in W_{2p}\tilde{\Omega}^{\cdot}(G_\cdot^p)/\textit{coboundaries}.$$

Proof. The proof follows from (8.2)(ii) using the evident spectral sequence argument. ■

We conclude this section by giving Falk's example [F] of an algebraic manifold with $q = 0$ that is a $K(\pi, 1)$, but not a rational $K(\pi, 1)$. Let X be the complement of the 6 lines $x = \pm 1$, $y = \pm 1$, $x = \pm y$ in $\mathbf{C}^2$.
This is a $K(\pi, 1)$ as can be seen using Deligne's theorem [D3] or directly as follows (cf. [FR]): Consider the linear system

$$s(x^2 - 1) - t(y^2 - 1) = 0, \quad [s,t] \in \mathbf{P}^1$$

---

*We have constructed a 4-logarithm by showing that G_2^4 is a rational 2-$K(\pi, 1)$.

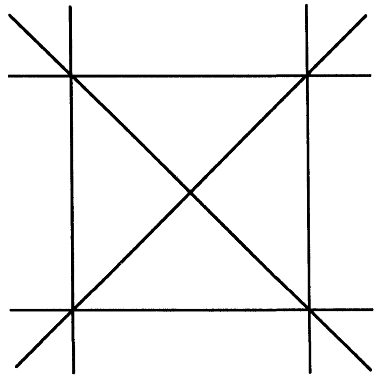


FIG. 10

of conics in $\mathbf{P}^2$. The base locus of this system consists of the 4 points $x = \pm 1$, $y = \pm 1$. The family has 3 singular fibers $x^2 = 1$, $y^2 = 1$ and $x^2 = y^2$. All smooth fibers intersect the line at infinity in 2 points. It follows that X is a fibration

$$\mathbf{P}^1 - (6\text{ points}) \to X \to \mathbf{P}^1 - \{0, 1, \infty\}$$

where each fiber is a smooth conic minus the 4 basepoints and the 2 points at infinity, and that X is a $K(\pi, 1)$.

To prove that X is not a rational $K(\pi, 1)$, we show that $X \times \mathbf{C}^*$ is not a rational $K(\pi, 1)$. To do this it suffices to show that $H^{\bullet}(X \times \mathbf{C}^*)$ is not a quadratic algebra. This follows by applying the following beautiful criterion to

$$X \times \mathbf{C}^* = \mathbf{C}^3 - \{x = \pm z, y = \pm z, x = \pm y, z = 0\}.$$

(8.10) FALK'S CRITERION. *Suppose that $\mathscr{H}$ is a set of codimension* 1 *linear subspaces of $\mathbf{C}^n$. Let $X = \mathbf{C}^n - \cup \mathscr{H}$. If there exist hyperplanes $\{L, H_1, \ldots, H_k\} \subseteq \mathscr{H}$ such that*

(i) $\operatorname{cod}(L \cap H_1 \cap \cdots \cap H_k) = k$,

(ii) *for each i, j satisfying $1 \leq i < j \leq n$, there is no hyperplane $H \in \mathscr{H} - \{H_i, H_j\}$ containing $H_i \cap H_j$,*

then $H^{\bullet}(X)$ is not a quadratic algebra.

Applying Falk's criterion with

$$H_1 = \{z = 0\}, \quad H_2 = \{x = y\}, \quad H_3 = \{x = -y\}, \quad L = \{x = z\}$$

shows that $X \times \mathbf{C}^*$ is not a rational $K(\pi, 1)$. Falk gives a similar argument to show that the complement of the arrangement of hyperplanes in $\mathbf{C}^4$ corresponding to the reflection group D_4, a $K(\pi, 1)$ by [D2], is not a rational $K(\pi, 1)$.

Another class of examples of algebraic manifolds with $q = 0$ that are $K(\pi, 1)$'s but not rational $K(\pi, 1)$'s arises as follows. Fix a smooth curve C in $\mathbf{P}^2$ of genus ≥ 1. Let X be any Zariski open subset of $\mathbf{P}^2 - C$. Then X is not a rational $K(\pi, 1)$ as $H^1(X)$ is pure of weight 2 and $H^2(X)$ contains a copy of $H^1(C, \mathbf{Z}(-1))$ which is of weight 3. Since the cup product preserves weights, $H^1(X)$ cannot generate $H^{\cdot}(X)$. So, by (8.2), X cannot be a rational $K(\pi, 1)$. By Artin's theorem, there are many Zariski open subsets X of $\mathbf{P}^2 - C$ that are $K(\pi, 1)$'s.

9. Symmetry

In this section we exploit the action of the symmetric group on G_q^p to construct a more canonical representative of the 3-logarithm. We also use symmetry to prove the vanishing of $A^* \operatorname{vol}_p$ for all p.

The symmetric group on $n + 1$ letters, Σ_{n+1}, acts on $\mathbf{P}^n$ by permuting the coordinates:

$$\sigma : [z_0 : z_1 : \ldots : z_n] \mapsto [z_{\sigma(0)} : z_{\sigma(1)} : \ldots : z_{\sigma(n)}].$$

When $n = p + q$, this action induces an action of Σ_{p+q+1} on G_q^p. This gives $\Omega^{\cdot}(G_q^p)$ the structure of a Σ_{p+q+1}-module which does not seem to lift to a Σ_{p+q+1} module structure on $(\tilde{\Omega}^{\cdot}(G_q^p), W_{\cdot})$.[5] Nonetheless, by (2.6), we have:

(9.1) Proposition. *The action of Σ_{p+q+1} on G_q^p induces a Σ_{p+q+1} action on the differential graded algebra*

$$\left(Gr_{\cdot}^W \tilde{\Omega}^{\cdot}(G_q^p), d\right).$$

The combinatorial differential

$$A^* : Gr_{\cdot}^W \tilde{\Omega}^{\cdot}(G_{q-1}^p) \to G_{\cdot}^W \tilde{\Omega}^{\cdot}(G_q^p)$$

behaves well with respect to the actions of Σ_{p+q} and Σ_{p+q+1} in a sense that we now make precise. Denote the character of Σ_m that takes a permutation σ to its signature $\operatorname{sgn}(\sigma) \in \{1, -1\}$ by sgn_m. Denote the isotypical component of a Σ_m-module V corresponding to sgn_m by $s(V)$ and its unique Σ_m-invariant complement by $r(V)$. The following result is a special case of (9.5) which is proved later in this section.

[5] However, the Σ_{p+q+1} action on $\Omega^{\cdot}(G_q^p)$ does lift to an action of the braid group B_{p+q+1} on $(\tilde{\Omega}^{\cdot}(G_q^p), W_{\cdot})$. This can be seen from the proof of (5.4).

(9.2) PROPOSITION. *A^* preserves the decomposition $V_t = s(V_t) \oplus r(V_t)$, where*

$$V_t = Gr_l^W \tilde{\Omega}^{\cdot}(G_t^p), \quad t = q-1, q.$$

In particular, if $v \in V_{q-1}$ spans a copy of sgn_{p+q}, then A^*v spans a copy of sgn_{p+q+1} provided that it is not zero.

Define the symbol of

$$Z_p = \left(Z_0^p, \dots, Z_{p-1}^p\right) \in \bigoplus_{q=0}^{p-1} W_{2p}\tilde{\Omega}^{p-q-1}\left(G_q^p\right).$$

to be its equivalence class $\overline{Z}_p = (\overline{Z}_0^p, \dots, \overline{Z}_{p-1}^p)$ in

$$\bigoplus_{q=0}^{p-1} Gr_{2p}^W \tilde{\Omega}^{p-q-1}\left(G_q^p\right).$$

Since the volume form $\mathrm{vol}_p \in \Omega^p(G_0^p)$ spans a copy of sgn_{p+1}, we have the following refined version of (8.9) which follows from an argument using (7.8).

(9.3) THEOREM. *Fix an integer $p > 0$. If each G_q^p is a rational $(p-q)$-$K(\pi, 1)$ for $1 \le q \le p$, then there exists a p-logarithm Z_p whose symbol*

$$\overline{Z}_p = \left(\overline{Z}_0^p, \dots, \overline{Z}_{p-1}^p\right), \quad \overline{Z}_q^p \in Gr_{2p}^W \tilde{\Omega}^{p-q-1}\left(G_q^p\right)$$

has the property that, for $0 \le q \le p$, $\sigma(\overline{Z}_q^p) = \mathrm{sgn}(\sigma)\overline{Z}_q^p$, for all $\sigma \in \Sigma_{p+q+1}$. ∎

The representative of the dilogarithm that we constructed in (6.4) is of this form.

We now prove (9.2). It is convenient to abstract the setting. Consider Σ_{n+1} as the group of automorphisms of the set $[n] = \{0, 1, \dots, n\}$. Suppose that M^t is a $\mathbf{C}\Sigma_{t+1}$ module for $t = m-1$ and m. Suppose that we have $\mathbf{C}$-linear maps A_j: $M^{m-1} \to M^m$ for $0 \le j \le m$ which satisfy

$$(i-1, i) \circ A_j = \begin{cases} A_j \circ (i-2, i-1), & j < i-1, \\ A_i, & j = i-1, \\ A_{i-1}, & j = i, \\ A_j \circ (i-1, i), & j > i, \end{cases} \tag{9.4}$$

for $j = 0, \dots, m$ and $i = 1, \dots, m-1$. Define A^*: $M^{m-1} \to M^m$ to be $\sum_{j=0}^m (-1)^j A_j$.

(9.5) LEMMA. *If V is a Σ_m-submodule of M^{m-1}, then the Σ_{m+1}-submodule A^*V of M^m generated by A^*V is a quotient of* $\operatorname{Ind} V$, *the Σ_{m+1}-module induced from V. Moreover, A^* preserves the decompositions*

$$M^l = s(M^l) \oplus r(M^l).$$

In particular, $A^*s(V)$ is a Σ_{m+1}-submodule of M^m.

Proposition (9.2) is a special case of this lemma because of the next proposition.

(9.6) PROPOSITION. *The modules*

$$M^t = Gr_l^W \tilde{\Omega}^{\cdot}(G_{t-p}^p)$$

and the maps A_j: $M^{t-1} \to M^t$ satisfy (9.4) *for all $t > p$.*

Proof. This is a direct consequence of the fact that the face maps

$$A_j = \mathbf{P}(d_j)\colon \mathbf{P}^{t-1} \to \mathbf{P}^t$$

satisfy (9.4), where the $\mathbf{P}(d_j)$ are the face maps defined at the beginning of Section 5. ■

Proof of (9.5). By replacing M^{m-1} by V if necessary, we may assume that $M^{m-1} = V$. The condition (9.4) implies that

$$A_j = (j, j-1, \ldots, 1, 0) \circ A_0, \quad j = 0, \ldots, m.$$

Let

$$\tilde{M}^m = \operatorname{Ind} M^{m-1} = \mathbf{C}\Sigma_{m+1} \otimes_{\Sigma_m} M^{m-1}.$$

Define $\tilde{A}_j$: $M^{m-1} \to \tilde{M}^m$ to be the composite

$$M^{m-1} \to (j, j-1, \ldots, 1, 0) \otimes M^{m-1} \hookrightarrow \tilde{M}^m.$$

By the universal mapping property of induced representations, there is a unique Σ_{m+1} module homomorphism φ: $\tilde{M}^m \to M^m$ such that the diagram

$$\begin{tikzcd} & \tilde{M}^m \arrow[d, "\varphi"] \\ M^{m-1} \arrow[ur, "\tilde{A}_j"] \arrow[r, "A_j"'] & M^m \end{tikzcd}$$

commutes when $j = 0$. By construction of the $\tilde{A}_j$, it commutes for all j, from which it follows that the image of φ contains $\overline{A^*M^m}$. This establishes the first claim.

If $v \in M^{m-1}$ spans a copy of sgn_m, then by (9.4),

$$\begin{aligned}(i-1,i)\circ A^*v &= (i-1,i)\sum_{j=0}^{m}(-1)^jA_jv \\ &= \sum_{j=0}^{i-2}(-1)^jA_j\circ(i-2,i-1)v + (-1)^{i-1}A_iv \\ &\quad +(-1)^iA_{i-1}v + \sum_{j=i+1}^{m}(-1)^iA_i\circ(i-1,i)v \\ &= -A^*v.\end{aligned}$$

Since the transpositions $(i-1,i)$ generate Σ_{m+1}, it follows that A^*v, if non-zero, spans a copy of sgn_{m+1}. Consequently, $A^*s(V) \subseteq s(M^{m+1})$.

We will abuse notation and not distinguish between a representation and its character. Since $\operatorname{Ind} V \to \overline{A^*V}$ is surjective,

$$0 \le \langle \chi, \overline{A^*V}\rangle \le \langle \chi, \operatorname{Ind} V\rangle = \langle \operatorname{Res}\chi, V\rangle$$

for all characters χ of Σ_{m+1}. (Here $\operatorname{Res}\chi$ denotes the restriction of χ to Σ_m.) In particular,

$$0 \le \langle \mathrm{sgn}_{m+1}, \overline{A^*V}\rangle \le \langle \mathrm{sgn}_m, V\rangle.$$

Taking V to be $r(M^{m-1})$, we see that $A^*r(M^{m-1}) \subseteq r(M^m)$. ■

Now we establish the vanishing of $A^*\mathrm{vol}_p$ using symmetric groups.

(9.7) THEOREM. *For all $p \ge 1$, $A^*\mathrm{vol}_p = 0$ in $\Omega^p(G_1^p)$.*

Since vol_p spans a copy of sgn_{p+1} in $\Omega^p(G_0^p)$, to prove (9.7), it suffices to show that $\Omega^p(G_1^p)$ does not contain a copy of sgn_{p+2}. We will prove this by taking residues.

(9.8) LEMMA. *If D is a divisor in a smooth variety X, then there is a linear mapping*

$$\operatorname{Res}_D\colon W_l\Omega^k(X-D) \to W_{l-2}\Omega^{k-1}(D^*),$$

where D^* is the nonsingular part of D.

This can be proved by using resolution of singularities [Hk] to reduce to the case where D is a divisor with normal crossings in a smooth, complete variety. The residue in the normal crossings case is constructed in [D1]. One then checks that the result is independent of the resolution.

The residue has the following well known properties:

(9.9) PROPOSITION. (i) *If D is a divisor in X, then the sequence*

$$0 \to \Omega^k(X) \to \Omega^k(X - D) \xrightarrow{\mathrm{Res}_D} \Omega^{k-1}(D^*)$$

is exact.

(ii) *If $f\colon (X, D) \to (X, D)$ is an isomorphism, then the diagram*

$$\begin{tikzcd} \Omega^k(X - D) \arrow[r] \arrow[d, "f^*"'] & \Omega^{k-1}(D^*) \arrow[d, "f^*"] \\ \Omega^k(X - D) \arrow[r] & \Omega^{k-1}(D^*) \end{tikzcd}$$

commutes.

Since $\Omega^p(X) = 0$ for all smooth, complete rational varieties X and all $p > 0$, repeated application of (9.9)(i) yields the next result.

(9.10) COROLLARY. *Suppose that $X = \overline{X} - \cup D_i$ and that $\cup D_i = \coprod S_\alpha$, is a stratification of $\cup D_i$ satisfying the condition that the closure $\overline{S}_\alpha$ of each stratum is a union of strata. If X and each S_α are rational, then $\omega \in \Omega^p(X)$ is zero if and only if*

$$\mathrm{Res}_{S_{\alpha_p}} \mathrm{Res}_{S_{\alpha_{p-1}}} \cdots \mathrm{Res}_{S_{\alpha_1}} \omega = 0$$

for all sequences $(\alpha_1, \dots, \alpha_p)$ for which codim $S_{\alpha_j} = j$ *and* $S_{\alpha_{j+1}} \subseteq \overline{S}_{\alpha_j}$.

Our next task is to construct such a stratification of $G(1, \mathbf{P}^{p+1})$, the grassmannian of lines in $\mathbf{P}^{p+1}$. We will call a linearly imbedded $\mathbf{P}^{q+1}$ in $\mathbf{P}^{p+1}$ a *coordinate flat* if it is the intersection of coordinate hyperplanes.

(9.11) LEMMA. *There exists a stratification*

$$G(1, \mathbf{P}^{p+1}) = \coprod S_\alpha$$

of $G(1, \mathbf{P}^{p+1})$ with the following properties.

(i) *Each S_α is smooth.*
(ii) *The closure of each stratum is a union of strata.*
(iii) *The top stratum is G_1^p.*

(iv) *Each stratum is rational.*

(v) *The stratification is natural with respect to the inclusions* $G(1, \mathbf{P}^{q+1}) \to G(1, \mathbf{P}^{p+1})$ *induced by the inclusions* $\mathbf{P}^{q+1} \to \mathbf{P}^{p+1}$ *of coordinate flats.*

(vi) *The stratification is invariant under the action of* Σ_{p+2} *on* $G(1, \mathbf{P}^{p+1})$.

(vii) *For each chain of closed strata* $\overline{S}_{\alpha_1} \supseteq \overline{S}_{\alpha_2} \supseteq \cdots \supseteq \overline{S}_{\alpha_p}$ *satisfying* $\operatorname{cod} S_{\alpha_j} = j$, *there exists a transposition* $\sigma \in \Sigma_{p+2}$ *that stabilizes the whole chain, i.e.,* $\sigma(S_{\alpha_j}) = S_{\alpha_j}$ *for every* j.

Remark. These strata of $G(1, \mathbf{P}^{p+1})$ are the grassmannian strata of [GGMS].

Proof. We prove the result by induction on p. When $p = 1$ the proposition is true. Suppose that it is true for all $q < p$. By induction and naturality, we can write

$$Y = \{l \in G(1, \mathbf{P}^{p+1}) : l \subseteq A, \text{where } A \text{ is a coordinate hyperplane}\}.$$

as a union of strata

$$Y = \coprod_{\alpha \in \mathscr{A}} S_\alpha$$

which satisfies (i), (ii), (iv), (v), (vi).

For each coordinate flat $A \subseteq \mathbf{P}^{p+1}$, we have the $p + \dim A$ dimensional variety

$$D_A = \{l \in G(1, \mathbf{P}^{p+1}) : l \cap A \neq \varnothing\}.$$

For each unordered pair of coordinate flats A, B that span $\mathbf{P}^{p+1}$ and are each of dimension $< p$, we have the $\dim A + \dim B$ dimensional subvariety

$$E_{A,B} = D_A \cap D_B.$$

Set

$$S_A = D_A - \left(Y \cup \bigcup_B E_{A,B}\right), \quad S_{A,B} = E_{A,B} - \left(Y \cup \bigcup_{\{C,D\} < \{A,B\}} E_{C,D}\right).$$

It is necessary to prove the disjointness of the $S_{A,B}$. This follows directly from the next Proposition. If $A \approx \mathbf{P}^m$ is a coordinate flat in $\mathbf{P}^n$, let

$$\operatorname{int} A = A - \bigcup B \approx (\mathbf{C}^*)^m$$

where B ranges over the proper coordinate flats of A.

(9.12) PROPOSITION. *Suppose that A, B, C are distinct coordinate flats in $\mathbf{P}^n$, each of dimension $\leq n-2$. If l is a line in $\mathbf{P}^n$ that intersects each of* $\operatorname{int} A, \operatorname{int} B, \operatorname{int} C$, *then l is contained in a proper coordinate flat of $\mathbf{P}^n$.*

Proof. Since l is contained in the span $\langle A, B\rangle, \langle B, C\rangle, \langle C, A\rangle$ of each pair of A, B, C, we are done if any of $\langle A, B\rangle$, $\langle B, C\rangle$ or $\langle C, A\rangle$ has dimension $< n$. So we assume that each pair of A, B, C spans $\mathbf{P}^n$. Since each of A, B, C has codimension ≥ 2, there are 6 vertices (a vertex is a coordinate flat of dimension 0) $v_0, v_1, v_2, v_3, v_4, v_5$ such that each of the intersections

$$\langle v_0, v_1\rangle \cap A, \quad \langle v_2, v_3\rangle \cap B, \quad \langle v_4, v_5\rangle \cap C$$

is trivial. Denote the unique $n-6$ dimensional coordinate flat complementary to $\langle v_0, \dots, v_5\rangle$ by D. If $l \subseteq D$ we are done. Otherwise project l to $\langle v_0, \dots, v_5\rangle \approx \mathbf{P}^5$. Since D intersects each of $\operatorname{int} A, \operatorname{int} B, \operatorname{int} C$ non-trivially, l projects to a line $\bar{l}$ in $\mathbf{P}^5$. Since each pair of A, B, C span $\mathbf{P}^n$, the projections of A, B, C to $\mathbf{P}^5$ are

$$\overline{A} = \langle v_2, v_3, v_4, v_5\rangle, \quad \overline{B} = \langle v_0, v_1, v_4, v_5\rangle, \quad \overline{C} = \langle v_0, v_1, v_2, v_3\rangle$$

respectively. By elementary linear algebra, there is no line in $\mathbf{P}^5$ that intersects each of $\operatorname{int} \overline{A}, \operatorname{int} \overline{B}, \operatorname{int} \overline{C}$. This proves the proposition. ■

(9.13) COROLLARY.

$$G(1, \mathbf{P}^{p+1}) = G_1^p \amalg \coprod_{\dim A < p} S_A \amalg \coprod_{\substack{A, B \\ \dim A, B < p}} S_{A,B} \amalg \coprod_{\alpha \in \mathscr{A}} S_\alpha. \qquad ■$$

This stratification clearly satisfies (i)–(vi). It remains to prove (vii). For this we need a more complete description of the strata. By induction on p,

$$G(1, \mathbf{P}^{p+1}) = \coprod_{C} S_C \amalg \coprod_{(A;C)} S_{A;C} \amalg \coprod_{(A,B;C)} S_{A,B;C} \tag{9.14}$$

where

• C ranges over all coordinate flats of $\mathbf{P}^{p+1}$ and S_C is dense in the grassmannian of lines in C;

$$\dim S_C = 2 \dim C - 2;$$

• $(A; C)$ ranges over all pairs of coordinate flats $A \subseteq C$ where A has codimension ≥ 2 in C;

$$\dim S_{A;C} = \dim A + \dim C - 1;$$

• $(A, B; C)$ ranges over all unordered pairs A, B of coordinate flats of a coordinate flat C where A and B span C and A and B both have codimension ≥ 2 in C;

$$\dim S_{A,B;C} = \dim A + \dim B.$$

(9.15) PROPOSITION. (1) *All strata of codimension* 1 *in* $\overline{S}_C$ *are of the form* $S_{A;C}$, *where* A *has codimension* 2 *in* C.

(2) *All strata of codimension* 1 *in* $\overline{S}_{A;C}$ *are of one of the following forms*:

$S_{A;D}$, *where* D *is a hyperplane in* C *that contains* A *and* $\operatorname{cod}_C A \geq 3$,
$S_{A,B;C}$ *where* B *has codimension* 2 *in* C, A *and* B *span* C,
S_D, *where* D *is a hyperplane in* C *and* $\operatorname{cod}_C A = 2$.

(3) *All strata of codimension* 1 *in* $\overline{S}_{A,B;C}$ *are of one of the following forms*:

$(A, D; C)$, *where* D *is a hyperplane in* B *and* A *and* D *span* C,
$(A, D; E)$, *where* E *is a hyperplane in* C *that contains* A, $D = B \cap E$ *and* $\operatorname{cod}_C A \geq 3$
$(D; E)$, *where* E *is a hyperplane in* C *that contains* A, $D = B \cap E$, *and* $\operatorname{cod}_C A = 2$.

Denote the set of vertices of $\mathbf{P}^{p+1}$ by $[p+1] = \{0, 1, \ldots, p+1\}$. To each coordinate flat we can associate its set of vertices, a subset of $[p+1]$.

A decomposition

$$[p+1] = \coprod_j F_{\alpha,j}$$

can be associated to each stratum S_α of (9.14): To S_α, associate the 1, 2 or 3 coordinate flats that index it. Each of these corresponds to a subset of $[p+1]$. The decomposition above is the coarsest decomposition of $[p+1]$ for which each of these subsets is a union of the $F_{\alpha,j}$. For example, the decomposition associated to the stratum

$$(\langle 0, \ldots, q-1\rangle, \langle 2, \ldots, q+1\rangle, \langle 0, \ldots, q+1\rangle)$$

is

$$\{0,1\} \cup \{2,\dots,q-1\} \cup \{q,q+1\} \cup \{q+2,\dots,p+1\}.$$

The following assertions are direct consequences of (9.15).

(9.16) PROPOSITION. *Suppose that the decomposition associated to a stratum S_α divides $[p+1]$ into m subsets. If S_β has codimension 1 in $\bar{S}_\alpha$, then the common refinement of the decompositions associated to S_α and S_β divides $[p+1]$ into m or $m+1$ subsets.*

(9.17) PROPOSITION. *If the decomposition of $[p+1]$ associated to the stratum S_α is*

$$[p+1] = \coprod_{j=1}^{m} F_j,$$

then the stablizer of S_α in Σ_{p+1} contains

$$\prod_{j=1}^{m} \operatorname{Aut}(F_j).$$

We can now prove (9.11)(vii). Suppose that $\bar{S}_{\alpha_1} \supseteq \bar{S}_{\alpha_2} \supseteq \cdots \supseteq \bar{S}_{\alpha_p}$ is a chain of closed strata of (9.14), where S_{α_j} has codimension j in $G(1,\mathbf{P}^{p+1})$. By (9.16) the common refinement of the decomposition associated to the S_{α_j} divides $[p+1]$ into at most $p+1$ nonempty subsets F_j. Since $[p+1]$ has cardinality $p+2$, one of the F_j must have at least 2 elements. By (9.17) there is an involution $\sigma \in \Sigma_{p+2}$ that stabilizes the chain. This completes the proof of (9.11). ■

Proof of (9.7). Suppose that $\omega \in \Omega^p(G_1^p)$ satisfies $\sigma \cdot \omega = \operatorname{sgn}(\sigma)\omega$ for all $\sigma \in \Sigma_{p+2}$. By (9.11)(vii), for each chain

$$\bar{S}_{\alpha_1} \supseteq \bar{S}_{\alpha_2} \supseteq \cdots \supseteq \bar{S}_{\alpha_p}$$

of closed strata satisfying cod $S_{\alpha_j} = j$, there is a transposition $\sigma \in \Sigma_{p+2}$ that stablizes it. Denote the operator

$$\operatorname{Res}_{S_{\alpha_1}} \operatorname{Res}_{S_{\alpha_2}} \dots \operatorname{Res}_{S_{\alpha_p}} : \Omega^k(G_1^p) \to \Omega^{k-p}(S_{\alpha_p})$$

by R. Since $R\omega \in \Omega^0(S_{\alpha_p}) \cong \mathbf{C}$, $R\omega$ is invariant under σ, so that

$$R\omega = \sigma^* R\omega = R\sigma^*\omega = -R\omega.$$

This implies that $R\omega = 0$. So, by (9.10) and (9.11), $\omega = 0$. This completes the proof of (9.7). ■

10. Non-triviality and indecomposability of the 3-logarithm

In this section we prove that each representative of the 3-logarithm function is non-zero, not a product of logarithm and dilogarithm type functions, and does not trivially satisfy the functional equation. More precisely, we shall prove:

(10.1) THEOREM. *Let $L_3 \in W_6\tilde{\mathscr{O}}(G_2^3)$ be a 3-logarithm function.*

(i) *L_3 is not decomposable; that is, L_3 cannot be expressed as a sum*

$$L_3 = \sum_{i=1}^{N} F_i G_i,$$

where $F_i, G_i \in W_4\tilde{\mathscr{O}}(G_2^3)$;

(ii) *The residue class of L_3 in $W_6\tilde{\mathscr{O}}(G_2^3)/A^*W_6\tilde{\mathscr{O}}(G_1^3)$ is not zero.*

Here is our strategy for proving (10.1).

1. Calculate $Gr_6^W\tilde{\mathscr{O}}$ as a representation of the symmetric group for G_1^3 and G_2^3.
2. Calculate explicitly the symbol $\overline{L}_3$ of one particular representative of the 3-log function.
3. Show that $\overline{L}_3$ spans a copy of the sign representation in $Gr_6^W\tilde{\mathscr{O}}(G_2^3)$ whose projection to the indecomposable elements of $Gr_6^W\tilde{\mathscr{O}}(G_2^3)$ is non-zero. This will yield the indecomposability assertion.
4. Show that the indecomposable part of $Gr_6W\tilde{\mathscr{O}}(G_1^3)$ contains no copy of the sign representation. An application of (9.2) will yield the second assertion of (10.1).

We begin with some comments about the graded ring $Gr_\bullet^W\tilde{\mathscr{O}}$ and its indecomposables. Let X be an algebraic manifold with $q(X) = 0$. As in Section 3, we denote the algebra of iterated line integrals generated by $\Omega^1(X)$ by $A(X)$ and the subspace of relatively closed iterated integrals in $A(X)$ by $H^0(A(X))$. These algebras are graded. For each $x \in X$, the canonical homomorphism

$$H^0(A(X)) \to \tilde{\mathscr{O}}(X, x), \quad I \mapsto \left\{ p \mapsto \int_x^p I \right\}$$

is filtration preserving and induces a canonical isomorphism

$$H^0(A(X)) \cong Gr_\bullet^W\tilde{\mathscr{O}}(X)$$

which is natural with respect to morphisms $X \to Y$.

The diagonal

$$\Delta\colon \int \omega_1 \dots \omega_r \mapsto \sum_{i=0}^{r} \int \omega_1 \dots \omega_i \otimes \int \omega_{i+1} \dots \omega_r$$

imparts the structure of a commutative Hopf algebra to $A(X)$ and $H^0(A(X))$ [C1]. By the dual of the Poincaré-Birkhoff-Witt Theorem (see, for example [H3]) $H^0(A(X))$ is canonically isomorphic, as an algebra, to the polynomial ring $\mathbf{C}[Q]$ generated by its space of indecomposable elements $Q = I/I^2$. Here I denotes the maximal ideal of iterated integrals with no constant terms, and I^2 denotes its square. Denote the idempotent $I \to I$ corresponding to the canonical splitting $s\colon Q \to I$ of the quotient map $I \to Q$ by γ.

(10.2) PROPOSITION [H3]. *The idempotent γ is given by the formula*

$$\gamma\left(\int \omega_1 \dots \omega_n\right) = \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} \sum_{r_1 + \cdots + r_m = n} \sum_{\sigma \in \mathrm{sh}(r_1, \dots, r_m)} \int \omega_{\sigma(1)} \dots \omega_{\sigma(n)}$$

where $\mathrm{sh}(r_1, \dots, r_m)$ *denotes the shuffles of* $\{1, \dots, n\}$ *of type* $(r_1, \dots, r_m)$. *In particular, when* $n = 3$,

$$\gamma \int \omega_1\omega_2\omega_3 = \tfrac{1}{2}\int (\omega_1\omega_2\omega_3 + \omega_3\omega_2\omega_1) - \tfrac{1}{6}\sum_{\Sigma_3} \omega_{\sigma(1)}\omega_{\sigma(2)}\omega_{\sigma(3)}.$$

Alternatively, one can think of the Hopf algebra structure of $Gr_\bullet^W\tilde{\mathcal{O}}(X)$ as coming from the coordinate ring $H^0(A(X))$ of the proalgebraic group $G = \varprojlim G_s$ associated to X in Section 7.

The space of indecomposables Q of $Gr_\bullet^W\tilde{\mathcal{O}}(X)$ is graded:

$$Q = \bigoplus_{l=1}^{\infty} Q^l.$$

From either of the descriptions of $Gr_\bullet^W\tilde{\mathcal{O}}(X)$ above, it follows, when $q(X) = 0$, that

$$Q^l = \mathrm{Hom}\left(Gr_{-l}^W \mathfrak{g}, \mathbf{C}\right),$$

where $\mathfrak{g}$ is the Malcev Lie algebra associated to $\pi_1(X)$ in Section 7. Since $\mathfrak{g}$ has only even weights, and since the homomorphism $\mathscr{E}(\mathfrak{g}) \to \Omega^\bullet(X)$ induces an isomorphism on H^1 and an injection on H^2, we have:

(10.3) PROPOSITION. *If $q(X) = 0$, then $Q^l = 0$ when l is odd,*

$$Q^2 = H^1(X), \quad Q^4 = \ker\{\Lambda^2 H^1(X) \xrightarrow{\mu} H^2(X)\}$$

and

$$Q^6 = \ker\{Q^2 \otimes Q^4 \xrightarrow{\nu} \Lambda^3 H^1(X)\},$$

where μ is the cup product and ν is induced by the natural map

$$H^1 \otimes \Lambda^2 H^1 \to \Lambda^3 H^1.$$

(*All cohomology groups above are with complex coefficients.*)

Our next task is to compute the symbol

$$\overline{Z}_3 \in Gr_6^W\left[\tilde{\Omega}^2(G_0^3) \oplus \tilde{\Omega}^1(G_1^3) \oplus \tilde{\mathscr{O}}(G_2^3)\right]$$

of a representative of the 3-logarithm. The representative Z_3 will have the property that each component of $\overline{Z}_3$ will span a copy of the sign representation (cf. (9.3)). It is first necessary to understand the first cohomology of G_1^3 and G_2^3.

Denote the free **Z**-module spanned by the $q+1$ element subsets of $\{0,\ldots,p+q\}$ by $M^{[p,q+1]}$. The action of Σ_{p+q+1} on the $q+1$ element subsets gives $M^{[p,q+1]}$ the structure of a Σ_{p+q+1}-module, called the *Specht module* for the partition $[p, q+1]$. Taking each canonical basis vector to 1 defines a Σ_{p+q+1}-module homomorphism

$$M^{[p,q+1]} \to \mathbf{Z}, \tag{10.4}$$

where **Z** is considered as a trivial Σ_{p+q+1}-module.

(10.5) PROPOSITION. *As a Σ_{p+q+1}-module, $H^1(G_q^p)$ is canonically isomorphic to the kernel of* (10.4).

Proof. From the description of G_q^p and the grassmannian $G(q, \mathbf{P}^{p+q})$ given in (5.5) and (5.6), it follows that

$$G_q^p = G(q, \mathbf{P}^{p+q}) - \bigcup_S D_S,$$

where S ranges over the $q+1$ element subsets of $\{0,\ldots,p+q\}$ and D_S denotes the divisor corresponding to those $p+q+1$ tuples of vectors $(v_0,\ldots,v_{p+q})$ where the vectors $\{v_j\colon\ j \notin S\}$ are linearly dependent. The result follows directly from the equivariance of the Gysin sequence

$$0 \to H^1(G_q^p) \to \bigoplus_S H_{2(pq+p)-2}(D_S) \to H^2(G(q, \mathbf{P}^{p+q})) \to 0$$

as the middle term is isomorphic to the Specht module $M^{[p,q+1]}$ as a Σ_{p+q+1}-module. ■

By Young's correspondence, every representation of Σ_n corresponds to a formal sum of partitions of n.

(10.6) COROLLARY. *If* $\mathbf{F}$ *is a field of characteristic zero, then the* Σ_{p+q+1}-*module* $H^1(G_q^p, \mathbf{F})$ *corresponds to the formal sum of partitions*

$$[p, q+1] + [p+1, q] + \cdots + [p+q, 1]$$

and $H^1(Y_q^p, \mathbf{F})$ *to* $H^1(G_q^p, \mathbf{F}) - [p+q, 1]$.

Proof. This follows immediately from (5.9) and (10.5) using Young's rule (see [J; 14.1]). ■

(10.7) COROLLARY. *The first Betti numbers of* G_q^p *and* Y_q^p *are given by*

$$b_1(G_q^p) = \binom{p+q+1}{p} - 1, \quad b_1(Y_q^p) = \binom{p+q+1}{p} - (p+q+1).$$

Observe that the Specht module $M^{[p,q+1]}$ occurs as $H^1(L_q^p)$, where L_q^p is the restriction to G_q^p of the $\mathbf{C}^*$ bundle associated to the line bundle $\mathscr{O}(D_S)$ over the grassmannian. It is convenient to write elements of

$$\Lambda^{\cdot} H^1(G_q^p) \quad \text{and} \quad \otimes^{\cdot} H^1(G_q^p)$$

in terms of the natural basis of the larger modules

$$\Lambda^{\cdot} H^1(L_q^p) \quad \text{and} \quad \otimes^{\cdot} H^1(L_q^p).$$

We shall denote the generator of $H^1(L_q^p)$ corresponding to the subset $S = \{s_0 < s_1 < \cdots < s_q\}$ of $\{0, \ldots, p+q\}$ by $s_0 s_1 \ldots s_q$.

Suppose that v is an element of a Σ_n-module V. Define

$$\mathrm{Alt}_n v = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \sigma v.$$

(10.8) PROPOSITION. *In the notation above, the volume form* vol_p *is given by*

$$\mathrm{vol}_p = \frac{1}{p!} \mathrm{Alt}_{p+1} 1 \wedge 2 \wedge \cdots \wedge p.$$

(10.9) PROPOSITION. *If X is an algebraic manifold with $q(X) = 0$ and $H^2(X)$ pure of weight* 4 (*e.g.*, $\Lambda^2 H^1(X) \to H^2(X)$ *onto*), *then*

$$Gr_6^W \tilde{\Omega}^2(X) \cong H^1(X) \otimes H^2(X).$$

(10.10) COROLLARY. *The symbol of the element η of $W_6\tilde{\Omega}^2(G_0^3)$, constructed in* (6.8), *that satisfies $d\eta = \mathrm{vol}_3$ is*

$$\bar{\eta} = -\frac{1}{3!}\mathrm{Alt}_4 0 \otimes 1 \wedge 2$$

Before proceeding, it is necessary to describe the face maps in terms of the canonical bases.

(10.11) PROPOSITION. *The map*

$$A_j \colon H^1(G_q^p) \to H^1(G_{q+1}^p)$$

takes S to $d_j(S) \cup \{j\}$, where d_j is the j^{th} face map.

$$d_j \colon \{0, \dots, p+q\} \to \{0, \dots, p+q+1\}.$$

(10.12) LEMMA. *Suppose M^{n-1} is a Σ_n-module, that M^n is a Σ_{n+1}-module and that*

$$A_j \colon M^{n-1} \to M^n \quad 0 \le j \le n$$

are maps satisfying (9.4). *If $v \in M^{n-1}$, then*

$$A^*(\mathrm{Alt}_n v) = \mathrm{Alt}_{n+1}(A_0 v).$$

Proof. Denote the isotropy group of j in Σ_{n+1} by $\Sigma(j)$. Then

$$\Sigma(j) = (j, j-1, \dots, 1, 0)\Sigma(0)(0, 1, \dots, j).$$

As in the proof of (9.5)

$$A_j = (j, j-1, \dots, 1, 0) \circ A_0.$$

Since id, $(1,0), (2,1,0), \dots, (n, n-1, \dots, 1, 0)$ is a set of left coset representative of $\Sigma(0)$ in Σ_{n+1},

$$\begin{aligned}\mathrm{Alt}_{n+1}(A_0 v) &= \sum_{j=0}^{n} \sum_{(j,\dots,0)\Sigma(0)} \mathrm{sgn}(\sigma)\sigma A_0 v \\ &= \sum_{j=0}^{n} (-1)^j \sum_{\Sigma(j)} \mathrm{sgn}(\sigma)\sigma(j,\dots,0)A_0 v \\ &= \sum_{j=0}^{n} (-1)^j \sum_{\Sigma(j)} \mathrm{sgn}(\sigma)\sigma A_j v \\ &= \sum_{j=0}^{n} (-1)^j A_j(\mathrm{Alt}_n v) \\ &= A^*(\mathrm{Alt}_n v). \qquad \blacksquare\end{aligned}$$

(10.13) COROLLARY. *The symbol of $A^*\eta$ is*

$$A^*\overline{\eta} = -\frac{1}{3!}\,\mathrm{Alt}_5\, 01 \otimes 02 \wedge 03 \in Gr_6^W \tilde{\Omega}^2(G_1^3).$$

Our next task is to understand the relations in $H^2(G_1^3)$. As a corollary of the proof of (8.6) we have:

(10.14) PROPOSITION. *The Poincaré series of G_1^p and Y_1^p are*

$$g_1^p(t) = (1+t)^p \prod_{j=1}^{p}(1+jt) \quad \textit{and} \quad y_1^p(t) = \prod_{j=2}^{p}(1+jt).$$

Since G_1^p and Y_1^p are rational $K(\pi, 1)$ spaces, their cohomology rings are generated by H^1 (see (8.2)). Since $G_1^p = Y_1^p \times (\mathbf{C}^*)^{p+1}$,

$$\mathfrak{g}(G_1^p) = \mathfrak{g}(Y_1^p) \oplus \mathbf{C}^{p+1} \quad \text{and} \quad Q^l(G_1^p) = Q^l(Y_1^p)$$

whenever $l > 2$.

(10.15) PROPOSITION. *We have* $\dim Q^4(G_1^3) = \dim Q^4(Y_1^3) = 4$. *Moreover, as a Σ_5-module, Q^4 is isomorphic to the irreducible representation corresponding to the partition* [2, 1, 1, 1].

Proof. The first assertion follows from (10.3) and (10.14). To see the second, note that $H^1(Y_1^3)$ corresponds to the partition [3, 2] and its second exterior power to [3, 1, 1] + [2, 1, 1, 1]. The result follows as [3, 1, 1] has rank 6 and [2, 1, 1, 1] rank 4. $\blacksquare$

As in the proof of (10.12), we denote the stabilizer of $j \in \{0,1,2,3,4\}$ in Σ_5 by $\Sigma(j)$.

(10.16) PROPOSITION. *The relation module $Q^4(G_1^3)$ is spanned by the elements*

$$\begin{aligned} R_0 &= \tfrac{1}{2}\operatorname{Alt}_{\Sigma(0)} 12 \wedge 13 \\ R_1 &= \tfrac{1}{2}\operatorname{Alt}_{\Sigma(1)} 02 \wedge 03 \\ R_2 &= \tfrac{1}{2}\operatorname{Alt}_{\Sigma(2)} 01 \wedge 03 \\ R_3 &= \tfrac{1}{2}\operatorname{Alt}_{\Sigma(3)} 01 \wedge 02 \\ R_4 &= \tfrac{1}{2}\operatorname{Alt}_{\Sigma(4)} 01 \wedge 02. \end{aligned}$$

These are subject to the relation $\sum_{j=0}^4 (-1)^j R_j = 0$.

Proof. The single relation in $H^2(G_1^2)$ is $A^*\operatorname{vol}_2$. By (10.8), $\operatorname{vol}_2 = 1/2\operatorname{Alt}_3 0 \wedge 1$. So by (10.12), the relation in $H^2(G_1^2)$ is $A^*\operatorname{vol}_2 = 1/2\operatorname{Alt}_4 01 \wedge 02$. Viewing G_1^3 as G_2^2, we obtain 5 face maps $B_j\colon G_1^3 \to G_1^2$. The elements R_j are given by

$$R_j = B_j^*(A^*\operatorname{vol}_2).$$

The second assertion follows by a direct calculation, or by applying (9.5) and (10.15) to

$$B^*\colon Q^4(G_1^2) \to Q^4(G_1^3). \qquad \blacksquare$$

The next step is to find the symbol

$$\bar{\xi} \in Gr_6^W \tilde{\Omega}^1(G_1^3) \cong \left(S^2 H^1(G_1^3) \oplus Q^4\right) \otimes H^1(G_1^3)$$

of an element ξ of $W_6\tilde{\Omega}^1(G_1^3)$ that satisfies $d\xi + A^*\eta = 0$. Here S^2 denotes the second symmetric power.

To find such an element ξ, it suffices to find its symbol, for suppose that

$$u \in Gr_6^W \tilde{\Omega}^1(G_1^3)$$

satisfies $du + A^*\bar{\eta} = 0$ in $Gr_6^W\tilde{\Omega}^2(G_1^3)$. Choose as a lift $\tilde{u}$ of u to $W_6\tilde{\Omega}^1(G_1^3)$. Then $d\tilde{u} + A^*\eta$ is a closed element of $W_4\tilde{\Omega}^4(G_1^3)$. By (8.7) and (8.6), there exists $v \in W_4\tilde{\Omega}^1(G_1^3)$ satisfying $dv = d\tilde{u} + A^*\eta = 0$. The element $\xi = \tilde{u} - v$ has symbol u and satisfies $d\xi + A^*\eta = 0$.

It is convenient to view

$$Gr_6^W\tilde{\Omega}^1 \cong K \otimes H^1$$

as a subspace of $\otimes^3 H^1$. Here K denotes the kernel $Q^4 \oplus S^2H^1$ of the cup product $\otimes^2 H^1 \to H^2$.

(10.17) By (3.9) and the preceding discussion, $\bar{\xi} \in Gr_6^W \tilde{\Omega}^1(G_1^3)$ is the symbol of an element ξ of $W_6\tilde{\Omega}^1(G_1^3)$ satisfying $d\xi + A^*\eta = 0$ if and only if the image of ξ under

$$\mathrm{id} \otimes \mathrm{cup}: \otimes^3 H^1(G_1^3) \to H^1(G_1^3) \otimes H^2(G_1^3) \cong Gr_6^W \tilde{\Omega}^2(G_1^3)$$

is $-A^*\bar{\eta}$.

(10.18) PROPOSITION. *The element*

$$\tau = \tfrac{1}{72} \mathrm{Alt}_5[6(01 \otimes 02 \otimes 03) - 2(01 \otimes 12 \otimes 23) + 01 \otimes 23 \otimes 12 + 12 \otimes 01 \otimes 23]$$

of $\otimes^3 H^1(G_2^3)$ *lies in* $K \otimes H^1$ *and satisfies* (10.17). *Consequently, it is the symbol* $\bar{\xi}$ *of an element* ξ *of* $W_6\tilde{\Omega}^1(G_1^3)$ *which satisfies* $d\xi + A^*\eta = 0$.

(10.19) COROLLARY. *There is a 3-logarithm function* $L_3 \in W_6\tilde{\mathcal{O}}(G_2^3)$ *with symbol*

$$\bar{L}_3 = \tfrac{1}{72} \mathrm{Alt}_6[-2(012 \otimes 023 \otimes 034) + 012 \otimes 034 \otimes 023 + 023 \otimes 012 \otimes 034].$$

Proof. This follows from (10.18) using (6.2) and (10.12). ■

(10.20) PROPOSITION. *If* $\bar{L}_3$ *is as in* (10.19), *then* $\gamma(\bar{L}_3) = \bar{L}_3$ *so that* $\bar{L}_3$ *is canonically indecomposable.*

This proves that at least one 3-logarithm function is non-zero and indecomposable. To prove that all representatives are indecomposable, we need more explicit information about the multiplicities of the sign representation in $Gr_6^W \mathcal{O}$ of G_1^3 and G_2^3.

(10.21) LEMMA. *The multiplicities of the sign representation in the modules* S^3H^1, $Q^4 \otimes H^1$, *and* Q^6 *of* G_0^3, G_1^3, G_2^3 *and* G_3^3 *are given by the following table*:

	S^3H^1	$Q^4 \otimes H^1$	Q^6
G_0^3	0	0	0
G_1^3	1	1	0
G_2^3	2	1	1
G_3^3	1	0	0

Proof. In all cases, S^3H^1 was calculated from (10.6) with the aid of a computer. In the case of G_0^3, Q^4 and Q^6 are zero, as $\mathfrak{g}(G_0^3)$ is abelian.

By the discussion preceding (10.15), $Q^6(Y_1^3) = Q^6(G_1^3)$. From (10.6) and (10.15) it follows that $Q^4(Y_1^3) \otimes H^1(Y_1^3)$ contains no copies of the sign representation. This and (10.3) imply that the sign representation has multiplicity zero in $Q^6(Y_1^3)$. Another calculation shows that the sign representation occurs in $Q^4(G_1^3) \otimes H^1(G_1^3)$ with multiplicity 1.

To compute the entries for G_3^3, note that

$$Gr_6^W \tilde{\mathscr{O}} = S^3H^1 \oplus \left(Q^4 \otimes H^1\right) \oplus Q^6 \subseteq \bigotimes{}^3 H^1.$$

Machine calculations using (10.6) show that the sign representation has multiplicity 1 in S^3H^1 and $\otimes^3H^1$, from which we obtain the multiplicities for G_3^3.

To calculate the multiplicities for G_2^3 we need more information about its topology.

(10.22) PROPOSITION. *The Poincaré series of G_2^3 and Y_2^3 are*

$$y_2^3(t) = 1 + 14t + 72t^2 + 159t^3 + 126t^4 \quad \textit{and} \quad g_2^3(t) = (1+t)^5 y_2^3(t).$$

Moreover, the cohomology rings $H^\cdot(Y_2^3)$ and $H^\cdot(G_2^3)$ are both generated in dimension 1.

Proof. Since $G_2^3 = Y_2^3 \times (\mathbf{C}^*)^5$, it suffices to prove the result for Y_2^3. Forgetting one point defines a projection $\pi\colon Y_2^3 \to Y_1^3$ whose fiber F_P over the point P of $Y_1^3 = (\mathbf{C} - \{0,1\})^2 - \Delta$ is $\mathbf{C}^2$ minus the configuration in Fig. 11. The topology of the fiber of π is independent of P, and π is a bundle projection. Since the cohomology of the fiber [Br] and the base (8.2), (8.6) are

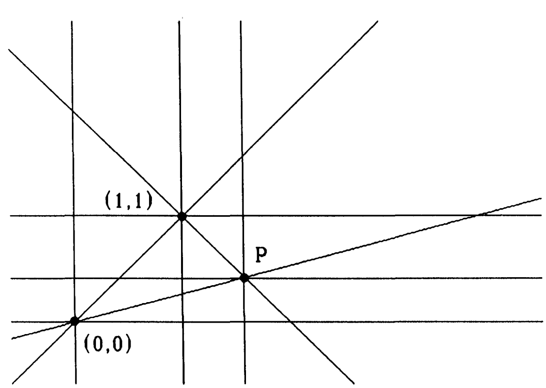


FIG. 11

generated by H^1, and since the Leray spectral sequence of π degenerates at E_2 (by Leray-Hirsch Theorem), the cohomology of Y_2^3 is also generated by H^1.

The degeneration of the Leray spectral sequence also implies that

$$y_2^3(t) = y_1^3(t)p_F(t),$$

where $p_F(t)$ is the Poincaré series of the fiber. Since F is affine of dimension 2, $p_F(t)$ has degree 2. The first Betti number of F is just the number of lines in the configuration, which is 9. To compute the second Betti number, choose the point P to be real. Then, by a theorem of Orlik and Solomon [OS], $p_F(1)$ equals the number of connected components of the real points of F_P, which is 31. Thus, $p_F(t) = 1 + 9t + 21t^2$, from which the result follows. ■

Now we are ready to complete the proof of (10.21). From the discussion preceding (10.15),

$$Q^4(Y_2^3) = Q^4(G_2^3) \quad \text{and} \quad Q^6(Y_2^3) = Q^6(G_2^3).$$

From (10.3), (10.7) and (10.22) it follows that $Q^4(Y_2^3)$ is a rank 19 submodule of the Σ_6-module $\Lambda^2 H^1(Y_2^3)$. There are 3 possibilities, namely

$$(10.23) \qquad [3,1,1,1] + [2,2,1,1], \quad [5,1] + [3,3] + [2,2,1,1], \quad [4,1,1] + [2,2,1,1].$$

By (10.3), $Q^6(Y_2^3)$ is the kernel of

$$Q^4(Y_2^3) \otimes H^1(Y_2^3) \to \Lambda^3 H^1(Y_2^3).$$

Since $\Lambda^3 H^1(Y_2^3)$ contains no copy of the sign representation, and since, for each of the possibilities for Q^4, $Q^4 \otimes H^1$ contains one copy of the sign representation, $Q6$ has exactly one copy of sgn_6. This completes the proof of (10.21). ■

To complete the proof of (10.1), we have to prove assertion (ii). To do this, it suffices to show that the symbol $\overline{L}_3$ of a 3-logarithm function is not of the form $A^*\beta$ for some $\beta \in Gr_6^W \tilde{\mathcal{O}}(G_1^3)$. But this follows from the fact that

$$Gr_6^W \tilde{\mathcal{O}} \cong S^3 H^1 \oplus (Q^4 \otimes H^1) \oplus Q^6$$

and that this isomorphism is natural, so that

$$A^*\colon Gr_6^W \tilde{\mathcal{O}}(G_1^3) \to Gr_6^W \tilde{\mathcal{O}}(G_2^3)$$

preserves this decomposition. The assertion (10.1)(ii) now follows from (10.20), (10.21) using (9.2).

(10.24) *Remark.* The columns of the table in (10.21) may be regarded as complexes with differential A^*. The first two of these columns are exact. This can be seen by noting that $sS^3H^1(G_1^3)$ is spanned by

$$\sigma_1 = \mathrm{Alt}_5[01 \otimes 12 \otimes 23 + 01 \otimes 23 \otimes 12 + 12 \otimes 01 \otimes 23],$$

that $sQ^4(G_1^3) \otimes H^1(G_1^3)$ is spanned by

$$\theta = \mathrm{Alt}_5[2(01 \otimes 02 \otimes 03) - 01 \otimes 23 \otimes 12 + 12 \otimes 01 \otimes 23],$$

and that $sS^3H^1(G_2^3)$ is spanned by $A^*\sigma_1$ and

$$\sigma_2 = \mathrm{Alt}_6[012 \otimes 234 \otimes 045].$$

This implies that the homology of the complex

$$0 \to Gr_6^W\tilde{\mathscr{O}}(G_0^3) \to Gr_6^W\tilde{\mathscr{O}}(G_1^3) \to Gr_6^W\tilde{\mathscr{O}}(G_2^3) \to Gr_6^W\tilde{\mathscr{O}}(G_3^3) \to 0$$

is one dimensional and spanned by the class of a trilogarithm.

11. Real Albanese manifolds and generalized Block-Wigner functions

In this section we construct a real valued, real analytic function

$$D_3\colon Y_2^3 \to \mathbf{R}$$

whose pullback to G_2^3 bears the same relation to the 3-logarithm function as the Bloch-Wigner function does to the classical dilogarithm.

It is first necessary to construct the ring of *Bloch-Wigner functions* associated to a smooth variety X. Denote by $\tilde{\mathscr{O}}_{\mathbf{R}}(X)$ the ring of multivalued, real valued functions on the object X of $\tilde{\mathscr{A}}$ consisting of polynomials with real coefficients of the real and imaginary parts of elements of $\tilde{\mathscr{O}}(X)$. The *ring of Bloch-Wigner functions of* X, $\mathscr{BW}(X)$, is defined to be the subring of $\tilde{\mathscr{O}}_{\mathbf{R}}(X)$ of single valued functions. Equivalently, $\mathscr{BW}(X)$ is the subring of $\tilde{\mathscr{O}}_{\mathbf{R}}(X)$ invariant under monodromy:

$$\mathscr{BW}(X) = \tilde{\mathscr{O}}_{\mathbf{R}}(X)^{\pi_1(X)}.$$

The weight filtration $W.$ of $\tilde{\mathscr{O}}(X)$ induces weight filtrations on $\tilde{\mathscr{O}}_{\mathbf{R}}(X)$ and $\mathscr{BW}(X)$.

PROPOSITION. *The assignment of* $(\mathscr{BW}(X), W.)$ *to* X *defines a functor from the category* $\mathscr{A}$ *of complex algebraic manifolds into the category of filtered* $\mathbf{R}$*-algebras.*

(11.2) *Examples.* (a) If $f, g \in \mathscr{O}^*(X)$, then $\log|f|, \log|g| \in W_2\mathscr{BW}(X)$ and $\log|f| \log|g| \in W_4\mathscr{BW}(X)$.

(b) The Bloch-Wigner function $\operatorname{Im} \ln_2(z) + \arg(1 - z)\log|z|$ is in $W_4\mathscr{BW}(\mathbf{C} - \{0, 1\})$. Likewise, Ramakrishnan's analogues $R_k(z)$ of the Bloch-Wigner function for the classical k-logarithms $\ln_k(z)$ [R3] satisfy $R_k(z) \in W_{2k}\mathscr{BW}(\mathbf{C} - \{0, 1\})$. ■

To simplify the discussion, we now assume that $q(X) = 0$. As was pointed out in the discussion preceding (7.8), an element of $\tilde{\mathscr{O}}(X)$ may be viewed as a multivalued function obtained by pulling back along θ_s a multivalued function on $\mathrm{Alb}^s(X)$ which is polynomial on the universal over $\mathfrak{g}_s$ of $\mathrm{Alb}^s(X)$, provided that s is sufficently large.

$$\begin{array}{ccc} & & \mathfrak{g}_s \\ & & \downarrow \\ X & \xrightarrow[\theta_s]{} & \mathrm{Alb}^s(X) = \Gamma_s \backslash G_s \end{array}$$

By the same token, an element of $\tilde{\mathscr{O}}_{\mathbf{R}}(X)$ corresponds to a real polynomial $\mathfrak{g}_s \to \mathbf{R}$ on the real vector space underlying $\mathfrak{g}_s$, for s sufficiently large.

The group G_s is, in fact, the set of complex points of a $\mathbf{Q}$-algebraic group $\mathscr{G}_s$ associated to $\pi_1(X, x)$. One way to see this is to note that the Baker-Campbell-Hausdorff formula implies that the $\mathbf{Q}$ vector space spanned by $\{\log \gamma \in \mathfrak{g}_s \colon \gamma \in \Gamma_s\}$ is a $\mathbf{Q}$ Lie algebra $\mathfrak{g}_s(\mathbf{Q})$ whose complexification is $\mathfrak{g}_s$. Alternatively, one may use Quillen's description of $\mathscr{G}_s(K)$, K a field of characteristic zero, as the set of group like elements of the truncated group algebra $K\pi_1(X, x)/W_{-s-1}$ [Q; Appendix A].

For a scheme $\mathscr{X}$ over k and a field extension K/k, we denote the set of K points $\mathscr{X}(K)$ of $\mathscr{X}$ viewed as a scheme over k by restriction of scalars by $R_{K/k}\mathscr{X}$.

Elements of $\mathscr{BW}(X)$ arise as real polynomial functions on the real variety $R_{\mathbf{C}/\mathbf{R}}\mathscr{G}_s$ which are Γ_s, and hence $\mathscr{G}_s(\mathbf{R})$, invariant. In other words, they arise as real polynomial functions on the homogeneous space $\mathscr{G}_s(\mathbf{R}) \backslash \mathscr{G}_s(\mathbf{C})$, which is isomorphic to the real affine space $\mathbf{A}(\mathfrak{g}_s(\mathbf{R}))$.

(11.3) PROPOSITION. *Pulling back elements of the coordinate ring* $\mathbf{R}[\mathscr{G}_s(\mathbf{R}) \backslash \mathscr{G}_s(\mathbf{C})]$ *along the composite*

$$X \to \mathrm{Alb}^s(X) \to \mathscr{G}_s(\mathbf{R}) \backslash \mathscr{G}_s(\mathbf{C})$$

defines an injective ring homomorphism

$$\mathbf{R}[\mathscr{G}_s(\mathbf{R}) \backslash \mathscr{G}_s(\mathbf{C})] \to \mathscr{B}\mathscr{W}(X).$$

These induce a filtered ring isomorphism

$$\mathbf{R}[\mathscr{G}(\mathbf{R}) \backslash \mathscr{G}(\mathbf{C})] := \varinjlim \mathbf{R}[\mathscr{G}_s(\mathbf{R}) \backslash \mathscr{G}_s(\mathbf{C})] \to \mathscr{B}\mathscr{W}(X).$$

(11.4) COROLLARY. *As a filtered ring, $\mathscr{B}\mathscr{W}(X)$ is isomorphic to*

$$\operatorname{Sym}(\mathfrak{g}(\mathbf{R})^*) := \varinjlim \operatorname{Sym}(\mathfrak{g}_s(\mathbf{R})^*).$$

(11.5) COROLLARY. *There is a graded algebra isomorphism*

$$Gr_\bullet^W \mathscr{B}\mathscr{W}(X) \otimes \mathbf{C} \cong Gr_\bullet^W \tilde{\mathscr{O}}(X).$$

(11.6) *Example.* This is a continuation of (7.4)(b). Here $X = \mathbf{C} - \{0, 1\}$,

$$\Gamma_2 = \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathscr{G}_s(\mathbf{R}) = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} \\ 0 & 1 & \mathbf{R} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathscr{G}_2(\mathbf{C}) = \begin{pmatrix} 1 & \mathbf{C} & \mathbf{C} \\ 0 & 1 & \mathbf{C} \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote the coordinate functions of $\mathscr{G}_2(\mathbf{R})$ and $\mathscr{G}_2(\mathbf{C})$ by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+ay+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix},$$

the function $\psi = \operatorname{Im} z - \operatorname{Re} x \operatorname{Im} y$ is $\mathscr{G}_2(\mathbf{R})$ invariant, and thus a function on $\mathscr{G}_2(\mathbf{R}) \backslash \mathscr{G}_2(\mathbf{C})$. Composing the albanese mapping

$$\theta_2 \colon \mathbf{C} - \{0, 1\} \to \Gamma_2 \backslash \mathscr{G}_2(\mathbf{C}),$$

$$x \mapsto \begin{pmatrix} 1 & \log(1-x)/2\pi i & -\ln_2(x)/(2\pi i)^2 \\ 0 & 1 & \log x/2\pi i \\ 0 & 0 & 1 \end{pmatrix}$$

with the composite

$$\Gamma_2 \backslash \mathscr{G}_2(\mathbf{C}) \to \mathscr{G}_2(\mathbf{R}) \backslash \mathscr{G}_2(\mathbf{C}) \xrightarrow{-(2\pi i)^2\psi} \mathbf{R}$$

yields the Bloch-Wigner function. ■

Suppose that $\mathscr{G}$ is an $\mathbf{R}$-algebraic group whose $\mathbf{R}$ points form a simply connected nilpotent Lie group. Denote the lower central series of $\mathscr{G}$ by

$$\mathscr{G} = \mathscr{G}^1 \geq \mathscr{G}^2 \geq \dots$$

Let $L_{\bullet}$ be the filtration of the coordinate ring of $\mathscr{G}$ dual to the lower central series of $\mathscr{G}$. This gives $K[\mathscr{G}(K)]$ the structure of a filtered ring. When $\mathscr{G}$ is the group $\mathscr{G}_s$ associated to the fundamental group of $\pi_1(X, x)$, then

$$L_p\mathbf{C}[\mathscr{G}_s(\mathbf{C})] = \mathbf{C}[\mathscr{G}_s(\mathbf{C})] \cap W_{2p}\tilde{\mathscr{O}}(X).$$

(11.7) LEMMA. *There exists a canonical, graded* $\mathbf{R}$*-algebra homomorphism*

$$p \colon Gr_{\bullet}^L\mathbf{C}[\mathscr{G}(\mathbf{C})] \to Gr_{\bullet}^L\mathbf{R}\left[R_{\mathbf{C}/\mathbf{R}}\mathscr{G}\right]$$

with the properties:

(i) *The image of* p *consists of the elements of* $Gr_{\bullet}^L\mathbf{R}[R_{\mathbf{C}/\mathbf{R}}\mathscr{G}]$ *invariant under the left action of* $\mathscr{G}(\mathbf{R})$ *on* $\mathscr{G}(\mathbf{C})$.

(ii) *Modulo decomposables*, p *is the projection*

$$\begin{aligned} QGr_{\bullet}^L\mathbf{C}[\mathscr{G}(\mathbf{C})] &\cong QGr_{\bullet}^L\mathbf{R}[\mathscr{G}(\mathbf{R})] \otimes \mathbf{C} \\ &\xrightarrow{\mathrm{Im}} QGr_{\bullet}^L\mathbf{R}[\mathscr{G}(\mathbf{R})] \hookrightarrow QGr_{\bullet}^L\mathbf{R}\left[R_{\mathbf{C}/\mathbf{R}}\mathscr{G}\right]. \end{aligned}$$

For example, when $\mathscr{G}$ is as in (11.6), then $x \mapsto \mathrm{Im}\, x$, $y \mapsto \mathrm{Im}\, y$ and $z \mapsto \mathrm{Im}\, z - \mathrm{Re}\, x \,\mathrm{Im}\, y$. The result is proved by induction on the length of $\mathscr{G}$, observing that $\mathscr{G}(\mathbf{R})$ acts trivially on

$$QGr_{\bullet}^L\mathbf{C}[\mathscr{G}(\mathbf{C})] / QGr_{\bullet}^L\mathbf{R}[\mathscr{G}(\mathbf{R})].$$

Applying (11.7) to (11.3), we obtain:

(11.8) COROLLARY. *There is a canonical graded* $\mathbf{R}$*-algebra homomorphism*

$$p \colon Gr_{\bullet}^W\tilde{\mathscr{O}}(X) \to Gr_{\bullet}^W\mathscr{B}\mathscr{W}(X).$$

We are now ready to state our main theorem. Denote the canonical projection $G_2^3 \to Y_2^3$ by π.

(11.9) THEOREM. *If L_3 is a 3-logarithm function whose symbol $\overline{L}_3 \in Gr_6^W \tilde{\mathcal{O}}(G_2^3)$, satisfies*

$$\sigma^* \overline{L}_3 = \operatorname{sgn}(\sigma) \overline{L}_3 \quad \textit{for all } \sigma \in \Sigma_6,$$

then there exists a unique function $D_3 \in \mathscr{B}\mathscr{W}(Y_2^3)$ whose symbol $\overline{D}_3 \in Gr_6^W \mathscr{B}\mathscr{W}(Y_2^3)$ satisfies:

(i) $\pi^* \overline{D}_3 = p(\overline{L}_3) \in Gr_6^W \mathscr{B}\mathscr{W}(G_2^3)$;

(ii) $\sigma^* D_3 = \operatorname{sgn}(D_3)$ *for all* $\sigma \in \Sigma_6$;

(iii) $\sum_{j=0}^6 (-1)^j A_j^* D_3 = 0$, *where* A_j: $Y_3^3 \to Y_2^3$ $j = 0, \ldots, 6$ *are the 7 face maps.*

Moreover, if $F \in W_6 \mathscr{B}\mathscr{W}(Y_2^3)$ satisfies (ii) *and* (iii), *then there exist $a \in \mathbf{R}$ and $G \in W_6 \mathscr{B}\mathscr{W}(Y_1^3)$ such that $F = aD_3 + A^* G$.*

As a warm up, we show that the symmetry property of the Bloch-Wigner function and its symbol uniquely determine it and imply that it satisfies the 5 term functional equation.

(11.10) THEOREM. *There is a unique element D_2 of $W_4 \mathscr{B}\mathscr{W}(\mathbf{C} - \{0, 1\})$ with symbol*

$$\overline{D}_2 = p(\overline{\ln_2}) = p(\overline{\phi}),$$

where ϕ denotes Rogers' function, and which satisfies:

(i) *The symmetry condition*

$$\sigma^* D_2 = \operatorname{sgn}(\sigma) D_2 \quad \textit{for all } \sigma \in \Sigma_4;$$

(ii) *the functional equation*

$$\sum_{j=0}^{4} (-1)^j A_j^* D_2 = 0,$$

where A_j: $Y_2^2 \to Y_1^2 = \mathbf{C} - \{0, 1\}$, $j = 0, \ldots, 4$, *are the 5 face maps.*

Proof. By (11.4),

$$W_4 \mathscr{B}\mathscr{W} \cong \mathbf{R} \oplus H_{\mathbf{R}}^1 \oplus S^2 H_{\mathbf{R}}^1 \oplus Q_{\mathbf{R}}^4, \tag{11.11}$$

and by (10.6), $H_{\mathbf{R}}^1$ and $Q_{\mathbf{R}}^4$ are the Σ_4 modules $[2, 2]$ and $[1, 1, 1, 1]$, so it follows that

$$W_4 \mathscr{B}\mathscr{W}(\mathbf{C} - \{0, 1\})$$

is the Σ_4-module

$$2[4] + 2[2,2] + [1,1,1,1].$$

Consequently, up to a scalar multiple, there is exactly one element of $W_4\mathscr{BW}(\mathbf{C} - \{0,1\})$ with the symmetry property, and this element is determined by its symbol. Since $p(\overline{\ln_2}) = p(\overline{\phi})$, since $\overline{\phi}$ has the desired symmetry, and since p commutes with the Σ_4 action, $p(\overline{\ln_2})$ determines a unique element D_2 of $W_4\mathscr{BW}(\mathbf{C} - \{0,1\})$ satisfying the symmetry property.

To see that D_2 necessarily satisfies the functional equation, note that, by (10.6), (10.15), and (11.11),

$$W_4\mathscr{BW}(Y_2^2) = 2[5] + 2[3,2] + [4,1] + [2,2,1] + [2,1,1,1].$$

In particular, $W_4\mathscr{BW}(Y_2^2)$ contains no copy of sgn_5. That D_2 satisfies the functional equation follows from (9.5). ■

To prove (11.9), we first construct a function $\tilde{D}_3 \in W_6\mathscr{BW}(G_2^3)$ which has the desired symmetry property and which satisfies the natural functional equation. The descent of $\tilde{D}_3$ to Y_2^3 then follows from the following result and the injectivity of $\mathscr{BW}(Y_3^3) \to \mathscr{BW}(G_3^3)$. Recall that if M is a Σ_n-module, then $s(M)$ denotes the sgn_n isotypical component of M.

(11.12) Lemma. *The mapping*

$$s\big(W_6\mathscr{BW}(Y_2^3)\big) \to s\big(W_6\mathscr{BW}(G_2^3)\big)$$

induced by the projection $G_2^3 \to Y_2^3$ is an isomorphism.

The proof begins with the following result which is proved by direct calculation using (10.6) and (10.23).

(11.13) Proposition. $s(W_4\mathscr{BW}(Y_2^3)) = s(W_4\mathscr{BW}(G_2^3)) = 0.$

(11.14) Corollary. *The natural surjections*

$$W_6\mathscr{BW}(G_2^3) \to Gr_6^W\mathscr{BW}(G_2^3), \quad W_6\mathscr{BW}(Y_2^3) \to Gr_6^W\mathscr{BW}(Y_2^3)$$

induce isomorphisms on the sgn_6 isotypical components. Consequently, there are canonical splittings

$$s\big(Gr_6^W\mathscr{BW}(G_2^3)\big) \to W_6\mathscr{BW}(G_2^3), \quad s\big(Gr_6^W\mathscr{BW}(Y_2^3)\big) \to W_6\mathscr{BW}(Y_2^3).$$

Lemma (11.12) now follows directly from (10.21) and (11.14).

To prove (11.9), suppose that L_3 is a 3-logarithm function whose symbol spans a copy of sgn_6 in $Gr_6^W\tilde{\mathscr{O}}(G_2^3)$. From the naturality of

$$p\colon Gr_6^W\tilde{\mathscr{O}}(G_2^3) \to Gr_6^W\mathscr{BW}(G_2^3)$$

it follows that $p(\bar{L}_3)$ spans a copy of sgn_6 in $Gr_6^W\mathscr{BW}(G_2^3)$. Let $\tilde{D}_3 \in s(W_6\mathscr{BW}(G_2^3))$ be the lift of $p(\bar{L}_3)$ to $W_6\mathscr{BW}(G_2^3)$ given by (11.14). This function clearly possesses the symmetry property. To prove that it satisfies the functional equation, we need the following result.

(11.15) PROPOSITION. *The module $W_4\mathscr{BW}(G_2^3)$ contains no copy of* sgn_7.

Proof. This follows from a machine calculation using (10.6) and (11.11). It is not necessary to compute $H^1 \otimes Q^4$ as this is a submodule of $H^1 \otimes \Lambda^2 H^1$ which contains no copy of sgn_7. ∎

(11.16) COROLLARY. *If $F \in s(W_6\mathscr{BW}(G_3^3))$, then $F = 0$ if and only if its symbol*

$$\bar{F} \in Gr_6^W\mathscr{BW}(G_3^3)$$

is zero.

Thus, by (9.5), to show that $\tilde{D}_3$ satisfies the functional equation, we need only show that

$$\sum_{j=0}^{6}(-1)^j A_j^* p(\bar{L}_3) = 0.$$

But this follows from the fact that L_3 satisfies the functional equation and the naturality of p. As previously explained, $\tilde{D}_3$ descends to a function $D_3 \in W_6\mathscr{BW}(Y_2^3)$ with the desired symmetry property and which satisfies the desired functional equation. This proves the first assertion of (11.9). The second follows from (10.24) using a similar argument.

(11.17) *Remark.* It is interesting to note that $Gr_6^W(\mathscr{BW}(\mathbf{C} - \{0,1\})$ contains no copy of sgn_4. So there is no classical Bloch-Wigner trilogarithm with the symmetry property.

12. Epilogue

As mentioned in the introduction, the definition of p-logarithms given in Section 6 is only part of a more complicated definition of p-logarithms as Deligne cohomology classes of the simplicial variety $G_\cdot^p$. We now describe this in more detail. An exposition of the role of these higher logarithms in algebraic K-theory is given in [BMS].

Multivalued Deligne cohomology. Suppose that X is a complex algebraic manifold. The most refined version of the Deligne cohomology $H^{\cdot}_{\mathscr{D}}(X, \mathbf{Q}(p))$ of X is Beilinson's *absolute Hodge cohomology* [Be2] which is defined to be the cohomology of the complex

$$D(X, \mathbf{Q}(p)) = \operatorname{cone}\left[W_{2p}A^{\cdot}_{\mathbf{Q}}(X) \to W_{2p}A^{\cdot}_{\mathbf{C}}(X)/F^{p}W_{2p}A^{\cdot}_{\mathbf{C}}(X)\right][-1],$$

where $(A^{\cdot}_{\mathbf{Q}}(X), W_{\cdot}) \to (A^{\cdot}_{\mathbf{C}}(X), F^{\cdot}, W_{\cdot})$ is a natural mixed Hodge complex[6] whose cohomology is Deligne's natural mixed Hodge structure on $H^{\cdot}(X)$ [D1].

We now introduce the multivalued Deligne complex of X: Let $\mathfrak{g}(X)$ be the Malcev Lie algebra associated to $\pi_1(X)$, topologized by its weight filtration (or equivalently, by its lower central series). The complex of multivalued forms $\tilde{\Omega}^{\cdot}(X)$ on X is a continuous $\mathfrak{g}(X)$ module. Let

$$\mathscr{C}(\mathfrak{g}(X), \tilde{\Omega}^{\cdot}(X))$$

be the double complex of continuous $\tilde{\Omega}^{\cdot}(X)$ valued cochains on $\mathfrak{g}(X)$. That is,

$$\mathscr{C}(\mathfrak{g}, \tilde{\Omega}^{\cdot}) = \varinjlim \mathscr{C}(\mathfrak{g}/W_{-l-1}, W_l\tilde{\Omega}^{\cdot}),$$

where, for a module V over the Lie algebra $\mathfrak{h}$, $\mathscr{C}(\mathfrak{h}, V)$ denotes the Chevalley-Eilenberg complex $\Lambda(\mathfrak{h}^*) \otimes V$ with the usual differential.

Define a Hodge filtration on $\mathscr{C}(\mathfrak{g}, \tilde{\Omega}^{\cdot})$ by

$$F^{p}\mathscr{C}(\mathfrak{g}, \tilde{\Omega}^{\cdot}) = \bigoplus_{q \geq p} \mathscr{C}(\mathfrak{g}, \tilde{\Omega}^{q}).$$

The weight filtrations on $\mathfrak{g}$ and $\tilde{\Omega}^{\cdot}$ induce a weight filtration on $\mathscr{C}(\mathfrak{g}, \tilde{\Omega}^{\cdot})$.

The assignment of

$$\left(\mathscr{C}(\mathfrak{g}(X), \tilde{\Omega}^{\cdot}(X)), W_{\cdot}, F^{\cdot}\right)$$

to X defines a functor from $\tilde{\mathscr{A}}$ into the category of bifiltered differential graded algebras.

Let X be an object of $\tilde{\mathscr{A}}$. Denote the $\mathbf{Q}$-form of the Malcev Lie algebra associated to $\pi_1(X)$ by $\mathfrak{g}_{\mathbf{Q}}(X)$ and the continuous $\mathbf{Q}$ valued cochains on it by $\mathscr{C}(\mathfrak{g}_{\mathbf{Q}}(X), \mathbf{Q})$.

---

[6]Note that the weight filtration on $A^{\cdot}(X)$ is the *filtration décalée* of the commonly used weight filtration (see [D1]).

Consider the complex

$$M(X,\mathbf{C}/\mathbf{Q}(p)) := \operatorname{cone}\left[W_{2p}\mathscr{C}(\mathfrak{g}_\mathbf{Q}(X),\mathbf{Q}) \to W_{2p}\mathscr{C}(\mathfrak{g}(X),\tilde{\Omega}^{\cdot}(X))\right][-1].$$

This may be viewed as a double complex ($q = 0$ case):

$$\begin{array}{ccccccccccc}
 & 0 & & 0 & & & & 0 & & & \\
 & \uparrow & & \uparrow & & & & & & & \\
p & \Lambda^p H^1(\mathfrak{g}_\mathbf{Q}) & \to & \Lambda^p H^1(\mathfrak{g}) & & & & & & & \\
 & \uparrow & & \uparrow & & & & & & & \\
\vdots & \vdots & & \vdots & & & & & & & \\
 & \uparrow & & & & & & & & & \\
r & W_{2p}\Lambda^r\mathfrak{g}_\mathbf{Q}^* & \to & \cdots & \to & W_{2p}\Lambda^r(\mathfrak{g}^*)\otimes\tilde{\Omega}^{s-1} & \to & & & & \\
 & \uparrow & & & & \uparrow & & & & & \\
\vdots & \vdots & & \vdots & & \vdots & & 0 & & & \\
 & \uparrow & & & & & & \uparrow & & & \\
1 & W_{2p}\mathfrak{g}_\mathbf{Q}^* & \to & & & & \cdots \to & W_{2p}\mathfrak{g}^*\otimes\tilde{\Omega}^{p-1} & \to & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
0 & \mathbf{Q} & \xrightarrow{(2\pi i)^p} & W_{2p}\tilde{\Omega}^0 & \to \cdots \to & W_{2p}\tilde{\Omega}^{s-1} & \to \cdots \to & W_{2p}\tilde{\Omega}^{p-1} & \to & \Omega^p & \to 0. \\
 & 0 & & 1 & & s & & p & & p+1 &
\end{array}$$

The *multivalued Deligne complex* of X is defined to be the quotient complex

$$MD(X,\mathbf{Q}(p)) = M(X,\mathbf{C}/\mathbf{Q}(p))/F^p\tilde{\Omega}^{\cdot}(X).$$

This can be viewed as a double complex ($q = 0$ case):

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & & & 0 & \\
 & \uparrow & & \uparrow & & & & & \\
p & \Lambda^p H^1(\mathfrak{g}_\mathbf{Q}) & \to & \Lambda^p H^1(\mathfrak{g}) & & & & & \\
 & \uparrow & & \uparrow & & & & & \\
\vdots & \vdots & & \vdots & & & & & \\
 & \uparrow & & & & & & & \\
r & W_{2p}\Lambda^r\mathfrak{g}_\mathbf{Q}^* & \to & \cdots & \to & W_{2p}\Lambda^r(\mathfrak{g}^*)\otimes\tilde{\Omega}^{s-1} & \to & & \\
 & \uparrow & & & & \uparrow & & & \\
\vdots & \vdots & & \vdots & & \vdots & & 0 & \\
 & \uparrow & & & & & & \uparrow & \\
1 & W_{2p}\mathfrak{g}_\mathbf{Q}^* & \to & & & & \cdots \to & W_{2p}\mathfrak{g}^*\otimes\tilde{\Omega}^{p-1} & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \mathbf{Q} & \xrightarrow{(2\pi i)^p} & W_{2p}\tilde{\Omega}^0 & \to \cdots \to & W_{2p}\tilde{\Omega}^{s-1} & \to \cdots \to & W_{2p}\tilde{\Omega}^{p-1} & \to 0. \\
 & 0 & & 1 & & s & & p &
\end{array}$$

Define the *multivalued Deligne cohomology* of the object X of $\tilde{\mathscr{A}}$ by

$$H^{\cdot}_{\mathscr{M}\mathscr{D}}(X,\mathbf{Q}(p)) = H^{\cdot}(MD(X,\mathbf{Q}(p))).$$

This defines a functor form $\tilde{\mathscr{A}}$ into the category of graded rings. (Actually, one should sheafify this construction and define $H^{\cdot}_{\mathscr{M}\mathscr{D}}(X, \mathbf{Q}(p))$ to be the Čech hypercohomology of the sheaf $\mathscr{M}\mathscr{D}_X(\mathbf{Q}(p))$. This complication will not concern us here.)

One can construct a natural homomorphism

$$(12.1) \qquad H^{\cdot}_{\mathscr{M}\mathscr{D}}(X, \mathbf{Q}(p)) \to H^{\cdot}_{\mathscr{D}}(X, \mathbf{Q}(p)).$$

(12.2) THEOREM. *If X is a rational n-$K(\pi, 1)$, then* (12.1) *is an isomorphism in dimensions $\leq n$.*

If $X.$ is a simplicial object of $\tilde{\mathscr{A}}$, one can define a triple complex $MD(X., \mathbf{Q}(p))$ in the obvious way. The multivalued Deligne cohomology of $X.$ is defined to be the cohomology of this complex. The next result is the analogue of (12.2) for simplicial varieties.

(12.3) THEOREM. *Suppose that $X.$ is a simplicial object of $\tilde{\mathscr{A}}$. If each X_m is a rational $(n - m)$-$K(\pi, 1)$, then the natural map*

$$H^k_{\mathscr{M}\mathscr{D}}(X., \mathbf{Q}(p)) \to H^k_{\mathscr{D}}(X., \mathbf{Q}(p))$$

is an isomorphism when $k \leq n$.

Higher logarithms as Deligne cohomology classes. Applying the multivalued Deligne complex functor $MD(\ , \mathbf{Q}(p))$ to $G^p_{\cdot}$, we obtain a triple complex in which the double complex $(5.1\bar{0})$ is imbedded in the ground floor.

We also have the triple complex $M(G^p_{\cdot}, \mathbf{C}/\mathbf{Q}(p))$. The short exact sequence of complexes

$$0 \to \Omega^p(G^p_{\cdot})[-p-1] \to M(G^p_{\cdot}, \mathbf{C}/\mathbf{Q}(p)) \to MD(G^p_{\cdot}, \mathbf{Q}(p)) \to 0$$

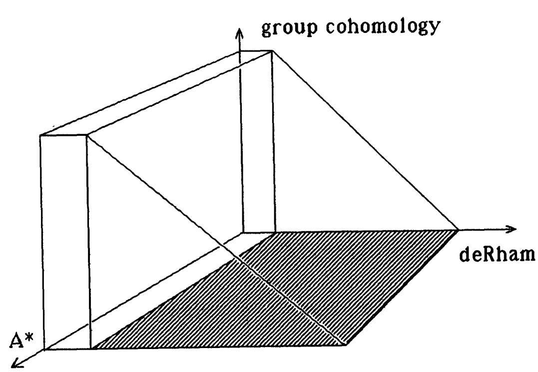


FIG. 12

gives rise to a long exact sequence of cohomology groups. Consider the connecting homomorphism

$$\delta\colon H^{2p}_{\mathcal{M}\mathcal{D}}(G^p_\cdot, \mathbf{Q}(p)) \to \Omega^p(G^p_0).$$

(12.4) Definition. *A generalized p-logarithm is an element* C_p *of* $H^{2p}_{\mathcal{M}\mathcal{D}}(G^p_\cdot, \mathbf{Q}(p))$ *satisfying* $\delta(C_p) = \mathrm{vol}_p$.

Equivalently, C_p is represented by a $2p-1$ cochain Z_p in the triple complex $M(G^p_\cdot, \mathbf{C}/\mathbf{Q}(p))$ which satisfies

$$DZ_p = \mathrm{vol}_p. \tag{12.5}$$

If Z_p represents a generalized p-logarithm, then the components of Z_p which lie in the ground floor of the complex $M(G^p_\cdot, \mathbf{C}/\mathbf{Q}(p))$ is a p-logarithm as defined in (6.1). Observe also that the conditions imposed on the component

$$L_p \in W_{2p}\tilde{\mathcal{O}}(G^p_{p-1})$$

of Z_p by (12.5) correspond to the three essential properties of the p-logarithm function discussed in the introduction; the analytic property corresponds to the de Rham differential

$$d\colon W_{2p}\tilde{\mathcal{O}}(G^p_{p-1}) \to \tilde{\mathcal{O}}(G^p_{p-1}) \otimes \Omega^1(G^p_{p-1});$$

the topological property to the group cohomology differential

$$\partial\colon W_{2p}\tilde{\mathcal{O}}(G^p_{p-1}) \to \mathcal{C}\big(\mathfrak{g}(G^p_{p-1}), \tilde{\mathcal{O}}\big);$$

and the algebraic property to the combinatorial differential

$$A^*\colon W_{2p}\tilde{\mathcal{O}}(G^p_{p-1}) \to \tilde{\mathcal{O}}(G^p_p).$$

The next result is the generalization of (8.8) to the current setting.

(12.6) Theorem. *When* $p = 1, 2, 3$,

$$\delta\colon H^{2p}_{\mathcal{M}\mathcal{D}}(G^p_\cdot, \mathbf{Q}(p)) \to \Omega^p(G^p_0)$$

is an isomorphism. In particular, there is a canonical generalized p-logarithm.

The first step in the proof is to replace multivalued Deligne-Beilinson cohomology by ordinary Deligne-Beilinson cohomology by applying (12.3).

(12.7) LEMMA. *Fix p. If each G_q^p is a rational $(p-q)$-$K(\pi,1)$, then the natural homomorphism*

$$H^{2p}_{\mathscr{M}\mathscr{D}}(G_\cdot^p,\mathbf{Q}(p)) \to H^{2p}_{\mathscr{D}}(G_\cdot^p,\mathbf{Q}(p))$$

is an isomorphism.

By (8.6), the hypotheses of (12.7) are satisfied when $p = 1,2,3$, so we need only calculate the Deligne cohomology of $G_\cdot^p$. This we do with the help of the standard short exact sequence [Be2]

$$0 \to \mathrm{Ext}^1_{\mathscr{H}}(\mathbf{Q}, H^{2p-1}(G_\cdot^p,\mathbf{Q}(p))) \to H^{2p}_{\mathscr{D}}(G_\cdot^p,\mathbf{Q}(p)) \\ \to \mathrm{Hom}_{\mathscr{H}}(\mathbf{Q}, H^{2p}(G_\cdot^p,\mathbf{Q}(p))) \to 0,$$

where $\mathscr{H}$ is the category of mixed Hodge structures. The Ext^1 term is isomorphic to

$$W_{2p}H^{2p-1}(G_\cdot^p,\mathbf{C})/F^pW_{2p}H^{2p-1}(G_\cdot^p,\mathbf{C}) + W_{2p}H^{2p-1}(G_\cdot^p,\mathbf{Q})).$$

Theorem (12.6) follows from (12.3) once one has shown that

$$W_{2p}H^{2p-1}(G_\cdot^p) = 0 \quad \text{and} \quad W_{2p}H^{2p}(G_\cdot^p) = \Omega^p(G_0^p).$$

These assertions can be checked by direct calculation using the observation that the spectral sequence with

$$E_1^{s,t} = H^t(G_{s-p}^p) \quad \text{and} \quad d_1 = A^*, \tag{12.8}$$

which converges to $H^\cdot(G_\cdot^p)$, degenerates at E_2 in degrees $\le 2p$. This follows by a weight argument using the fact that (12.8) is a spectral sequence of mixed Hodge structures and the fact that $H^t(G_{s-p}^p)$ is pure of weight $2t$ in low degrees by (8.2).

REFERENCES

[A1] K. AOMOTO, *A generalization of Poincaré normal functions on a polarized manifold*, Proc. Japan Acad., vol. 55 (1979), pp. 353–358.

[A2] ______, *Addition theorems of Abel type for hyperlogarithms*, Nagoya Math. J., vol. 88 (1982), pp. 55–71.

[A3] ______, *Special values of hyperlogarithms and linear difference schemes*, Illinois J. Math., vol. 34 (1989), pp. – (this issue).

[Be1] A. BEILINSON, *Higher regulators and values of L-functions*, J. Soviet Math., vol. 30 (1985), pp. 2036–2070 (Translated from Sovr. Probl. Mat., vol. 24, Mosc. VINITI (1984), pp. 181–238).

[Be2] ______, *Notes on absolute Hodge cohomology, in applications of algebraic K-theory to algebraic geometry and number theory, part I*, Contemp. Math., vol. 55 (1986), pp. 35–68.

[BG] A. Beilinson and V. Ginsberg, *Mixed categories, Ext-duality and representations (results and conjectures)*, preprint, 1988.

[BGSV] A. Beilinson, A.B. Goncharov, V.V. Schechtman, and A.N. Varchenko, *Aomoto integrals, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane*, preprint, 1988.

[BMS] A. Beilinson, R. MacPherson, and V. Schechtman, *Notes on motivic cohomology*, Duke Math. J., vol. 54 (1987), pp. 679–710.

[B1] S. Bloch, *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry*, International Symposium on Algebraic Geometry, Kyoto, 1977, pp. 103–114.

[B2] ______, *Higher regulators, Algebraic K-theory, and zeta functions of elliptic curves*, unpublished manuscript, 1978.

[B3] ______, *The dilogarithm and extensions of Lie algebras, algebraic K-theory*, edited by M. Stein and E. Friedlander, Lecture Notes in Math., no. 854, Springer-Verlag, New York, 1984, pp. 1–23.

[Br] E. Brieskorn, *Sur les groupes des tresses (d'aprés V.I. Arnold)*, Séminaire Bourbaki, 24^e année. 1971/1972, Lecture Notes in Math., no. 317, Springer-Verlag, New York, 1973.

[Ca] P. Cartier, to appear.

[C1] K.-T. Chen, *Algebras of iterated path integrals and fundamental groups*, Trans. Amer. Math. Soc., vol. 156 (1971), pp. 359–379.

[C2] ______, *Iterated path integrals*, Bull. Amer. Math. Soc., vol. 83 (1977), pp. 831–879.

[C3] ______, *Extensions of C^∞ function algebra by integrals and Malcev completion of π_1*, Advances in Math., vol. 23 (1977), pp. 181–210.

[Da] D. Damiano, *Vanishing of the characteristic forms on the orbits of the Cartan subgroup*, Advances in Math., vol. 44 (1982), pp. 313–317.

[D1] P. Deligne, *Theorie de Hodge II*, Publ. Math. IHES., vol. 40 (1971), pp. 5–57.

[D2] ______, *Les immeubles des groupes de tresses généralisés*, Invent. Math., vol. 17 (1972), pp. 272–302.

[D3] ______, Letter to Wojtkowiak, October 25, 1983.

[D4] ______, Letter to Spencer Bloch, April 3, 1984.

[Dp1] J. Dupont, *The dilogarithm as a characteristic class for flat bundles*, J. Pure Appl. Alg., vol. 44 (1987), pp. 137–164.

[Dp2] ______, *On polylogarithms*, Aarhus Universitet preprint series 1987/88, no. 17, 1987.

[DS] J. Dupont and C.H. Sah, *Scissors congruences, II*, J. Pure Appl. Alg., vol. 25 (1982), pp. 159–195.

[F] M. Falk, *The minimal model of the complement of an arrangement of hyperplanes*, Trans. Amer. Math. Soc., vol. 309 (1988), pp. 543–556.

[FR] M. Falk and R. Randall, *The lower central series of a fiber-type arrangement*, Invent. Math., vol. 82 (1985), pp. 77–88.

[GGL] A.M. Gabrielov, I.M. Gelfand and M.H. Losik, *Combinatorial calculation of characteristic classes*, Functsional Anal. i Prilozheniya, vol. 9 (1975), no. 1, pp. 54–55; no. 2, pp. 12–28; no. 3, pp. 5–26.

[GM] I.M. Gelfand and R. MacPherson, *Geometry in grassmannians and a generalization of the dilogarithm*, Advances in Math., vol. 44 (1982), pp. 279–312.

[GGMS] I. Gelfand, M. Goresky, R. MacPherson and V. Serganova, *Geometries, convex polyhedra and Schubert cells*, Advances in Math., vol. 63 (1987), pp. 301–316.

[H1] R. Hain, "The geometry of the mixed Hodge structure on the fundamental group" in *Algebraic geometry*, 1985, Proc. Symp. Pure Math., vol. 46 (1987), pp. 247–282.

[H2] ______, "Higher albanese manifolds" in *Hodge Theory*, 1985, Lecture Notes in Math., no. 1246, Springer-Verlag, New York, 1987, pp. 84–91.

[H3] ______, *On the indecomposables of the bar construction*, Proc. Amer. Math. Soc., vol. 98 (1986), pp. 312–316.

[H4] ______, *Algebraic cycles and extensions of variations of mixed Hodge structure*, preprint, 1990.

[HZ] R. HAIN and S. ZUCKER, *Unipotent variations of mixed Hodge structure*, Invent. Math., vol. 88 (1987), pp. 83–124.

[Ha] S. HALPERIN, *Lectures on minimal models*, Mem. Math. Soc. France, nouvelle série No. 9–10, 1983.

[Hk] H. HIRONAKA, *Resolution of singularites of an algebraic variety over a field of characteristic zero*, Ann. of Math., vol. 79 (1964), pp. 109–326.

[J] G.D. JAMES, *The representation theory of the symmetric groups*, Lecture Notes in Math., no. 682, Springer-Verlag, New York, 1978.

[K] T. KOHNO, *Séries de Poincaré-Kozul associée aux groupes de tresse pure*, Invent. Math., vol. 82 (1985), pp. 57–75.

[L] L. LEWIN, *Polylogarithms and associated functions*, North-Holland, New York, 1981.

[Li] S. LICHTENBAUM, *The construction of weight-two arithmetic cohomology*, Invent. Math., vol. 88 (1987), pp. 183–215.

[Lo] J.-L. LODAY, *Comparaison des homologies du groupe linéaire et de son algébre de Lie*, Ann. Inst. Fourier, vol. 37 (1987), pp. 167–190.

[Mv] A. MALCEV, *On a class of homogeneous spaces*, Izvestja Akademii Nauk SSSR ser. Math., vol. 13 (1949) (russian); Amer. Math. Soc. Translations, vol. 42, 1951.

[M1] J. MILNOR, *Hyperbolic geometry: the first 150 years*, Bull., Amer. Math. Soc., vol. 6 (1982), pp. 9–24.

[M2] ______, *On polylogarithms, Hurwitz zeta functions, and the Kubert identities*, Enseign. Math., tome 29 (1983), pp. 281–322.

[N] K. NOMIZU, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math., vol. 59 (1954), pp. 531–538.

[OS] P. ORLIK and L. SOLOMON, *Combinatorics and topology of complements of hyperplanes*, Invent. Math., vol. 56 (1980), pp. 167–189.

[Q] D. QUILLEN, *Rational homotopy theory*, Ann. of Math., vol. 90 (1969), pp. 205–295.

[R1] D. RAMAKRISHNAN, *A regulator for curves via the Heisenberg group*, Bull. Amer. Math. Soc., vol. 5 (1981), pp. 191–195.

[R2] ______, *On the monodromy of higher logarithms*, Proc. Amer. Math. Soc., vol. 85 (1982), pp. 596–599.

[R3] ______, "Analogs of the Bloch-Wigner function for higher polylogarithms" in *Applications of algebraic K-theory to algebraic geometry and number theory, part I*, Contemp. Math., vol. 55 (1986), pp. 371–376.

[Ro] L.J. ROGERS, *On function sum theorems connected with the series* $\Sigma_1^\infty 1/n^2$, Proc. London Math. Soc., vol. 4 (1907), pp. 169–189.

[S1] D. SULLIVAN, *Topology of manifolds and differential forms*, Proceedings of a conference on manifolds, Tokyo, 1973.

[S2] ______, *Infinitesimal computations in topology*, Publ. Math. IHES., vol. 47 (1977), pp. 269–331.

[Z1] D. ZAGIER, *The remarkable dilogarithm*, Max-Planck Institute preprint, 1988.

[Z2] ______, *The Bloch-Wigner-Ramakrishnan polylogarithm function*, preprint, 1988.

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS