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TRANSCENDENTAL ASPECTS OF THE RIEMANN-HILBERT CORRESPONDENCE[1]

BY

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In memory of Kuo-Tsai Chen

1. Introduction

Given a system of ordinary differential equations, locally with coefficients which are holomorphic functions on a Riemann surface, one obtains a representation of the fundamental group. If P is a chosen base point, and if γ is a path beginning and ending at P, then there is a matrix $m(\gamma)$ which expresses the transformation effected on a basis of solutions at P, by the process of continuation around γ. Thus there is a map from the set of systems of differential equations to the set of representations—a map which has come to be known as the Riemann-Hilbert correspondence. The purpose of this paper is to describe some properties of this map which reflect on its essentially transcendental nature. The main technique is Kuo-Tsai Chen's expansion of the solution of a system of differential equations as a sum of iterated integrals [3], [4]. I never met K-T. Chen, but learned about his work from Richard Hain. I hope that this paper may make a contribution toward showing the influence of Chen's ideas.

In order to illustrate the types of problems to be considered, let us discuss the case of systems of rank one on a compact Riemann surface X of genus g. A system of rank one consists of a line bundle L and a connection ∇ on L. The set of these objects forms a group under tensor product, and we will denote this group by U. It is an algebraic group. There is a map to the Jacobian of line bundles on X of degree zero, and the kernel is the set of connections on the trivial bundle:

$$0 \to H^0\left(\Omega^1_X\right) \to U \to \mathrm{Jac}(X) \to 0.$$

On the other hand, the set of one dimensional representations of the fundamental group of X is $\mathrm{Hom}(\pi_1, \mathbf{G}_m)$, which is isomorphic to $\mathbf{G}_m^{2g}$ after a choice of generators $\gamma_1, \dots, \gamma_{2g}$. The Riemann-Hilbert correspondence in this case is

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an isomorphism of complex manifolds

$$\Psi : U^{an} \cong (\mathbf{C}^*)^{2g}.$$

Suppose X is defined over $\overline{\mathbf{Q}} \subset \mathbf{C}$. Then the algebraic group U is defined over $\overline{\mathbf{Q}}$. On the other hand, the group $\mathbf{G}_m^{2g}$ is certainly defined over $\overline{\mathbf{Q}}$. The prototype of the questions considered in §3 is the statement of Theorem 1, that the only points in $U(\overline{\mathbf{Q}})$ which are mapped to $(\overline{\mathbf{Q}}^*)^{2g}$ by Ψ are the points of finite order, mapped to points whose coordinates are roots of unity. In §3 we will discuss some conjectures about the general question of which local systems defined over $\overline{\mathbf{Q}}$ are mapped to monodromy representations defined over $\overline{\mathbf{Q}}$. Some results will be obtained in the special cases of irreducible systems of rank 2 on $\mathbf{P}^1 - \{0, 1, \infty\}$ (Theorem 2), and unipotent systems of rank 3 on subsets of $\mathbf{P}^1$ (Theorem 3).

In the rank one case, one part of the map Ψ is easy to write down: the map from $H^0(\Omega_X^1)$ to $(\mathbf{C}^*)^{2g}$ is given by

$$\Psi : \alpha \mapsto \left(\ldots, \exp \int_{\gamma_i} \alpha, \ldots\right).$$

In §4 we will try to find and describe similar exponential behaviour in the Riemann-Hilbert correspondence for higher ranks. The results are asymptotic expansions for the monodromy matrices of certain families of systems of equations (Corollary 4.3 and Theorem 5).

The discussion in §3 will make use of some of the principal results of transcendence theory: the theorem of Gelfond-Schneider, which says that if α, β, and α^β are algebraic, then β is rational; its generalization, Baker's theorem, which says that if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers, and $\log \alpha_1, \ldots,$ $\log \alpha_n$ are determinations of the logarithms which are linearly independent over $\mathbf{Q}$, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\overline{\mathbf{Q}}$; and a consequence due to Waldschmidt of the criterion of Schneider-Lang. Our application of Baker's theorem was inspired by a somewhat similar application due to Hoffman. The technique which we use in §4 is based on the classical method of the stationary phase. The general version, although analytic in nature, is modeled on Laumon's l-adic interpretation of the Fourier transform.

I would like to thank J. Bernstein, B. Gross, G. Laumon, and I would particularly like to thank R. Hain for introducing me to Chen's beautiful ideas.

2. Differential equations and iterated integrals

Throughout this paper, X will denote a smooth quasiprojective algebraic curve, and $\overline{X}$ its smooth projective completion. The field of definition of X

will always be of characteristic zero, sometimes and sometimes not with a chosen embedding in $\mathbf{C}$. If X is defined over $\mathbf{C}$, then X^{an} will denote the corresponding Riemann surface in the analytic category. From now on, P will denote a fixed base point in X. The universal cover of X^{an} will be denoted by Z, with a chosen base point which will also be denoted P.

A *system of algebraic differential equations* on X will mean the following data: A locally free sheaf E on X; and a *connection*, in other words a first order algebraic differential operator

$$\nabla\colon E \to E \otimes \Omega^1_X$$

satisfying Leibniz's rule $\nabla(ae) = d(a)e + a\nabla(e)$. Very often, our bundle E will be a trivial bundle $E = \mathcal{O}^n_X$ (in fact this is always the case after going to a Zariski open subset). In this case, the exterior derivative d provides one example of a connection, and any other example can be derived from d by adding a matrix of holomorphic one forms. In other words, a connection ∇ on $E = \mathcal{O}^n_X$ is an expression of the form

$$\nabla = d - A$$

where A is an $n \times n$ matrix with coefficients in Ω^1_X.

If X is not compact, a system (E, ∇) has *regular singularities* if for every $s \in \overline{X} - X$, a trivialization of the bundle E may be chosen over a neighborhood of s such that the connection matrix $A(z)$ has a pole of order ≤ 1 at $z = s$. The choice of trivialization amounts to choice of an extension $\overline{E}$ of the bundle over s, although not every extension will do. See [6].

Suppose E is a system of algebraic differential equations. Suppose γ is an element of $\pi_1(X, P)$. Then we obtain a matrix $m(\gamma)$ in $Gl(E_P)$ as follows. For any vector e_0 in E_P, there is a unique solution of the differential equation $\nabla(e(z)) = 0$ with initial conditions $e(P) = e_0$, defined along the path γ. At the end of the path, the solution is a new vector $m(\gamma)e_0$. Another way of looking at this is to consider the fundamental solution matrix $m(z)\colon E_P \to E_z$ satisfying $\nabla m(z) = 0$ and $m(P) = 1$ (it is really a function on the universal cover Z). Then $m(\gamma)$ is the value of this solution matrix after being continued along γ. The matrices $m(\gamma)$ combine together to form a representation $\Psi_{(E,\nabla)}$ of the fundamental group. This map Ψ from the set of regular singular systems of differential equations to the set of representations of the fundamental group is an equivalence known as the *Riemann-Hilbert correspondence*. The inverse construction is provided by Serre's GAGA theorems—see [6].

One of the features of the study of ordinary differential equations which should not be overlooked is the fact that a formula for the solution can be written: the expression is a convergent sum of *iterated integrals*. Suppose $W_1, \ldots, W_k$ are one-forms on X (possibly with coefficients in a matrix algebra) and $\gamma\colon [0,1] \to X$ is a path. Write $\gamma^* W_i = w_i(t)\, dt$. The iterated integral of

$W_1 \dots W_k$ over γ is defined to be

$$\int_{\Delta_k \gamma} W_1 \dots W_k = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-1}} w_1(t_1) w_2(t_2) \dots w_k(t_k)\, dt_k \dots dt_2\, dt_1.$$

This definition was made by K.-T. Chen in [3], [4]. The importance of iterated integrals is indicated by the following proposition.

PROPOSITION 2.1 (CHEN). *Suppose that $\nabla = d - A$ is a connection on a trivial bundle $\mathcal{O}_X^n$, where A is an $n \times n$ matrix of one-forms. Suppose γ is a path in X, with $\gamma(0) = P$ and $\gamma(1) = Q$. Let $m(z)$ denote the unique solution matrix with $\nabla m(z) = 0$, and $m(P) = 1$, defined along γ. Then*

$$m(Q) = 1 + \int_\gamma A + \int_{\Delta_2 \gamma} AA + \int_{\Delta_3 \gamma} AAA + \cdots.$$

The right hand expression is absolutely convergent.

Proof ([3], [4], and see also [8]). This can be seen by differentiating both sides with respect to Q. The series converges because for the k-th term, the size of the region of integration $0 \leq t_k \leq \cdots \leq t_1 \leq 1$ is $1/(k!)$, whereas the size of the integrand $AA \dots A$ is bounded by C^k.

If the $A(x)$ are upper triangular $n \times n$ matrices with zeros along the diagonal, then $A(z_1) \dots A(z_k) = 0$ for $k \geq n$. In this case, the expansion is a finite sum.

3. Algebraic values of the Riemann-Hilbert Correspondence

The first of the two types of problems connected with the transcendence of the Riemann-Hilbert correspondence is the following general question:

Suppose X is defined over $\overline{\mathbf{Q}} \subset \mathbf{C}$. What are all of the differential equations (E, ∇) defined over $\overline{\mathbf{Q}}$, such that the associated monodromy representations $\Psi_{(E,\nabla)}$ can be defined over $\overline{\mathbf{Q}}$?

Before going to consider some special cases of this problem, let us take note of some situations in which such points automatically arise, and in view of these situations, formulate a standard conjecture.

Suppose $Y \to X$ is a family of varieties, which locally over X varies in a topologically trivial way. Then the fundamental group of X acts on the cohomology of the fiber Y_P, and the resulting representation is the monodromy of a system of differential equations known as the *Gauss-Manin connection*. This is a system of equations with regular singularities, and if $Y \to X$ is defined over $\overline{\mathbf{Q}}$, then the Gauss-Manin system is defined over $\overline{\mathbf{Q}}$. Furthermore, the representation is defined over $\mathbf{Q}$. Any subquotient defined by geometric maps can be defined over $\overline{\mathbf{Q}}$ and hence satisfies the conditions of the general problem. More generally, there may be other subquotients which are algebraic for both $\overline{\mathbf{Q}}$ structures, although it does not seem to be automatically the case for every subquotient. In keeping with the usual kinds of conjectures made about periods, we have the following.

STANDARD CONJECTURE. *Any regular singular system of differential equations which is defined over $\overline{\mathbf{Q}}$ and such that the monodromy representation can be defined over $\overline{\mathbf{Q}}$, comes as a subquotient of a Gauss-Manin system.*

It is somewhat disingenuous to call this a conjecture. There is certainly no more reason to believe it is true than to believe the Hodge conjecture, and whether or not it is true, it is evidently impossible to prove with any methods which are now under consideration. However, it is an appropriate motivation for some easier particular examples, and it leads to some conjectures which might in some cases be more tractable. These (essentially) weaker conjectures are based on the philosophy that if a system comes from a Gauss-Manin system, then there are certain properties it satisfies. A conjecture derived from the standard conjecture is obtained by asking if any solution to the principle problem must automatically satisfy some property known for Gauss-Manin systems. These conjectures are analogous to Deligne's conjectures about absolute Hodge cycles [7].

CONJECTURE (Absoluteness I). *Suppose $i\colon \overline{\mathbf{Q}} \to \mathbf{C}$ and suppose E is a regular singular system defined over $\overline{\mathbf{Q}}$ such that the representation corresponding to $i(E)$ can be defined over $\overline{\mathbf{Q}}$. Then for any other embedding $\sigma\colon \overline{\mathbf{Q}} \to \mathbf{C}$, the representation corresponding to $\sigma(E)$ can be defined over $\overline{\mathbf{Q}}$.*

CONJECTURE (Absoluteness II). *Suppose $j\colon \overline{\mathbf{Q}} \to \mathbf{C}$ and suppose ρ is a representation defined over $\overline{\mathbf{Q}}$ such that the system of equations with regular singularities corresponding to $j(\rho)$ can be defined over $\overline{\mathbf{Q}}$. Then for any other embedding $\tau\colon \overline{\mathbf{Q}} \to \mathbf{C}$, the system of equations corresponding to $\tau(\rho)$ can be defined over $\overline{\mathbf{Q}}$.*

CONJECTURE (Variation of Hodge structure). *Suppose E is a regular singular system defined over $\overline{\mathbf{Q}} \subset \mathbf{C}$ such that the corresponding representation can be defined over $\overline{\mathbf{Q}}$. Then the representation is a subquotient of the monodromy of a good variation of mixed Hodge structure on X.*

CONJECTURE (Galois-type). *Suppose E is a regular singular system defined over a number field $K \subset \mathbf{C}$ such that the corresponding representation ρ can be defined over $\overline{\mathbf{Q}}$. Let l be a prime where the coefficients of ρ are integral, and let ρ_l denote the completion of ρ to a representation*

$$\rho_l\colon \pi_1^{\mathrm{alg}}(X \otimes_K \overline{\mathbf{Q}}) \to Gl(n, \overline{\mathbf{Q}}_l).$$

Then ρ_l extends to a representation of $\pi_1^{\mathrm{alg}}(X \otimes_K L)$ for some finite extension L/K.

Remark. These conjectures hold for any geometric subquotient of a Gauss-Manin system. Admittedly, it is not clear that they hold for any doubly algebraic subquotient, so they do not follow from the standard conjecture. Perhaps one should strengthen the standard conjecture by asking for geometric subquotients. My guess is that there may be counterexamples to this statement.

The standard conjecture must be considered as a conjecture in transcendence theory and algebraic geometry, because it asks for an actual family of varieties. The absoluteness and variation of Hodge structure conjectures can be thought of as conjectures in transcendence theory only. The Galois-type conjecture is a problem in number theory—we will not discuss it any further, but have included it for comparison.

We will treat the following special cases: the standard conjecture for systems of equations of rank 1 on any smooth variety; the standard conjecture for irreducible systems of rank 2 on $\mathbf{P}^1 - \{0, 1, \infty\}$; and the absoluteness (I and II) and variation of Hodge structure conjectures for unipotent systems of rank three on $\mathbf{P}^1 - \{s_1, \dots, s_k\}$. This last case is the most interesting—it uses iterated integrals and Baker's theorem.

Before giving these theorems, let us recall a proof of the monodromy theorem using transcendence theory.

LEMMA 3.1. *If E is a system of equations on X defined over $\overline{\mathbf{Q}}$, with regular singularities, such that the associated monodromy representation has a $\overline{\mathbf{Q}}$ structure, then the eigenvalues of the monodromy transformations around the points at infinity are roots of unity.*

Proof. This is due to Brieskorn [2]. Let z be a local parameter at a point of $\overline{X} - X$. By choosing an appropriate frame for the bundle, the system of equations becomes

$$\nabla = d - A\frac{dz}{z} - B(z)\,dz$$

with A a constant matrix and B regular at $z = 0$. We can assume that A has coefficients in $\overline{\mathbf{Q}}$ since ∇ is defined over $\overline{\mathbf{Q}}$. The monodromy transformation is conjugate to $e^{2\pi i A}$. If the eigenvalues of A are α_j then the assumption that the associated local system has a $\overline{\mathbf{Q}}$ structure implies that $e^{2\pi i \alpha_j} \in \overline{\mathbf{Q}}$. By the Gelfond-Schneider theorem, $\alpha_i \in \mathbf{Q}$, so the eigenvalues of the monodromy transformation are roots of unity.

Remark. If one assumes the regularity of the Gauss-Manin connection then this lemma proves the monodromy theorem, that the eigenvalues of the

Picard-Lefschetz transformations around the singular fibers of a family of varieties are roots of unity [6].

THEOREM 1. *Suppose X is defined over $\overline{\mathbf{Q}}$. The standard conjecture holds for regular singular systems of equations of rank one on X.*

Proof. The idea is to show that the monodromy representation is a character which takes values in the group of roots of unity. Any such character comes from geometry, by the following construction. The character is a finite cyclic quotient G of $\pi_1(X)$, so it gives an algebraic finite cyclic Galois covering space $f\colon Y \to X$. The Gauss-Manin system for this map has monodromy representation equal to the regular representation of the cyclic Galois group G. The component which transforms according to the canonical character $G \to \mathbf{C}^*$ is isomorphic to the given rank one system.

By Lemma 3.1, the monodromy transformations around the points $s \in S = \overline{X} - X$ are roots of unity. Therefore there is a finite abelian cover $\pi\colon Y \to X$ branched at the singularities, so that $\pi^*(L, \nabla)$ has removable singularities. Note that (L, ∇) is a direct summand of $\pi_*\pi^*(L, \nabla)$, so we may assume that (L, ∇) has removable singularities. Choose a trivialization of the line bundle over an open set U, so that now our system is $(\mathcal{O}_U, d - a)$ where a is a one form with poles of order ≤ 1 and residues which are integer multiples of $2\pi i$. The monodromy transformations are $\exp(\int_{\gamma_i} a)$. Waldschmidt proves that if these are not all roots of unity, then one of them is transcendental ([13] Corollary 5.2.7, and remark on pp. 92–93). Note that the basic result from transcendence theory which is behind Waldschmidt's theorem is the criterion of Schneider-Lang.

THEOREM 2. *The standard conjecture holds for irreducible regular singular systems of rank two on $\mathbf{P}^1 - \{0, 1, \infty\}$.*

Proof. The idea behind this proof is that the hypergeometric differential equation essentially covers all possible systems of rank 2 on $\mathbf{P}^1 - \{0, 1, \infty\}$ (there are some unipotent cases left over) whereas Riemann's integral representation formula expresses the hypergeometric system as a direct image of a rank one system (cf. Messing [15]). Let D denote the divisor of the function $z(z-1)w(w-1)(z-w)$ in $\mathbf{C}^2$, and let $X = \mathbf{C}^2 - D$. Let $\pi\colon X \to \mathbf{P}^1 - \{0, 1, \infty\}$ be the projection to the z-axis. Consider the rank one system on X given by

$$\nabla = d - a\frac{dw}{w} - b\frac{dw}{w-1} - c\frac{d(w-z)}{w-z} - u\frac{dz}{z} - v\frac{dz}{z-1}.$$

The monodromy transformations around the divisors (w), $(w-1)$, $(w-z)$, (z) and $(z-1)$ are

$$\alpha = e^{2\pi i a}, \quad \beta = e^{2\pi i b}, \quad \gamma = e^{2\pi i c}, \quad \mu = e^{2\pi i u} \quad \text{and} \quad \nu = e^{2\pi i v}$$

respectively. Let $E(a, b, c, u, v) = R^1\pi_*(\mathcal{O}_X, \nabla)$ be the direct image system on $\mathbf{P}^1 - \{0, 1, \infty\}$. Assume that c is not an integer. Then E has rank 2. Note that the system of differential equations E has regular singularities, and depends algebraically on the coefficients a, b, c, u, v (the relevant formulas may be found in [14] for example). On the other hand, the monodromy transformations may be calculated topologically from the monodromy transformations for $(\mathcal{O}_X, \nabla)$. The fundamental group of $\mathbf{P}^1 - \{0, 1, \infty\}$ is free on two generators, and one calculates that the monodromy representation of E is given by the two matrices

$$\varphi_0 = \begin{pmatrix} \alpha\gamma\mu & 0 \\ \mu(\beta - 1) & \mu \end{pmatrix} \quad \text{and} \quad \varphi_1 = \begin{pmatrix} \nu & \gamma\nu(\alpha - 1) \\ 0 & \beta\gamma\nu \end{pmatrix}.$$

Now suppose (M, ∇) is an irreducible regular singular rank 2 system of equations, defined over $\overline{\mathbf{Q}}$ and such that the monodromy is defined over $\overline{\mathbf{Q}}$. The monodromy representation is given by two matrices ρ_0 and ρ_1 with coefficients in $\overline{\mathbf{Q}}$. Since the representation is irreducible, there is a matrix $S \in PGl(2, \overline{\mathbf{Q}})$ such that $S\rho_0 S^{-1}$ is upper triangular and $S\rho_1 S^{-1}$ is lower triangular. There are less than four choices for S, up to left multiplication by a diagonal matrix. There are at least one and at most finitely many choices of this diagonal matrix, and nonzero numbers $(\alpha, \beta, \gamma, \mu, \nu)$, such that $\varphi_0 = S\rho_0 S^{-1}$ and $\varphi_1 = S\rho_1 S^{-1}$. In particular, $\alpha, \beta, \gamma, \mu, \nu \in \overline{\mathbf{Q}}$. (Note that $\gamma \neq 1$, for otherwise the matrices expressed by φ_0 and φ_1 would have a common eigenvector, so the representation would be reducible.) There are finitely many choices of a, b, c, u, v (modulo integers) such that the resulting direct image E is isomorphic to M. By a specialization argument, this implies that the a, b, c, u, v are in $\overline{\mathbf{Q}}$. But now we may apply the Gelfond-Schneider theorem to conclude that a, b, c, u, v are rational, and $\alpha, \beta, \gamma, \mu, \nu$ are roots of unity. Thus the system of equations $(\mathcal{O}_X, \nabla)$ came from geometry (from an abelian covering space of X), so E comes from geometry.

THEOREM 3. *Suppose $s_1, \ldots, s_n$ are $\overline{\mathbf{Q}}$ rational points in $\mathbf{P}^1$. The conjectures of absoluteness I and II, and the conjecture of variation of Hodge structure, hold for unipotent regular singular systems of equations of rank* 3 *on* $\mathbf{P}^1 - \{s_1, \ldots, s_n\}$.

Proof. Suppose for now that an embedding $\overline{\mathbf{Q}} \subset \mathbf{C}$ is fixed. Assume that (V, ∇) is unipotent and $\operatorname{rank}(V) = 3$. One checks that $V \cong \mathcal{O}^3_{\mathbf{P}^1 - S}$ and the connection is given by

$$\nabla = d - w = d - A_1 \frac{dz}{z - s_1} - \cdots - A_n \frac{dz}{z - s_n},$$

where

$$A_k = \begin{pmatrix} 0 & a_k & c_k \\ 0 & 0 & b_k \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\gamma_1, \ldots, \gamma_n$ be paths in $\mathbf{P}^1 - S$ going out from P to $s_1, \ldots, s_n$ respectively, around counterclockwise, and back to P. Let $M_1, \ldots, M_n \in GL(3)$ be the monodromy transformations around $\gamma_1, \ldots, \gamma_n$ respectively. These monodromy transformations may be expressed in terms of iterated integrals:

$$M_k = 1 + \int_{\gamma_k} w + \int_{\Delta_2 \gamma_k} ww.$$

The series terminates after these terms. Fortunately, the iterated integrals in question can be calculated in terms of logarithms:

$$\int_{\Delta_2 \gamma_k} \left(\frac{dz}{z - s_j}\right)\left(\frac{dz}{z - s_k}\right) = \begin{cases} (2\pi i)\big(\log(P - s_k) - \log(s_j - s_k)\big), & i = j \neq k, \\ (2\pi i)\big(\log(s_k - s_j) - \log(P - s_j)\big), & i = k \neq j, \\ (2\pi i)^2, & i = j = k, \\ 0, & i \neq j,\ i \neq k. \end{cases}$$

In each formula, the determinations of the logarithms are related by the paths joining P to s_j.

Thus the monodromy matrices become

$$M_k = \begin{pmatrix} 1 & (2\pi i)\, a_k & (2\pi i)^2 m_k \\ 0 & 1 & (2\pi i)\, b_k \\ 0 & 0 & 1 \end{pmatrix}$$

where m_k is defined by

$$(2\pi i)(m_k - a_k b_k/2) = c_k + \sum_j (a_k b_j - a_j b_k)\big(\log(P - s_j) - \log(s_k - s_j)\big).$$

We first isolate the degenerate case of depth 2, in other words when the system of equations has a filtration of length two with trivial quotients. In this case either $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$. If the system of equations is defined over $\overline{\mathbf{Q}}$, then after conjugating by the matrices $\operatorname{diag}(1, 1, 2\pi i)$ or $\operatorname{diag}(1, 2\pi i, 2\pi i)$ respectively, the monodromy is always defined over $\overline{\mathbf{Q}}$. This proves the conjecture absoluteness I; conversely if the monodromy is defined over $\overline{\mathbf{Q}}$ then the system of equations can be defined over $\overline{\mathbf{Q}}$, for absoluteness II. From now on, we will assume that neither a_k nor b_k all vanish.

The conditions are that a_k, b_k, and c_k are in $\overline{\mathbf{Q}}$, and that we may conjugate M_k so that the entries are in $\overline{\mathbf{Q}}$. It is easy to see that the matrix of conjugation must be upper triangular. Furthermore, the diagonal of R must be

$1, 2\pi i, (2\pi i)^2$ up to algebraic multiples which we can ignore. After conjugating by $\mathrm{diag}(1, 2\pi i, (2\pi i)^2)$, the monodromy matrices become

$$M_k' = \begin{pmatrix} 1 & a_k & m_k \\ 0 & 1 & b_k \\ 0 & 0 & 1 \end{pmatrix}.$$

Now R is upper triangular with 1's on the diagonal, say

$$R = \begin{pmatrix} 1 & v & * \\ 0 & 1 & -u \\ 0 & 0 & 1 \end{pmatrix}.$$

Conjugating by R has the effect of changing M_k' to

$$\begin{pmatrix} 1 & a_k & m_k + ua_k + vb_k \\ 0 & 1 & b_k \\ 0 & 0 & 1 \end{pmatrix}.$$

The conditions of the problem are that we can choose u and v so that $m_k + ua_k + vb_k = p_k \in \overline{\mathbf{Q}}$. A similar analysis works the other way, going from the monodromy representation to the local system. Thus we have the following criteria (assuming that neither a_k nor b_k all vanish):

(1) The system of equations determined by (a_k, b_k, c_k) has monodromy representation defined over $\overline{\mathbf{Q}}$ if and only if there exist $u, v \in \mathbf{C}$ such that

$$c_k + \sum_j (a_k b_j - a_j b_k)\big(\log(P - s_j) - \log(s_k - s_j)\big) + ua_k + vb_k \in 2\pi i \overline{\mathbf{Q}}.$$

(2) The monodromy representation determined by (a_k, b_k, p_k) comes from a local system defined over $\overline{\mathbf{Q}}$ if and only if there exist $u, v \in \mathbf{C}$ such that

$$(2\pi i)(p_k - a_k b_k/2) - \sum_j (a_k b_j - a_j b_k)\big(\log(P - s_j) - \log(s_k - s_j)\big)$$
$$- ua_k - vb_k \in \overline{\mathbf{Q}}.$$

Absoluteness I. There is a universal $\overline{\mathbf{Q}}$ vector space $\mathscr{L}$ with inclusion $\overline{\mathbf{Q}} \hookrightarrow \mathscr{L}$ and map $\log\colon \overline{\mathbf{Q}} \to \mathscr{L}$ satisfying $\log(ab) = \log(a) + \log(b)$. As a consequence of Baker's theorem we get the following statement: for any embedding $\sigma\colon \overline{\mathbf{Q}} \to \mathbf{C}$, the resulting map also denoted

$$\sigma\colon \mathscr{L} \to \mathbf{C}/2\pi\sqrt{-1}\,\overline{\mathbf{Q}}$$

is injective. Now we may prove absoluteness I. For a given embedding

$\sigma\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, the representation corresponding to local system given by (a_k, b_k, c_k) has a $\overline{\mathbf{Q}}$-structure if and only if there exist $u, v \in \mathbf{C}$ such that

$$\sigma(c_k) + \sum_j \sigma(a_k b_j - a_j b_k)\big(\log \sigma(P - s_j) - \log \sigma(s_k - s_j)\big)$$
$$+ u\sigma(a_k) + v\sigma(b_k) \in 2\pi i \overline{\mathbf{Q}}$$

for all k. We may assume that $u, v \in \sigma(\mathscr{L}) \subset \mathbf{C}/2\pi\sqrt{-1}\,\overline{\mathbf{Q}}$. Then because of the injectivity in the above statement, the representation has a $\overline{\mathbf{Q}}$-structure if and only if

$$c_k + \sum_j (a_k b_j - a_j b_k)\big(\log(P - s_j) - \log(s_k - s_j)\big) + a_k u + b_k v = 0$$

in $\mathscr{L}$, for all k. This condition is clearly independent of the embedding σ.

Absoluteness II: In this case, the complex numbers

$$\big(\log(P - s_j) - \log(s_k - s_j)\big)$$

are fixed. Let $\mathscr{L}^{\wedge}$ denote the $\mathbf{Q}$ vector space spanned by these numbers and $2\pi i$. Baker's theorem says that

$$\mathscr{L}^{\wedge} \otimes \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}/\overline{\mathbf{Q}}$$

is injective. Suppose that a representation ρ with coefficients in $\overline{\mathbf{Q}}$ is given by (a_k, b_k, p_k). If $\tau\colon \overline{\mathbf{Q}} \to \mathbf{C}$ is an embedding, the condition that $\tau(\rho)$ corresponds to a system of equations defined over $\overline{\mathbf{Q}}$ is that there exist u, v such that

$$(2\pi i)\tau(p_k - a_k b_k/2) - \sum_j \tau(a_k b_j - a_j b_k)\big(\log(P - s_j) - \log(s_k - s_j)\big)$$
$$- \tau(a_k)u - \tau(b_k)v \in \overline{\mathbf{Q}}.$$

This condition is equivalent to the condition that there exist $u, v \in \mathscr{L}^{\wedge} \otimes \overline{\mathbf{Q}}$ such that

$$(2\pi i)(p_k - a_k b_k/2) - \sum_j (a_k b_j - a_j b_k)\big(\log(P - s_j) - \log(s_k - s_j)\big)$$
$$- a_k u - b_k v = 0$$

in $\mathscr{L}^{\wedge} \otimes \overline{\mathbf{Q}}$. Again this is independent of the choice of τ.

Variation of Hodge structure. From the description of good unipotent variations of mixed Hodge structure on subsets of $\mathbf{P}^1$ given in [9], we can extract the following statement. A unipotent system of equations on $\mathbf{P}^1 - \{s_1, \dots, s_k\}$, such that the monodromy is defined over $\mathbf{Q}$, comes from a good variation of mixed Hodge structure if the system of equations can be put in the form

$$\nabla = d - A_1 \frac{dz}{z - s_1} - \cdots - A_n \frac{dz}{z - s_n}$$

where the matrices A_k have zeros everywhere except immediately above the diagonal.

In our case, we have a representation defined over $\overline{\mathbf{Q}}$. By adding together all of the Galois conjugates, we obtain a representation defined over $\mathbf{Q}$. By absoluteness II, each of these conjugates corresponds to a system of equations (a_k, b_k, c_k) defined over $\overline{\mathbf{Q}}$. This implies in particular that

$$c_i = a_i u + b_i v,$$

due to the part of Baker's theorem which says that 1 is linearly independent from the logarithms. Thus after conjugating by the upper triangular matrix with u and v immediately off the diagonal, we get

$$A_k = \begin{pmatrix} 0 & a_k & 0 \\ 0 & 0 & b_k \\ 0 & 0 & 0 \end{pmatrix}.$$

This holds for each system of equations corresponding to a conjugate of the representation, so the direct sum of these systems has the required form.

In the degenerate case of depth 2, note that any such system can be expressed as a subquotient of a direct sum of two systems of rank 2. Each of these rank 2 systems automatically satisfies the criterion to come from a variation of mixed Hodge structure. However, the subquotient of the direct sum might not be compatible with the Hodge and weight filtrations, so it might not itself carry a variation of mixed Hodge structure.

This completes the proof of Theorem 3.

Remark. The proof of Theorem 3 shows that there are restrictions placed on the possibilities for unipotent systems of rank 3 on subsets of $\mathbf{P}^1$ which satisfy the conditions of our problem. The fact that a Gauss-Manin system always satisfies those conditions means that there are restrictions on which systems may arise from Gauss-Manin systems. For example, a careful analysis of the case of four singularities shows the following.

PROPOSITION 3.2. *If (V, ∇) is a unipotent rank 3 system on $\mathbf{P}^1$ with regular singularities at 0, 1, τ, and ∞, satisfying the conditions of our general problem,* *then either*

(a) (V, ∇) *is abelian*,

(b) *one of the four singularities is removable*, *or*

(c) $\tau^a(1-\tau)^b = 1$ *for some integers* a, b.

COROLLARY 3.3. *If* τ *is a transcendental parameter, then there are no nonabelian unipotent rank* 3 *systems with nonremovable regular singularities at* 0, 1, τ, *and* ∞, *which come from geometry.*

4. Asymptotic behaviour of the monodromy

As described in the introduction, in the case of systems of rank one on a compact Riemann surface, the monodromy grew exponentially along certain algebraic paths of systems. For systems of higher ranks, we will consider some simple paths of systems which go to infinity. Namely, we look at linear paths of connections on a trivial bundle over a compact Riemann surface X. These are connections of the form

$$\nabla = d - tA - B$$

where A and B are fixed matrices of one forms, and t is a parameter which will tend to infinity (t is distinct from the variable of differentiation). Make the further simplifying assumption that A is a diagonal matrix, and then various types of matrices B will be treated. In all cases, we will assume that B has zeros along the diagonal.

The basic idea is to obtain an expression for the solution using iterated integrals, and then to investigate the asymptotic behaviour of the terms in the expression. The expansion of Proposition 2.1 in its basic form is not useful, because the terms will have higher and higher powers of t. Thus we first transform the equation. Let $E_t(z)$ be a diagonal matrix such that $d(E_t) = tAE_t$. If the diagonal entries of A are one forms $a_1, \dots, a_n$, let

$$g_i(z) = \int_P^z a_i.$$

Here g_i and hence $E_t = \operatorname{diag}(\dots, e^{tg_i}, \dots)$ are functions of points z in the universal cover of X, which we will denote by Z. Let $m_t(z)$ be the fundamental solution matrix with $m_t(P) = 1$. Let $h_t(z) = E_t^{-1}(z)m_t(z)$. We have

$$dh_t = -tAE_t^{-1}m_t + E_t^{-1}(tA + B)m_t = E_t^{-1}BE_th_t$$

(note that A and E_t commute). Therefore we can express $h_t(Q)$ as a sum of iterated integrals of $B^{E_t} = E_t^{-1}BE_t$. Then multiply by $E_t(Q)$ to obtain the expression

$$m_t(Q) = E_t(Q) + \int_\gamma E_t(Q)B^{E_t} + \int_{\Delta_2\gamma} E_t(Q)B^{E_t}B^{E_t} + \cdots.$$

Here γ is a path from P to Q in the universal cover Z. We will investigate the asymptotic behaviour of this series term by term.

Write the entries of the matrix B as b_{ij}. Then the entries of B^{E_t} are $b_{ij}e^{t(g_j-g_i)}$. The entries of the kth term in the above expansion are iterated integrals of the form

$$\int_{\Delta_k\gamma} b_{i_k i_{k-1}}(z_k)\dots b_{i_2 i_1}(z_2)\,b_{i_1 i_0}(z_1)\,e^{t(g_{i_k}(Q)-g_{i_k}(z_k)+\cdots+g_{i_0}(z_1)-g_{i_0}(P))}.$$

Here z_i denotes $\gamma(t_i)$. The term written here contributes to the (i_k, i_0) entry of the matrix $m_t(Q)$.

We will show that any finite sum of terms such as these has an asymptotic expansion. At the end of the section, we will state (without proof) a theorem to the effect that in some cases, our infinite sum of integrals has an asymptotic expansion, obtained formally by adding up the expansions for the terms.

The iterated integrals can be interpreted as follows. Inside the space Z^k are the subsets

$$S_\alpha = \{(z_1,\dots,z_k)\colon z_\alpha = z_{\alpha+1}\}.$$

Where α runs from 0 to k. The convention that $z_0 = P$ and $z_{k+1} = Q$ will be maintained throughout. Let $S = \bigcup S_\alpha$. It is a complex subvariety of Z^k. The path γ from P to Q leads to a singular k-simplex $\Delta_k\gamma$. It is the set of points of the form

$$(\gamma(t_1),\gamma(t_2),\dots,\gamma(t_k)) \quad \text{for } 0 \le t_1 \le t_2 \le \cdots \le t_k \le 1.$$

The boundary of $\Delta_k\gamma$ is contained in S, so this gives a relative homology class $\Delta_k \in H_k(Z^k, S)$. The product

$$\beta = b_{i_k i_{k-1}}(z_k)\dots b_{i_1 i_0}(z_1)$$

is a holomorphic k-form on Z^k. Finally, the function

$$g(z_1,\dots,z_k) = g_{i_k}(Q) - g_{i_k}(z_k) + \cdots + g_{i_0}(z_1) - g_{i_0}(P)$$

is a holomorphic function on Z^k. As such, for any t we can multiply to get another holomorphic k-form βe^{tg} on Z^k. The integral considered previously can now be written as

$$\int_{\Delta_k} \beta e^{tg}.$$

This integral does not depend on the choice of representative for the relative homology class Δ_k. The reason for this is that Z^k has complex dimension k, so any holomorphic k-form is automatically closed; and S has complex dimension $k-1$, so the restriction of a holomorphic k-form to S must vanish identically.

We will formalize the classical fact, known as the lemma of the stationary phase, that integrals of the form $\int \beta e^{tg}$ have asymptotic expansions as $t \to \infty$. Since we intend to add together several integrals of this form, there is the possibility that the asymptotic expressions would cancel out. In this case we want to be able to get an expansion with a smaller exponent. For this reason, we approach the lemma of the stationary phase via the Laplace transform.

Formally the *Laplace transform* of a function $m(t)$ is the integral

$$f(\zeta) = \int_0^\infty m(t) e^{-\zeta t}\, dt.$$

The integration is taken along a direction in which the integrand is rapidly decreasing. Such a direction exists for large values of ζ if m has exponential order, in other words if it satisfies an estimate $|m(t)| \le Ce^{R|t|}$. In this case $f(\zeta)$ is defined and holomorphic for $|\zeta| > R$. Furthermore f is holomorphic and vanishes at ∞. Conversely any such function $f(\zeta)$ determines a function $m(t)$ by the integral formula

$$m(t) = \frac{1}{2\pi i}\oint f(\zeta) e^{\zeta t}\, d\zeta$$

where the path of integration is a large circle. These constructions are inverses of each other.

If the function m is given as an integral

$$m(t) = \int b e^{gt}$$

over a space with measure b and a bounded function g, then m has exponential order and the Laplace transform is given by the integral

$$f(\zeta) = \int \frac{b}{g-\zeta}.$$

Fix R so that $f(\zeta)$ is defined for $|\zeta| \ge R$. Say that *f has an extension with locally finite branching* if for each M there exists a finite subset $S_M \subset \mathbf{C}$ such that for any path γ of length less than or equal to M in $\mathbf{C} - S_M$, such that $|\gamma(0)| \ge R$, f can be analytically continued as a holomorphic function along a neighborhood of γ.

Given that this is the case, suppose γ is a path of length $\le M - 2\varepsilon$ such that $\gamma(1) = s \in S_M$ and $\gamma(t)$ is in $\mathbf{C} - S_M$ for $0 \le t < 1$. Suppose that γ approaches s along a ray. Let $\Delta = \Delta(s, \varepsilon)$ be the disc of radius ε about s, and let Δ^* be the punctured disc. Then for any point in the universal cover of Δ^* there is a path of length $\leq M$ extending $\gamma|_{[0,1-\varepsilon]}$ and ending at that point. Thus f can be analytically continued to any point on the universal cover of Δ^*. In other words, f can be continued to a multivalued function on Δ^*.

Say that a multivalued function $f(\zeta)$ on the punctured disc $0 < |\zeta| < \varepsilon$ is *regular singular* if it has a convergent power series expansion

$$f(\zeta) = \sum_{j=J}^{\infty} \sum_{k=0}^{K} c_{jk}\zeta^{j/N}(\log \zeta)^k.$$

Let T be the monodromy operator on multivalued functions: $Tf(\zeta) = f(e^{2\pi i}\zeta)$.

LEMMA 4.1. *$f(\zeta)$ is regular singular if and only if it has polynomial growth along every ray, and there exist N and K such that $(T^N - I)^{K+1}f(\zeta)$ is single valued. The N and K in the expansion are the same as these.*

Proof. If $(T - I)f = 0$ then f is single valued, and hence meromorphic by the condition of polynomial growth. Suppose for example $(T - I)^2 f = 0$. Then $h_1 = (T - I)f/2\pi i$ is meromorphic. Set

$$f_1(\zeta) = (\log \zeta)\, h_1.$$

Then $(T - I)f_1 = (2\pi i)h = (T - I)f$, so $h_0 = f - f_1$ is meromorphic. Thus

$$f = h_0 + h_1 \log \zeta$$

gives the required expansion. Proceed like this in general, replacing ζ by $\zeta^{1/N}$ if necessary.

Say that our function f has *locally finite regular singularities* if it has an extension with locally finite branching, and if the multivalued functions on punctured discs around all singular points are regular singular. We remark that the sum or product of two functions with locally finite regular singularities again has locally finite regular singularities.

There is one comment which should be made as a clarification. The condition of locally finite regular singularities does not preclude the possibility that the set of all singularities of the function is dense for example. In that case it would imply that to get to most of the singularities, the analytic continuations must be taken over long paths winding back and forth.

The following lemma is the lemma of stationary phase for the inverse Laplace transform.

LEMMA 4.2. *Suppose that a function $f(\zeta)$ has an extension with locally finite regular singularities. Let $m(t)$ be the inverse Laplace transform, and suppose m*

is not identically zero. Then the function $m(t)$ has a nonzero asymptotic expansion for positive real $t \to \infty$.

Proof. Recall that $m(t)$ is given as a path integral of $f(\zeta)e^{\zeta t}$. Deform the path of integration until it is a sum of paths which go around critical points with real part ξ and back in the negative real direction from those points, and paths which are supported on points of real part strictly less than ξ. The lower paths will not contribute to the expansion. Integrate by parts to remove any poles of f at $\zeta = \lambda$ (in our case the function will already be bounded on every sector). The contribution from a path which goes around a singular point λ is equal to

$$\frac{1}{2\pi i}\int_{\lambda-\varepsilon}^{\lambda}(T-I)f(\zeta)e^{\zeta t}\,d\zeta.$$

This has an asymptotic expansion of the form $e^{t\lambda}$ times a series in $t^{-1/N}$ and $\log t$, given by the regular expansion for $(T-I)f$ at s. If the expansion is zero, then $(T-I)f = 0$, so f can be analytically continued across s. If all of the singular points vanish, then f is entire, so $m = 0$.

THEOREM 4. *The Laplace transform of the iterated integral*

$$\int_{\Delta_k\gamma} b_{i_k i_{k-1}}(z_k)\dots b_{i_2 i_1}(z_2)\,b_{i_1 i_0}(z_1)\,e^{t(g_{i_k}(Q)-g_{i_k}(z_k)+\cdots+g_{i_0}(z_1)-g_{i_0}(P))}.$$

has an extension with locally finite regular singularities.

COROLLARY 4.3. *Suppose A is a diagonal matrix of one forms with distinct entries, and B is an upper triangular matrix of one forms with zeros on the diagonal. Let $m_t(z)$ denote the fundamental solution matrix beginning at P, for the system $\nabla = d - tA - B$. Then for any point Q, the matrix $m_t(Q)$ has a nonzero asymptotic expansion in t:*

$$m(t) \sim \sum_{i=1}^{m} e^{\lambda_i t}\sum_{j=J}^{\infty}\sum_{k=0}^{K} c_{ijk}t^{-j/N}(\log t)^k$$

Proof. Since B is upper triangular, $m_t(Q)$ is a finite sum of integrals treated in Theorem 4. Therefore the Laplace transform of $m_t(Q)$ has locally finite regular singularities. Now apply Lemma 4.2.

Proof of Theorem 4. Keep the notation

$$\beta = b_{i_k i_{k-1}}(z_k)\dots b_{i_2 i_1}(z_2)\,b_{i_1 i_0}(z_1)$$

and

$$g = g_{i_k}(Q) - g_{i_k}(z_k) + \cdots + g_{i_0}(z_1) - g_{i_0}(P).$$

Let

$$f(\zeta) = \int_\Delta \frac{\beta}{g - \zeta}$$

be the Laplace transform of the integral in question. The basic idea is to use the method of steepest descent to move the cycle of integration around, making it possible to analytically continue f. Suppose that the support of Δ does not meet the fiber $Y_u = g^{-1}(u)$. Then the integral $f(\zeta)$ is well defined in a neighborhood of $\zeta = u$. Suppose γ is a path with $\gamma(0) = u$ (with no self intersections). Δ is a class in the relative homology $H_k(Y - Y_u, S - S_u)$. If Δ is homologous to a class Δ' in

$$H_k(Y - g^{-1}(\gamma), S - g^{-1}(\gamma)),$$

then

$$f(\zeta) = \int_{\Delta'} \frac{\beta}{g - \zeta}$$

can be analytically continued along γ. Intuitively, f can be analytically continued along any path γ such that Y is smooth over γ. We really only need smoothness in some compact subset, whose size depends on how far we are moving the chains, in other words, on the length of γ. In this compact subset, there are only finitely many components of critical points of g, so if the path γ misses the finitely many images of these critical points, then we will be able to analytically continue along γ. We will make this argument more precise.

Let $Y = Z^k$, and for a subset of indices I, let $Y_I = \bigcup_{\alpha \in I} S_\alpha$. These are the smooth closed strata for a stratification of (Y, S). In order to deal with relative homology classes, form the disjoint unions of the various closed strata:

$$Y_b = \coprod_{|I| = b} Y_I, \quad \tilde{Y} = \coprod Y_b.$$

Let $C_{a,b}(Y)$ denote the group of a-chains on Y_b. There is a map $e\colon C_{a,b} \to C_{a,b+1}$ obtained by considering the various inclusions $Y_I \subset Y_J$ and introducing appropriate minus signs (in the usual way of getting a boundary operator from a simplicial object). This has the properties that $e^2 = 0$ and $\partial e + e\partial = 0$. Notice that with the maps ∂ and e, we get a double complex $C_{\cdot,\cdot}$ which computes the relative homology. Given a chain Δ which is a relative cycle for (Y, S), we can extend it to an element $\tilde{\Delta}$ of the double complex, so that $(\partial + e)\tilde{\Delta} = 0$.

For each stratum Y_I, choose a complex vector field V lifting the vector field $\partial/\partial z$ on $\mathbf{C}$, in other words $dg_*(V) = \partial/\partial z$. The vector field V will have singularities at the critical points of g. However, we may choose a singular metric h on Y so that the norm of the vector field V with respect to the metric h is less than 1. Furthermore, we may assume that h is a complete metric. To see this, note that Y_I covers a compact space (a product of copies of our Riemann surface), and the differential dg is pulled back from the compact space. We may choose the vector fields and the metric on the compact space, and pull back.

Let $F_m \subset \tilde{Y}$ denote the set of points at distance $\leq m$ from P (using the metric h), and let $F_mC_{\cdot,\cdot}$ denote the subgroup of chains supported on the set F_m. Let S_m denote the set of images of critical points of g on $\tilde{Y} \cap F_m$.

The vector field V can now be used to move things around. Fix a stratum Y_I. Suppose $r(z,t)$ is a function from $\mathbf{C} \times [0,1]$ to $\mathbf{C}$, such that $r(z,0) = z$. Suppose $\int_0^1 |\partial r/\partial t| \leq \kappa$ for all z. Suppose r fixes a neighborhood of any point $z \in S_{m+\kappa}$, and furthermore that $r(z,t)$ is not in such a neighborhood if z is not. Then we may lift r to a flow $R(y,t)$ from $F_mY_I \times [0,1]$ to $F_{m+\kappa}Y_I$, using the vector fields V: set $R(y,0) = y$ and

$$\partial R/\partial t(y,t) = \operatorname{Re} \partial r/\partial t(g(y),t) V(R(y,t))$$

(taking the real part of the complex vector field on the right gives a real vector field as on the left). This differential equation has a solution due to the completeness of the metric h. Furthermore, the condition that the norm of V is ≤ 1 means that the distance from y to $R(y,1)$ is less than $\int_0^1 |\partial r/\partial t|$. Note that this also works if r is defined on a subset of $\mathbf{C}$. The lifted flow R is defined on the inverse image of this subset.

Suppose γ is a path of length $\leq \kappa$, without self intersection. If γ does not meet $S_{m+\kappa}$, then we may choose a map $r\colon (\mathbf{C} - u) \times [0,1] \to \mathbf{C} - u$ such that $r(z,1) \in \mathbf{C} - \gamma$. Lift to a map R on every stratum. Set $M(c) = R(c,1)$ and $K(c) = R(c,[0,1])$; these are a chain map and a homotopy

$$M\colon F_mC_{a,b}(Y - g^{-1}(u)) \to F_{m+\kappa}C_{a,b}(Y - g^{-1}(\gamma)),$$
$$K\colon F_mC_{a,b}(Y - g^{-1}(u)) \to F_{m+\kappa}C_{a+1,b}(Y - g^{-1}(u))$$

such that

$$K\colon F_mC_{a,b}(Y - g^{-1}(\gamma)) \to F_{m+\kappa}C_{a+1,b}(Y - g^{-1}(\gamma))$$

and such that $\partial K + K\partial = M - 1$. The existence of a constant κ and the chain homotopy K follows from the estimate for the gradient flows of $g|Y_I$.

Set

$$L = K + KeK + KeKeK + \cdots$$

and

$$N = \sum_{i, j \geq 0} (Ke)^i M(eK)^j.$$

One checks that $(\partial + e)L + L(\partial + e) = N - 1$. Note that the series for L and N have at most k terms. Thus if γ misses $S_{m+k\kappa}$, then

$$N\colon F_m C_{a,b}\big(Y - g^{-1}(u)\big) \to F_{m+k\kappa} C_{a,b}\big(Y - g^{-1}(\gamma)\big)$$

and

$$L\colon F_m C_{a,b}\big(Y - g^{-1}(u)\big) \to F_{m+k\kappa} C_{a+1,b}\big(Y - g^{-1}(u)\big).$$

Now extend the cycle of integration Δ to an element $\tilde{\Delta}$ with $(\partial + e)\tilde{\Delta} = 0$, and let Δ' be the $b = 0$ component of $N(\tilde{\Delta})$. The equation for the operators L and N shows that Δ' is homologous to Δ in relative homology. But Δ' does not meet $g^{-1}(\gamma)$. Thus we have constructed a finite set $S_{m+k\kappa}$ such that if γ is a path of length $\leq \kappa$, then $f(\zeta)$ can be analytically continued along γ. The restriction that γ does not intersect itself may be removed by doing this process several times in a row, and noting that the increase in the distance from P is linear in the length of γ. Thus f has an extension with locally finite branching.

Now we have to prove that f has regular singularities. Choose one singular point, with a path to it. Apply the criterion of Lemma 4.1. We have to show that $f(\zeta)$ has polynomial growth along any ray, and that $(T^N - I)^K f = 0$.

To show polynomial growth, we must show that when the cycle of integration is moved toward the fiber over the critical point, its size grows at most polynomially. We may assume that we have analytically continued to the neighborhood of a point u near the critical point s, and that the cycle of integration is contained in some F_m (with m independent of u). Let U be a disc around s containing u. Let

$$r(z, t) = s + (1 - t)z \quad \text{for } z \in U \text{ and } t \in [0, 1 - \varepsilon].$$

Follow the same procedure as before, choosing a lifting $R(y, t)$ on each stratum, and then combining to get a homotopy of relative cycles. This will allow us to analytically continue from u to εu.

Notice that a stratum Y_I is a product Z^m, and the function g has the form

$$g(z) = \sum_{i=0}^{m} g_{j_i j_{i-1}}(z_i), \quad \text{where } g_{j_i j_{i-1}} = g_{j_i} - g_{j_{i-1}}.$$

(The indices j_i are those corresponding to the ends of strings of coordinates which are set equal to get the stratum Y_I.) In particular, we may write

$Y_I = Z^a \times Z^b$ where the function g does not depend on the factors Z^b, and has isolated singularities considered as a function on Z^a. We may ignore the factors Z^b in choosing R. Near an isolated singularity in Z^a, the situation is the same as the usual algebraic case. Thus, in a compact set such as F_m we may resolve singularities, and, by making the choice of U small enough, use Clemens' method [5] to construct a retraction from $g^{-1}(U)$ to the singular fiber $g^{-1}(s)$ covering the retraction $r(z, t)$ from U to s. The retraction $R(x, t)$ will be continuous up to and including $t = 1$, so in particular the sizes of the cycles obtained by flowing with R are bounded. Thus the functions $f(\zeta)$ are in fact bounded in any sector.

Finally we must show that $(T^N - I)^K f = 0$. We will do this by interpreting the operation T as a monodromy transformation on cohomology classes, and then will invoke the well known result (first proved by Landman) that the monodromy is quasi-unipotent. This result from algebraic geometry applies here because we may reduce to looking at isolated singularities as remarked above. Note that strictly speaking, the monodromy transformation does not make sense, because there can be singular points in fibers arbitrarily near to the fiber over s. However, these other singularities are far away in Y, and we may restrict our attention to a compact subset of the form $F_{m+\kappa}$ without further mention.

Suppose we have analytically continued to a neighborhood of a point u near the singularity, with cycle of integration Δ. The process of continuing once around and back to u yields a new cycle of integration Δ' and a chain A such that $\partial A = \Delta' - \Delta$ in relative homology. But notice that

$$(T - I)f(\zeta) = \int_{\Delta' - \Delta} \frac{\beta}{g - \zeta}$$

for ζ in a small neighborhood of u. The map $\wedge dg$: $\Omega^{k-1}_{Y/C} \to \Omega^k_Y$ is an isomorphism outside of the critical points of g. Thus we may write $b = c \wedge dg$ with c a section of $\Omega^{n-1}_{Z/D}$, possibly meromorphic at the critical points. Now the formula

$$(T - I)f(\zeta) = \int_{\partial A} c \wedge \left(\frac{dg}{g - \zeta}\right)$$

implies by the calculus of residues that

$$(T - I)f(\zeta) = \int_{A_\zeta} c$$

where $A_\zeta = A \cap g^{-1}(\zeta)$. This is a cycle for almost all ζ (by Sard's theorem) and all of these cycles are homologous in the relative pair (Y, S). The analytic continuation of $(T - I)f(\zeta)$ is obtained by moving the cycle A_ζ. In particular, if $(T^N - I)^K A_\zeta = 0$ (here, T denotes the monodromy transformation of moving the cycle A_ζ once around the singular fiber), then $(T^N - I)^K(T - I)f(\zeta) = 0$ and we will be done. Thus we must show that the monodromy is quasi-unipotent. There is a spectral sequence beginning at the homology of the strata and converging to the relative homology—it is the spectral sequence of the double complex considered above. The monodromy transformation acts on this spectral sequence. Thus it suffices to show the quasi-unipotence at the beginning of the spectral sequence, in other words on the strata. Once again, on a stratum Y_I, we may assume that g has isolated singularities. Resolve singularities, and use any one of several proofs of the monodromy theorem [11], [5] to conclude that the monodromy transformation is quasi-unipotent. This completes the proof of the theorem.

Example. Consider the case of matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

where a and b are one forms on X. Let $g(z) = \int_P^z a$. Then the monodromy matrix is

$$m_t(Q) = \begin{pmatrix} e^{tg(Q)} & e^{-tg(Q)} \int_P^Q b(z) e^{2tg(z)} \\ 0 & e^{-tg(Q)} \end{pmatrix}.$$

The asymptotic expansion for the integral $\int_P^Q b e^{2tg}$ has a nice geometric interpretation. We can map X into its Jacobian variety, and then take the universal cover of the Jacobian (which is a vector space). Let Z denote the resulting covering space of X. It is a Riemann surface of infinite genus, contained in the vector space. This one form a gives a linear function g on the universal cover of the Jacobian, so $g(z)$ is naturally defined on Z. Thus we don't have to go all the way to the universal cover of X, but now the path γ from P to Q must be specified. In order to get the asymptotic expansion for positive real t, look at the real part $\Re(g)$. Let ξ be the smallest real number such that γ is homotopic to a path contained in $\Re g \leq \xi$. Either $\xi = 0$, $\xi = \Re g(Q)$, or else $\xi = \Re g(C)$ where C is a zero of a. In the third case, γ is homotopic to a path which goes over the "mountain pass" at C (if everything is generic, there will be only one critical point at level ξ). The asymptotic expansion is the expansion usually given by the one variable version of the method of stationary phase. The exponent is $e^{t(g(C)-g(Q))}$.

Question. What is the geometric picture for the asymptotic expansion of an iterated integral?

Again, we may consider the functions g_i as linear functions on the universal cover of the Jacobian, but it does not seem to be sufficient to move the path of the iterated integral. One must move the cycle of integration to one which is no longer a simplex coming from a path.

Non unipotent systems. Finally we will state a convergence result for adding up the expansions even if the series does not terminate.

THEOREM 5. *Suppose A is a diagonal matrix of one forms with distinct entries, and suppose B is a matrix of one forms with zeros along the diagonal. Let $m_t(z)$ be the fundamental solution matrix for the system $\nabla = d - tA - B$. Denote the various terms in our expansion for $m_t(Q)$ by $m_t^k(Q)$. Then sum of the asymptotic expansions for $m_t^k(Q)$ converges formally to an asymptotic expression, and this expression is an asymptotic expansion form $m_t(Q)$, of the form*

$$m_t(Q) \sim \sum e^{\lambda_i t} \sum_{j=J}^{\infty} \sum_{k=0}^{K(j)} c_{ijk} t^{-j/N} (\log t)^k.$$

The constants c_{ijk} are polynomials in $B \in H^0(\Omega_X^1)^{n^2}$ and they do not all vanish as polynomials. Thus for generic values of B, a nonzero expansion is obtained.

The proof will be presented elsewhere. One proves in fact that the Laplace transforms of the terms m_t^k converge outside a locally finite set of singularities, and that the local expressions for the Laplace transform converge to local asymptotic expressions for the sum. The hard part is to show that relative homology classes such as Δ_k can be moved around, with bounds on the sizes which don't depend too badly on k. One feature to notice is that there is no bound, valid for all j, on the powers of logarithms which occur in the series. This seems to be unavoidable.

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