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A_∞-ALGEBRAS AND THE CYCLIC BAR COMPLEX

BY

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In memory of Kuo-Tsai Chen

This paper arose from our use of Chen's theory of iterated integrals as a tool in the study of the complex of S^1-equivariant differential forms on the free loop space LX of a manifold X (see [2]). In trying to understand the behaviour of the iterated integral map with respect to products, we were led to a natural product on the space of S^1-equivariant differential forms $\Omega(Y)[u]$ of a manifold Y with circle action, where u is a variable of degree 2. This product is not associative but is homotopy associative in a precise way; indeed there is whole infinite family of "higher homotopies". It turns out that this product structure is an example of Stasheff's A_∞-algebras, which are a generalization of differential graded algebras (DGAs).

Using the iterated integral map, it is a straightforward matter to translate this product structure on the space of S^1-equivariant differential forms on LX into formulas on the cyclic bar complex of $\Omega(X)$. Our main goal in this paper is to show that in general, the cyclic bar complex of a commutative DGA A has a natural A_∞-structure and we give explicit formulas for this structure. In particular, this shows that the cyclic homology of A has a natural associative product, but it is a much stronger result, since it holds at the chain level. Thus, it considerably strengthens the results of Hood and Jones [3].

We also show how to construct the cyclic bar complex of an A_∞-algebra, and in particular define its cyclic homology. As hinted at in [2], this construction may have applications to the problem of giving models for the $S^1 \times S^1$-equivariant cohomology of double loop spaces $LL(X)$ of a manifold and, since the space of equivariant differential forms on a smooth S^1-manifold Y is an A_∞-algebra, to the problem of finding models for the space of $S^1 \times S^1$-equivariant differential forms on LY. Although the methods that we use were developed independently, they bear a strong resemblance with those of Quillen [6].

Finally, we discuss in our general context the Chen normalization of the cyclic bar complex of an A_∞-algebra. This is a quotient of the cyclic bar complex by a complex of degenerate chains which is acyclic if A is connected,

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[1] In the preprint of [2], the maps m and $\tilde{m}$ are exchanged, for which we beg the reader's forgiveness.

and which was shown by Chen to coincide with the kernel of the iterated integral map in the case $A = \Omega(X)$. This normalization is an important tool since it allows us to remove a large contractible subcomplex of the cyclic bar complex.

The first two sections of this paper are devoted to generalities concerning coalgebras and A_∞-algebras; a good reference for further background on coalgebras is the book of McCleary [5]. The cyclic bar complex of an A_∞-algebra is constructed in Section 3, the A_∞-structure on the cyclic bar complex of a commutative DGA in Section 4, and we discuss Chen normalization in Section 5.

All our algebra will be carried out over a fixed coefficient ring K; in fact nothing will be lost by thinking of the case where K is the integers $\mathbf{Z}$. In particular, all tensor products are taken over K unless explicitly stated otherwise. We will make use of the sign-convention in the category of $\mathbf{Z}_2$-graded K-modules, which may be phrased as follows: the canonical map S_{21} from $V_1 \otimes V_2$ to $V_2 \otimes V_1$ is defined by

$$S_{21}(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1.$$

Using the map S_{21}, we can associate to any permutation $\sigma \in S_n$ an isomorphism of K-modules

$$S_{\sigma_1 \dots \sigma_n} \colon V_1 \otimes \cdots \otimes V_n \to V_{\sigma_1} \otimes \cdots \otimes V_{\sigma_n}.$$

We use the convention that $K[x]$ is a symmetric algebra over K if x has even degree, an antisymmetric algebra if x has odd degree.

Many of the ideas of this paper arose during our collaboration with Scott Petrack; our joint paper [2] is in many ways an introduction, and for the moment the sole application, of this work. The completion of this paper has been assisted by grants to the first author by the NSF and to the second author by the SERC.

1. Differential coalgebras and A_∞-algebras

Recall the definition of a differential graded coalgebra (DGC):

DEFINITION 1.1. (1) A (graded) coalgebra over K is a (graded) K-module C with a comultiplication $\Delta\colon C \to C \otimes C$ of degree 0, such that the following diagram commutes (this is called co-associativity):

$$\begin{array}{ccc} C & \xrightarrow[\Delta]{} & C \otimes C \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle \Delta\otimes 1} \\ C \otimes C & \xrightarrow{1\otimes\Delta} & C \otimes C \otimes C \end{array}$$

(2) A coderivation on a coalgebra is a map $L: C \to C$ satisfying co-Leibniz's rule, that is, the diagram

$$\begin{tikzcd} C \arrow[r, "L"] \arrow[d, "\Delta"'] & C \arrow[d, "\Delta"'] \\ C \otimes C \arrow[r, "L\otimes 1 + 1 \otimes L"] & C \otimes C \end{tikzcd}$$

commutes.

(3) A differential graded coalgebra is a graded coalgebra with coderivation $b: C \to C$ of degree -1 such that $b^2 = 0$.

The basic example of a graded coalgebra is the cotensor coalgebra of a graded K-module:

$$\mathbf{T}(V) = \sum_{n=0}^{\infty} V^{\otimes n}.$$

The comultiplication is defined by

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

In fact, this is the universal example of a graded coalgebra; for every graded coalgebra C and linear map $C \to V$, there is a unique extension to a coalgebra map $C \to \mathbf{T}(V)$ such that the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathbf{T}(V) \arrow[d] \\ V \arrow[r, equal] & V \end{tikzcd}$$

commutes. We would like to classify all of the differentials that may be imposed on this coalgebra.

There is a simple characterization of coderivations on a contensor coalgebra, which is the dualization of a corresponding result for derivations on tensor algebras.

PROPOSITION 1.2. *Composition of a coderivation L: $\mathbf{T}(V) \to \mathbf{T}(V)$ with the projection map $\mathbf{T}(V) \to V$ induces an isomorphism*

$$\operatorname{Coder}(\mathbf{T}(V)) \to \operatorname{Hom}(\mathbf{T}(V), V).$$

The inverse of this map is given by the formula

$$L \mapsto \sum_{i=0}^{n} \sum_{j=0}^{n-i} 1^{\otimes j} \otimes L_i \otimes 1^{\otimes n-i-j},$$

where L_i denotes the image of L in $\mathrm{Hom}(V^{\otimes i}, V)$.

If b is a coderivation of degree -1 on $\mathbf{T}(V)$ with components $b_n\colon V^{\otimes n} \to V$, then its square is a coderivation of degree 2 with components

$$(b^2)_n = \sum_{i+j=n+1} \sum_{k=0}^{n-j} b_i \cdot \left(1^{\otimes k} \otimes b_j \otimes 1^{\otimes n-k-j}\right).$$

Obviously, the coderivation b will be a differential if and only if all of the maps $(b^2)_n$ vanish. In this way, we obtain a characterization of all differentials compatible with the coalgebra structure on $\mathbf{T}(V)$.

Let us write out the first few of these relations:

$$b_1 \cdot b_0 = 0,$$
$$b_1 \cdot b_1 + b_2 \cdot (b_0 \otimes 1 + 1 \otimes b_0) = 0,$$
$$b_1 \cdot b_2 + b_2 \cdot (b_1 \otimes 1 + 1 \otimes b_1) + b_3 \cdot (b_0 \otimes 1 \otimes 1 + 1 \otimes b_0 \otimes 1 + 1 \otimes 1 \otimes b_0) = 0.$$

Before attempting to unravel these formulas, we need one more definition. If A is a graded K-module, let sA be its suspension, that is, the graded K-module

$$(sA)_i = A_{i-1}.$$

We would like to rewrite the formulas $b^2 = 0$ on the cotensor algebra of a suspended graded K-module $\mathbf{T}(sA)$; this will introduce extra signs into the formulas. We will denote the element $sa_1 \otimes \cdots \otimes sa_n$ of $\mathbf{T}(sA)$ by Eilenberg and MacLane's notation $[a_1| \dots |a_n]$.

As a warm-up exercise, we have the following lemma, whose proof we leave to the reader.

LEMMA 1.3. *If $b_k\colon (sA)^{\otimes k} \to sA$ is a multilinear map of degree -1, let us define $m_k\colon A^{\otimes k} \to A$ by $m_k = s^{-1} \cdot b_k \cdot s^{\otimes k}$. Then the following formula is satisfied*:

$$b_k[a_1| \dots |a_k] = (-1)^{(k-1)|a_1| + (k-2)|a_2| + \cdots + 2|a_{k-2}| + |a_{k-1}| + k(k-1)/2} m_k(a_1, \dots, a_k).$$

We will denote by $\tilde{m}_k$ the multilinear map obtained from m_k by multiplying by the above sign, so that

$$s\tilde{m}_k(a_1, \ldots, a_k) = b_k[a_1| \ldots |a_k].$$

PROPOSITION 1.4. *If $a_i \in A$, $1 \leq i \leq n$, let $\varepsilon_i = |a_1| + \cdots + |a_i| - i$. Then the boundary b on $\mathbf{T}(sA)$ is given in terms of the maps m_k (or equivalently, $\tilde{m}_k$) by the formula*

$$\begin{aligned} & b[a_1| \ldots |a_n] \\ & \quad = \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} (-1)^{\varepsilon_{i-1}} [a_1| \ldots |a_{i-1}|\tilde{m}_k(a_i, \ldots, a_{i+k-1})|a_{i+k}| \ldots |a_n]. \end{aligned}$$

Proof. By definition, $b[a_1| \ldots |a_n]$ is

$$\begin{aligned} & \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} \left(1^{\otimes(i-1)} \otimes b_k \otimes 1^{\otimes(n-k-i+1)}\right)[a_1| \ldots |a_n] \\ & \quad = \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} (-1)^{\varepsilon_{i-1}} \left[a_1| \ldots |a_{i-1}|s^{-1}b_k[a_i| \ldots |a_{i+k-1}]|a_{i+k}| \ldots |a_n\right], \end{aligned}$$

where we use the fact that b_k has degree -1 and hence is odd, and that $sa_1 \otimes \cdots \otimes sa_{i-1}$ has degree ε_{i-1}. Inserting the definition of $\tilde{m}_k$, we see that

$$\begin{aligned} & b[a_1| \ldots |a_n] \\ & \quad = \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} (-1)^{\varepsilon_{i-1}} [a_1| \ldots |a_{i-1}|\tilde{m}_k(a_i, \ldots, a_{i+k-1})|a_{i+k}| \ldots |a_n], \end{aligned}$$

which is precisely what we wished to prove. ∎

The above formulas, which determine when $b^2 = 0$, may be thought of as generalizations of a DGA structure on A. We will use the notation u for the element of A defined by m_0. Let us write out the first few of these formulas in full.

(1) The first formula says that $m_1(u) = 0$.

(2) The second formula,

$$m_1(m_1(a_1)) = -m_2(u, a_1) + m_2(a_1, u),$$

says that m_1 is a differential up to a correction involving the operator ad u.

(3) The third says that m_1 is a derivation with respect to the product, again up to certain correction terms involving u:

$$m_1(m_2(a_1, a_2)) = m_2(m_1(a_1), a_2) + (-1)^{|a_1|} m_2(a_1, m_1(a_2)) \\ - m_3(u, a_1, a_2) + m_3(a_1, u, a_2) - m_3(a_1, a_2, u).$$

(4) The fourth says that the product on A, while not necessarily associative, is homotopy associative in a precise way, again up to terms involving u:

$$\begin{aligned} m_2(m_2(a_1, a_2), a_3) &- m_2(a_1, m_2(a_2, a_3)) \\ = m_1&(m_3(a_1, a_2, a_3)) \\ &+ m_3(m_1(a_1), a_2, a_3) + (-1)^{|a_1|} m_3(a_1, m_1(a_2), a_3) \\ &+ (-1)^{|a_1|+|a_2|} m_3(a_1, a_2, m_1(a_3)) \\ &- m_4(u, a_1, a_2, a_3) + m_4(a_1, u, a_2, a_3) \\ &- m_4(a_1, a_2, u, a_3) + m_4(a_1, a_2, a_3, u). \end{aligned}$$

The outcome of this discussion is that we are led to think of conditions like $d^2 = 0$ or Leibniz's rule as akin to associativity. Stasheff has defined a natural generalization of a DGA, which he calls an **A_∞-algebra**: this is a graded K-module A along with a differential b on the coalgebra $\mathbf{T}(sA)$. (In fact Stasheff assumes that $b_0 = 0$ but it is just as convenient to allow non-zero b_0, which gives a preferred element $u \in A_{-2}$.) We will call the differential graded coalgebra $\mathbf{B}(A) = \mathbf{T}(sA)$ of an A_∞-algebra the **bar complex** of A.

Before continuing, let us give some examples of A_∞-algebras.

(1) If A is a graded algebra with an element $D \in A_{-1}$, then we may set $m_0 = D^2$, $m_1 = \operatorname{ad} D$ and m_2 equal to the product on A, with all higher m_n equal to zero. In this case, the formulas express the fact that $\operatorname{ad} D$ is almost, but not quite, a differential.

(2) If A is a graded complex, we may simply take $m_n = 0$ except if $n = 1$, where we take the differential.

(3) If A is a DGA, then it satisfies the above formulas simply by letting m_n equal zero unless $n = 1$, where we take the differential d: $A \to A$, or $n = 2$, where we take the product $A \otimes A \to A$.

(4) The example of Stasheff which motivated the whole theory is the graded abelian group of singular chains on the based loop space of a topological space.

(5) Consider the complex $\Omega(M)[u]$ of differential forms on a manifold with smooth circle-action, with a variable u of degree 2 adjoined. Define the multilinear maps

$$P_k(\omega_1, \ldots, \omega_k) : \Omega(M)^{\otimes k} \to \Omega(M)$$

by the formula

$$P_k(\omega_1, \dots, \omega_k) = \int_{\Delta_k} \iota\omega_1(t_1) \wedge \cdots \wedge \iota\omega_k(t_k)\, dt_1 \dots dt_k,$$

where Δ_k is the k-simplex $\{(t_1, \dots, t_k) \in \mathbf{R}^k | 0 \le t_1 \le \cdots \le t_k \le 1\}$, and ι is the vector field which generates the circle action. Then the following maps define an A_∞-structure on $\Omega(M)[u]$:

$$m_k(\omega_1, \dots, \omega_k) = \begin{cases} da_1 + uP_1(a_1), & k = 1, \\ a_1 \wedge a_2 + uP_2(a_1, a_2), & k = 2, \\ uP_k(a_1, \dots, a_k), & \text{otherwise}. \end{cases}$$

Here, we have reversed the grading of $\Omega(M)[u]$, because the differential of $\Omega(M)$ raises degree. This is the example which motivated us to consider the theory of A_∞-algebras (see [2]).

We say that $e \in A_0$ is an identity in the A_∞-algebra A if for a, $a_i \in A$,

$$\begin{aligned} m_2(a, e) &= m_2(e, a) = a, \\ m_k(a_1, \dots, a_i, e, a_{i+2}, a_k) &= 0 \quad \text{if } k \ne 2. \end{aligned} \tag{1.5}$$

Just as for algebras, an identity, if it exists, must be unique; if e and f are both identities, then $m_2(e, f) = e = f$. If A is an A_∞-algebra, then its augmention A^+ is the A_∞-algebra with identity whose underlying space is $A \oplus K$, and where, denoting the basis element of K by e, we extend the maps m_k using (1.5).

The collection of all A_∞-algebras forms a category in a natural way: we define a homomorphism $A_1 \to A_2$ between two A_∞-algebras as a map of DGCs from $\mathbf{B}(A_1)$ to $\mathbf{B}(A_2)$. The only disadvantage of considering such a large class of homomorphisms is that it is difficult to write out explicitly what it means in terms of the generalized products $\tilde{m}_n$ on A_1 and A_2, since there are so many different associativity laws that have to be verified. However, it is at least possible to get some idea of what a homomorphism looks like by means of the following lemma, which characterizes the coalgebra homomorphisms from a graded coalgebra C to $\mathbf{T}(V)$, and which reflects the universal property of the cotensor algebra $\mathbf{T}(V)$ among coalgebras with projection $\mathbf{T}(V) \to V$.

LEMMA 1.6. *Composition of a homomorphism f: $C \to \mathbf{T}(V)$ with the projection map $\mathbf{T}(V) \to V$ induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{Coalg}}(C, \mathbf{T}(V)) \to \mathrm{Hom}(C, V).$$

The inverse of this map is given by associating to a map $f \in \mathrm{Hom}(C, V)$ the map with components

$$C \xrightarrow{\Delta_n} C^{\otimes n} \xrightarrow{f^{\otimes n}} V^{\otimes n};$$

here, $\Delta_n\colon C \to C^{\otimes n}$ is the $(n-1)$-th iterate of the comultiplication map Δ.

Thus, we see that a homomorphism $f\colon A_1 \to A_2$ between two A_∞-algebras is determined by a series of maps

$$f_n\colon A_1^{\otimes n} \to A_2$$

of degree $1-n$, and that $f_0 = 0$, since otherwise $f[\ ] = \Sigma_{n=0}^{\infty}[f_0|\cdots|f_0]$, which being an infinite sum does not lie in the tensor coalgebra $\mathbf{B}(A_2)$. In order that $f \in \mathrm{Hom}(\mathbf{B}(A_1), \mathbf{B}(A_2))$ respects the differentials b on $\mathbf{B}(A_1)$ and $\mathbf{B}(A_2)$, the maps f_n must satisfy a series of identities, the first few of which have the form

(1.7a) $$f_1(u) = u,$$

(1.7b) $$f_1(m_1(a_1)) = m_1(f_1(a_1)),$$

(1.7c) $$\begin{aligned} f_1(m_2(a_1, a_2)) - m_2(f_1(a_1), f_1(a_2)) \\ = m_1(f_2(a_1, a_2)) - f_2(m_1(a_1), a_2) \\ - (-1)^{|a_1|} f_2(a_1, m_1(a_2)). \end{aligned}$$

There is a more restrictive type of homomorphism between two A_∞-algebras: **a strict homomorphism** $A_1 \to A_2$ is a linear map from A_1 to A_2 such that the induced map $\mathbf{B}(A_1) \to \mathbf{B}(A_2)$ is a map of DGCs. Thus, a homomorphism f of A_∞-algebras is a strict homomorphism if all of the maps f_n vanish for n not equal to 1.

2. Homology of A_∞-algebras

If A is an A_∞-algebra, we can associate to it its homology algebra $H(A)$, which is an ordinary graded algebra, if we impose the following hypotheses on A:

$$m_k(a_1, \ldots, u, \ldots, a_k) = 0 \quad \text{for } k > 2,$$
$$m_2(u, a) = m_2(a, u) \quad \text{for all } a \in A.$$

Such an A_∞-algebra will be called **standard**. This assumption simplifies the formulas which define an A_∞-algebra; in particular, m_1 becomes a derivation with respect to the non-associative product m_2, and m_2 is homotopy associative:

$$\begin{aligned} m_2(m_2(a_1, a_2), a_3) &- m_2(a_1, m_2(a_2, a_3)) \\ &= m_1(m_3(a_1, a_2, a_3)) - m_3(m_1(a_1), a_2, a_3) \\ &\quad -(-1)^{|a_1|} m_3(a_1, m_1(a_2), a_3) - (-1)^{|a_1|+|a_2|} m_3(a_1, a_2, m_1(a_3)). \end{aligned}$$

The homology of a standard A_∞-algebra is defined as the homology of the differential m_1 on A:

$$H(A) = H(A, m_1).$$

The non-associative product on A descends to an associative product on $H(A)$ and $H(A)$ becomes a graded algebra.

PROPOSITION 2.1. *The operation of taking homology $A \mapsto H(A)$ is a functor from the category of standard A_∞-algebras to the category of graded algebras.*

Proof. Let $f\colon A_1 \to A_2$ be a homomorphism of standard A_∞-algebras, with components

$$f_n\colon (sA_1)^{\otimes n} \to A_2$$

of degree $1 - n$. The fact that f is a homomorphism implies, in particular, that if $a_1, a_2 \in A_1$, $m_1(f_1(a_1)) = f_1(m_1(a_1))$ and

$$\begin{aligned} m_2(f_1(a_1), f_1(a_2)) &- f_1(m_2(a_1, a_2)) \\ &= m_1(f_2(a_1, a_2)) - f_2(m_1(a_1), a_2) - (-1)^{|a_1|} f_2(a_1, m_1(a_2)), \end{aligned}$$

from which the result follows. ■

There is an important generalization of $H(A)$, whose definition depends upon the following result.

PROPOSITION 2.2. *Let A be a standard A_∞-algebra and let u be the element $m_0 \in A_{-2}$. Then A is a module over the polynomial ring $K[u]$, where the action of u is given by $a \mapsto m_2(u, a)$.*

Proof. To check that the algebra $K[u]$ acts on A, we must verify that if $a \in A$, then

$$m_2(m_2(u, u), a) = m_2(u, m_2(u, a)).$$

But the hypothesis that A is a standard A_∞-algebra gives the following formula:

$$\begin{aligned} &m_2(m_2(u,u),a) - m_2(u, m_2(u,a)) \\ &\quad = -m_1(m_3(u,u,a)) + m_3(u,u,m_1(a)) = 0. \quad \blacksquare \end{aligned}$$

Let W be a graded module over $K[u]$. Forming the graded $K[u]$-module $A \otimes_{K[u]} W$ with boundary map b, we define

$$H(A; W) = H(A \otimes_{K[u]} W).$$

The following result follows from standard homological algebra.

PROPOSITION 2.3. *Suppose that K is a field and let*

$$0 \to W_1 \xrightarrow{i} W_2 \xrightarrow{j} W_3 \to 0$$

be a short exact sequence of $K[u]$-modules. Then there is a long exact sequence of homology groups

$$\cdots \to H_n(A; W_1) \xrightarrow{i} H_n(A; W_2) \xrightarrow{j} H_n(A; W_3) \xrightarrow{\partial} H_{n+1}(A; W_1) \to \cdots \quad \blacksquare$$

Let Λ be the graded algebra $K[\varepsilon]$ generated by a single supercommuting variable ε of degree 1. A differential graded Λ-module (also called a mixed complex by C. Kassel) is just a graded K-module V with two supercommuting differentials b: $V_* \to V_{*-1}$ and B: $V_* \to V_{*+1}$. The homology of a dg-Λ-module V with coefficients in the $K[u]$-module W is defined to be

$$H(V; W) = H(V[\![u]\!] \otimes_{K[u]} W, b + uB).$$

This may be reduced to the homology of an A_∞-algebra whose underlying space is the graded K-module $V[\![u]\!] \oplus K[u]$; we set $m_0 = u$, $m_1 = b + uB$ on $V[\![u]\!]$ and zero on $K[u]$, m_2 is given by the action of $K[u]$ on $V[\![u]\!]$, and all other m_n are zero. It follows easily that the homology of this A_∞-algebra with coefficients in W is just

$$H(V \otimes W, b + uB) \oplus K[u] = H(V; W) \oplus K[u].$$

The next result expresses a basic invariance property of the homology of a dg-Λ-module.

PROPOSITION 2.4. *Let f: $(V_1, b_1, B_1) \to (V_2, b_2, B_2)$ be a map of dg-Λ-modules such that f induces an isomorphism $H(V_1, b_1) \to H(V_2, b_2)$. Then for any* *coefficients W of finite projective dimension over $K[u]$, $f\colon H(V_1; W) \to H(V_2; W)$ is an isomorphism.*

Proof. The main step in the proof is contained in the following lemma.

LEMMA 2.5. *If (V, b, B) is a dg-Λ-module such that $H(V, b) = 0$, then $H(V[\![u]\!], b + uB) = 0$.*

Proof. If $a(u) = \sum_{k=0}^{\infty} a_k u^k$ is a cycle in $V[\![u]\!]$ for the differential $b + uB$, so that

$$(b + uB) \sum_{k=0}^{\infty} a_k u^k = 0,$$

then it must follow that $ba_0 = 0$. Since $H(V, b) = 0$, there is an element $c_0 \in V$ such that $bc_0 = a_0$. It follows that

$$(b + uB)\left(\sum_{k=1}^{\infty} a_{k+1} u^k + (a_1 - Bc_0) \right) = 0.$$

By induction, we obtain a sequence of elements $c_k \in V$ such that

$$(b + uB) \sum_{k=0}^{n} c_k u^k = \sum_{k=0}^{n} a_k u^k + u^{n+1} Bc_n.$$

Taking the limit as $n \to \infty$, we see that $a(u)$ is exact. ■

We can now complete the proof of 2.4. By a standard use of the mapping cone it is sufficient to show that if (V, b, B) satisfies $H(V, b) = 0$, then $H(V; W) = 0$ for all $K[u]$-modules W of finite homological dimension. This follows if W is a free $K[u]$-module directly from 2.5. The general case follows using induction on the homological dimension of W over $K[u]$. ■

3. The Hochschild chain complex of an A_∞-algebra

If A is a DGA, the cyclic bar complex of A is the graded K-module

$$\mathbf{C}(A, A^+) = A^+ \otimes \mathbf{B}(A);$$

it is a dg-Λ-module with respect to the Hochschild differential b and Connes's coboundary operator B. If we denote the element $a_0 \otimes [a_1 | \dots | a_k]$ of

$\mathbf{C}(A, A^+)$ by $(a_0, a_1, \ldots, a_k)$, then the operator b is given by the formula

$$\begin{aligned} b(a_0, \ldots, a_k) = &- \sum_{i=0}^{k} (-1)^{\varepsilon_i}(a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_k) \\ &+ (-1)^{(|a_k|-1)\varepsilon_{k-1}}(a_k a_0, a_1, \ldots, a_{k-1}) \\ &- \sum_{i=0}^{k} (-1)^{\varepsilon_{i-1}}(a_0, \ldots, a_{i-1}, da_i, a_{i+1}, \ldots, a_k), \end{aligned}$$

where $\varepsilon_i = |a_0| + \cdots + |a_i| - i$. The formula for B is

$$B(a_0, \ldots, a_k) = \sum_{i=0}^{k} (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k - \varepsilon_{i-1})}(e, a_i, \ldots, a_k, a_0, \ldots, a_{i-1}),$$

where e is the identity adjoined to A^+, and it is understood that $B(e, a_1, \ldots, a_k) = 0$. These sign conventions take into account that the elements a_i for $i > 0$ occur with an implicit suspension which reduces their degree to $|a_i| - 1$. It is a standard calculation that the operators b and B are well defined, and that $(\mathbf{C}(A, A^+), b, B)$ is a dg-Λ-module. The homology of $(\mathbf{C}(A, A^+), b)$ is the Hochschild homology of A with coefficients in the bimodule A^+.

There is a map from the dg-Λ-module $(\mathbf{C}(A, A^+), b, B)$ to the dg-Λ-module $(K, 0, 0)$, given by sending (ze) to z for $z \in K$, and all other chains to zero. If W is a $K[u]$-module, this induces a map of cohomology groups

$$H(\mathbf{C}(A, A^+); W) \to W,$$

the kernel of which is called the cyclic homology of A with coefficients in W, and denoted $\mathrm{HC}(A; W)$. Let us list some examples with respect to different coefficients W.

(1) $W = K[u]$ gives the negative cyclic homology $\mathrm{HC}^-(A)$ of Goodwillie and Jones, which is the most fundamental theory;

(2) $W = K[u, u^{-1}]$ gives periodic cyclic homology $\mathrm{HP}(A)$, so called because

$$\mathrm{HP}_*(A) \cong \mathrm{HP}_{*+2}(A),$$

the isomorphism being implemented by multiplication by u;

(3) $W = K[u, u^{-1}]/uK[u]$ gives the cyclic homology theory $\mathrm{HC}(A)$ studied by Feigin and Tsygan, and Loday and Quillen;

(4) $W = K$ gives the Hochschild homology $\mathrm{HH}(A)$; if A has an identity this is just the usual Hochschild homology of A with coefficients A considered as a bimodule over itself $H(A, A)$.

If K is a field and we apply 2.3 to the short exact sequence

$$0 \to K[u, u^{-1}]/K[u] \to K[u, u^{-1}]/uK[u] \to K \to 0,$$

we obtain the fundamental exact sequence relating cyclic homology and Hochschild homology,

$$\cdots \to \mathrm{HC}_{n+2}(A) \to \mathrm{HC}_n(A) \to \mathrm{HH}_n(A) \to \mathrm{HC}_{n+1}(A) \to \cdots,$$

while applying it to the short exact sequence

$$0 \to uK[u] \to K[u, u^{-1}] \to K[u, u^{-1}]/uK[u] \to 0,$$

gives the sequence

$$\cdots \to \mathrm{HC}^-_{n+2}(A) \to \mathrm{HP}_n(A) \to \mathrm{HC}_n(A) \to \mathrm{HC}^-_{n+1}(A) \to \cdots$$

In this section, we will develop the analogue of the above homology theories when A is an A_∞-algebra.

If A is an A_∞-algebra, there is a natural notion of left and right modules, and of bimodules, over A. In order to define these, we will need the definition of left, right and bi-comodules over a DGC.

DEFINITION 3.1. A left comodule for a DGC C is a complex P and a linear map

$$\Delta^{\mathrm{L}}: P \to C \otimes P$$

such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta^{\mathrm{L}}} & C \otimes P \\ {\scriptstyle \Delta^{\mathrm{L}}}\downarrow & & \downarrow{\scriptstyle \Delta \otimes 1} \\ C \otimes P & \xrightarrow{1 \otimes \Delta^{\mathrm{L}}} & C \otimes C \otimes P \end{array}$$

commutes and such that the coaction respects the differentials

$$\begin{array}{ccc} P & \xrightarrow{\Delta^{\mathrm{L}}} & C \otimes P \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b \otimes 1 + 1 \otimes b} \\ P & \xrightarrow{\Delta^{\mathrm{L}}} & C \otimes P. \end{array}$$

A right comodule with coaction $\Delta^{\mathrm{R}}: P \to P \otimes C$ is defined in a similar way. Finally, a bi-comodule is a graded K-module P with both left and right coactions Δ^{L} and Δ^{R} such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\Delta^{\mathrm{L}}\otimes 1} & C \otimes P \\ {\scriptstyle \Delta^{\mathrm{R}}}\big\downarrow & & {\scriptstyle 1\otimes\Delta^{\mathrm{R}}}\big\downarrow \\ P \otimes C & \xrightarrow{\Delta^{\mathrm{L}}\otimes 1} & C \otimes P \otimes C. \end{array}$$

If M is a graded K-module and C is a graded coalgebra, there is a canonical left-coaction of C on the graded K-module $C \otimes M$, a canonical right-coaction of C on $M \otimes C$, and a canonical bi-coaction of C on $C \otimes M \otimes C$. For example, the left-coaction of C on $C \otimes M$ is defined by the equation

$$\Delta^{\mathrm{L}}(c \otimes m) = (\Delta c) \otimes m \in C \otimes C \otimes M.$$

If C is in addition a DGC, it is natural to ask what differentials may be imposed on the comodules $C \otimes M$, $M \otimes C$ and $C \otimes M \otimes C$ which are compatible with the differential on C. Motivated by the classical case in which C is the bar coalgebra of a DGA A, we maker the following definition for an arbitrary A_∞-algebra A.

DEFINITION 3.2. (1) If M is a graded K-module, a left-module structure for M over an A_∞-algebra A is a differential b on the left-comodule $\mathbf{B}(A) \otimes M$ over the coalgebra $\mathbf{B}(A)$ compatible with the differential on $\mathbf{B}(A)$. The definition of a right-module is similar, except that $\mathbf{B}(A) \otimes M$ is replaced by the right-comodule $M \otimes \mathbf{B}(A)$.

(2) If M is a graded K-module, a bimodule structure for M over an A_∞-algebra A is a differential b on the bi-comodule $\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)$ over the coalgebra $\mathbf{B}(A)$ compatible with the differential on $\mathbf{B}(A)$.

Observe that if M is a left-module and N is a right-module for an A_∞-algebra, then $N \otimes M$ is a bi-module; this follows from the fact that the tensor product

$$(\mathbf{B}(A) \otimes N) \otimes (M \otimes \mathbf{B}(A)),$$

which is a bi-comodule of $\mathbf{B}(A)$, is isomorphic to $\mathbf{B}(A) \otimes (N \otimes M) \otimes \mathbf{B}(A)$ as a bi-comodule.

Suppose M is a left-module for an A_∞ algebra A. It is easy to see that the differential on $\mathbf{B}(A) \otimes M$ is defined uniquely by a linear map b_M from $\mathbf{B}(A) \otimes M$ to M, which decomposes into a series of maps

$$m_k(a_1, \ldots, a_{k-1}; x)\colon A^{\otimes(k-1)} \otimes M \to M, \quad k \geq 0,$$

satisfying a series of equations analogous to those defining an A_∞-structure.

For example, we see that $m_1\colon M \to M$ is a differential which satisfies Leibniz's rule in the form

$$m_1(m_1(x)) = m_2(u; x),$$

where u is the element of degree -2 in A defined by m_0. As another example, the left action of A on M in the usual sense is the map $m_2\colon A \otimes M \to M$, of total degree 0, which satisfies the Leibniz relation

$$\begin{aligned} m_1(m_2(a; x)) = m_2(m_1(a); x) &+ (-1)^{|a|} m_2(a; m_1(x)) \\ &+ m_3(u, a; x) + m_3(a, u; x), \end{aligned}$$

and the associativity relation

$$\begin{aligned} &m_2(a_1; m_2(a_2; x)) - m_2(m_2(a_1, a_2); x) \\ &\quad = m_1(m_3(a_1, a_2; x)) + m_3(m_1(a_1), a_2; x) \\ &\qquad + (-1)^{|a_1|} m_3(a_1, m_1(a_2); x) + (-1)^{|a_1|+|a_2|} m_3(a_1, a_2; m_1(x)) \\ &\qquad - m_4(u, a_1, a_2; x) + m_4(a_1, u, a_2; x) - m_4(a_1, a_2, u; x). \end{aligned}$$

Similarly, a bimodule structure on a graded K-module M is determined by a series of maps

$$m_{ij}(a_1, \ldots, a_{i-1}; x; \bar{a}_1, \ldots, \bar{a}_{j-1})\colon A^{\otimes(i-1)} \otimes M \otimes A^{\otimes(j-1)} \to M,$$

satisfying certain equations.

One of the most important examples of a left-module for A is the graded K-module A itself, with b_A defined by the formula

$$\begin{aligned} m_{ij}(a_1, \ldots, a_{i-1}; a; \bar{a}_1, \ldots, \bar{a}_{j-1}) \\ = m_{i+j-1}(a_1, \ldots, a_{i-1}, a, \bar{a}_1, \ldots, \bar{a}_{j-1}). \end{aligned} \tag{3.3}$$

It is an easy task to check that this is compatible with the differential b on $\mathbf{B}(A)$. As a generalization of this construction, we have the following result.

PROPOSITION 3.4. *If $f\colon A_1 \to A_2$ is an A_∞-homomorphism, then A_2 becomes a bimodule over A_1.*

Proof. This is true simply because $\mathbf{B}(A_2)$ is made into a bi-comodule of $\mathbf{B}(A_1)$ by the homomorphism of coalgebras $f\colon \mathbf{B}(A_1) \to \mathbf{B}(A_2)$. ∎

An example, which we will need later in the construction of the cyclic bar complex, is the bimodule associated to the augmentation A^+ of the A_∞-algebra A; here, we use the fact that the inclusion $A \hookrightarrow A^+$ is an A_∞-homomorphism, and is even strict.

If M is a left-module over the A_∞-algebra A and N is a right-module, then we can form the two-sided bar-complex $\mathbf{B}(M, A, N)$, which is a graded complex, by taking the cotensor product of $\mathbf{B}(A)$-comodules:

$$\mathbf{B}(M, A, N) = (M \otimes \mathbf{B}(A)) \otimes^{\mathbf{B}(A)} (\mathbf{B}(A) \otimes N).$$

The differential on $\mathbf{B}(M, A, N)$ is determined by the differentials on $M \otimes \mathbf{B}(A)$ and $\mathbf{B}(A) \otimes N$ which define the left-module and right-module structures. As a K-module, the bar complex $\mathbf{B}(M, A, N)$ is isomorphic to $M \otimes \mathbf{B}(A) \otimes N$. Denoting a typical element as a sum of terms of the form $x \otimes [a_1| \dots |a_k] \otimes y$, the differential may be written as follows:

$$\begin{aligned}
&b(x \otimes [a_1| \dots |a_k] \otimes y) \\
&\quad = \sum_{i=0}^{k} \pm m_{i+1}(x; a_1, \dots, a_i) \otimes [a_{i+1}| \dots |a_k] \otimes y \\
&\qquad + \sum_{i=0}^{k} \sum_{j=0}^{k-i} \pm x \otimes \left[a_1| \dots |a_j| m_i(a_{j+1}, \dots, a_{j+i}) | a_{j+i-1}| \dots |a_k\right] \otimes y \\
&\qquad + \sum_{i=0}^{k} \pm x \otimes [a_1| \dots |a_{k-i}] \otimes m_{i+1}(a_{k-i+1}, \dots, a_k; y).
\end{aligned}$$

Here, the signs may be determined precisely if so desired, by means of the standard sign conditions. We will now generalize this construction by defining the bar complex for a bimodule over an A_∞-algebra, in such away that it will reduce to $\mathbf{B}(M, A, N)$ when applied to the bimodule $N \otimes M$.

If P is a differential graded bi-comodule over the DGC C, then the two coactions Δ^{L} and Δ^{R} of C on P give two maps Δ^{R} and $S_{21} \circ \Delta^{\mathrm{L}}$ from P to $P \otimes C$, which respect the differentials on these two spaces; here, S_{21} is the natural map S_{21}: $C \otimes P \to P \otimes C$ which implements the sign convention for $\mathbf{Z}_2$-graded objects. Thus, if we denote by $\Phi(P)$ the graded K-module obtained from P by taking the kernel of $\Delta^{\mathrm{R}} - S_{21} \circ \Delta^{\mathrm{L}}$: $P \to P \otimes C$, we see that $\Phi(P)$ inherits a differential from those of C and P. It is not hard to check that Φ is a functor from the category of C-bimodules to the category of complexes.

DEFINITION 3.5. The Hochschild complex of an A_∞-algebra A with coefficients in the bimodule M, denoted by $\mathbf{C}(A, M)$, is the complex $\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A))$. The homology of this complex, called the Hochschild homology of A with coefficients in M, is denoted by $H(A, M)$.

We would now like to identify the complex $\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A))$ more explicitly. In fact, as a K-module, it may be identified with $M \otimes \mathbf{B}(A)$, by means of the projection from $\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)$ to $M \otimes \mathbf{B}(A)$ given by the counit η: $\mathbf{B}(A) \to K$:

$$\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)) \hookrightarrow \mathbf{B}(A) \otimes M \otimes \mathbf{B}(A) \xrightarrow{\eta \otimes 1 \otimes 1} M \otimes \mathbf{B}(A).$$

To show that this is an isomorphism of K-modules, we have only to construct an inverse; we will use the map

$$S_{312}\colon M \otimes \mathbf{B}(A) \otimes \mathbf{B}(A) \to \mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)$$

which swaps the right-hand graded K-module $\mathbf{B}(A)$ past the left-hand one $M \otimes \mathbf{B}(A)$.

PROPOSITION 3.6. *The map $S_{312} \circ (1 \otimes \Delta)$ obtained by composing the arrows*

$$M \otimes \mathbf{B}(A) \xrightarrow{1 \otimes \Delta} M \otimes \mathbf{B}(A) \otimes \mathbf{B}(A) \xrightarrow{S_{312}} \mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)$$

identifies $M \otimes \mathbf{B}(A)$ with $\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A))$.

Proof. We will use the notation $(x, a_1, \dots, a_k)$ for the element obtained by applying the above map $S_{312} \circ (1 \otimes \Delta)$ to $x \otimes [a_1|\dots|a_k] \in M \otimes \mathbf{B}(A)$; in other words,

$$(x, a_1, \dots, a_k) = \sum_{i=0}^{k} (-1)^{(\varepsilon_i + |x|)(\varepsilon_k - \varepsilon_i)} [a_{i+1}|\dots|a_k] \otimes x \otimes [a_1|\dots|a_i].$$

From this formula, it is clear that the composition

$$M \otimes \mathbf{B}(A) \xrightarrow{S_{312} \circ (1 \otimes \Delta)} \mathbf{B}(A) \otimes M \otimes \mathbf{B}(A) \xrightarrow{\eta \otimes 1 \otimes 1} M \otimes \mathbf{B}(A)$$

is the identity. Furthermore, from the formula for $(x, a_1, \dots, a_k)$, it is clear that it lies in $\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A))$; more abstractly, this follows from the co-associativity of the comultiplication Δ. Finally, we must show that the composition

$$\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)) \xrightarrow{\eta \otimes 1 \otimes 1} M \otimes \mathbf{B}(A) \xrightarrow{S_{312} \circ (1 \otimes \Delta)} \Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A))$$

is the identity; this is true because η is a counit. ■

It follows easily from the formula for $(x, a_1, \dots, a_k)$ that the differential on $\mathbf{C}(A, M)$ is given by the formula

$$\begin{aligned} b(x, a_1, \dots, a_n) &= \sum_{i+j \le n} \pm \big(m_{ij}(a_{n-j+2}, \dots, a_n; x; a_1, \dots, a_{i-1}), a_i, \dots, a_{n-j+1}\big) \\ &\quad + \sum_{i=0}^{n} \sum_{k=0}^{n-k} \pm \big(x, a_1, \dots, a_k, m_i(a_{k+1}, \dots, a_{k+i}), a_{k+i+1}, \dots, a_n\big). \end{aligned}$$

Connes, in his theory of cyclic homology, has underlined the importance of the Hochschild complex with coefficients in the augmented comodule A^+, which we defined at the end of Section 1. The reason for the importance of this case is that there is a natural boundary B on $\mathbf{C}(A, A^+)$, which raises degree by one and supercommutes with b, so that $\mathbf{C}(A, A^+)$ is actually a dg-Λ-module. The operator B is defined by the formula

$$\begin{aligned} B(a_0, a_1, \dots, a_n) &= \sum_{i=0}^{n} (-1)^{(\varepsilon_{i-1}+1)(\varepsilon_k - \varepsilon_{i-1})}(e, a_i, \dots, a_n, a_0, a_1, \dots, a_{i-1}) \\ &\in \mathbf{C}_0(A, A^+), \end{aligned}$$

where the signs are determined, as usual, by the sign convention. Here, $\mathbf{C}_0(A, A^+)$ is the subspace of $\mathbf{C}(A, A^+)$ consisting of chains of the form $(e, a_1, \dots, a_n)$; in other words,

$$\begin{aligned} \mathbf{C}_0(A, A^+) &= \mathbf{B}(A) \otimes K \otimes \mathbf{B}(A) \cap \Phi(\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)) \\ &\subset \mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A). \end{aligned}$$

In particular, if the elements a_i are all even (this was the case originally considered by Connes), we have

$$B(a_0, a_1, \dots, a_n) = \sum_{i=0}^{n} (-1)^{in}(e, a_i, \dots, a_n, a_0, a_1, \dots, a_{i-1}).$$

It is obvious that $B^2 = 0$, since B maps into $\mathbf{C}_0(A, A^+)$, but vanishes when applied to an element of this space. To show that $[b, B] = bB + Bb = 0$, we will use a different formula for the B-operator, which uses the fact that we have identified $\mathbf{C}(A, A^+)$ with

$$\Phi(\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)).$$

We will introduce the maps φ: $\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A) \to \mathbf{B}(A)$, defined by the formulas

$$\begin{aligned} \varphi[a_1| \dots |a_i] \otimes a \otimes [b_1| \dots |b_j] &= [a_1| \dots |a_i|a|b_1| \dots |b_j], \\ \varphi[a_1| \dots |a_i] \otimes e \otimes [b_1| \dots |b_j] &= 0, \end{aligned}$$

and σ: $\mathbf{B}(A) \to \mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)$, which is defined by

$$\sigma[a_1| \dots |a_k] = \sum_{i=0}^{k} [a_1| \dots |a_i] \otimes e \otimes [a_{i+1}| \dots |a_k].$$

It is evident from these formulas that both φ and σ are maps of $\mathbf{B}(A)$-bi-comodules, so that on applying the functor Φ to them, we obtain maps

$$\Phi(\varphi)\colon \mathbf{C}(A, A^+) = \Phi(\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)) \to \Phi(\mathbf{B}(A)),$$
$$\Phi(\sigma)\colon \Phi(\mathbf{B}(A)) \to \mathbf{C}_0(A, A^+) = \Phi(\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)).$$

The operator B: $\mathbf{C}(A, A^+) \to \mathbf{C}_0(A, A^+)$ is equal to the composition

$$C(A, A^+) \xrightarrow{\Phi(\varphi)} \Phi(\mathbf{B}(A)) \xrightarrow{\Phi(\sigma)} \mathbf{C}_0(A, A^+);$$

indeed, this might have served as our definition of B.

To show that $[B, b] = 0$, we must check that both φ and σ are maps of complexes. In the case of φ, this follows immediately from the definition (3.3) of the A_∞-module A^+.

We now rewrite σ as the composition

$$\mathbf{B}(A) \xrightarrow{\Delta} \mathbf{B}(A) \otimes \mathbf{B}(A) \cong \mathbf{B}(A) \otimes K \otimes \mathbf{B}(A) \hookrightarrow \mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A).$$

Of course, since $\mathbf{B}(A)$ is a DGC, Δ is a map of complexes. It remains to show that the inclusion

$$\mathbf{B}(A) \otimes K \otimes \mathbf{B}(A) \hookrightarrow \mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)$$

is a map of complexes. This is the same thing as showing that the inclusion $K \hookrightarrow A^+$ is a homomorphism of A-bimodules, where K is given the trivial A-bimodule structure for which all maps m_{ij} vanish. Let us denote the differential on $\mathbf{B}(A) \otimes K \otimes \mathbf{B}(A)$ by b_K, and that on $\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)$ by b_+. Applying the difference $b_+ - b_K$ to a chain

$$[a_1| \dots |a_i] \otimes e[a_{i+1}| \dots |a_k],$$

we obtain

$$\sum_{j=0}^{k} \sum_{l=1}^{\min(i+1, k-j)} (-1)^{\varepsilon_l - 1}[a_1| \dots |a_{l-1}]$$
$$\otimes \tilde{m}_j(a_l, \dots, a_i, e, a_{i+1}, \dots, a_{l_j - 1}) \otimes \left[a_{l+j}| \dots |a_k\right].$$

However, all of the terms in this sum vanish except those with $j = 2$,

$$(-1)^{\varepsilon_{i-1}}[a_1| \dots |a_{i-1}] \otimes \tilde{m}_2(a_i, e) \otimes [a_{i+1}| \dots |a_k]$$
$$-(-1)^{\varepsilon_{i-1}}[a_1| \dots |a_i] \otimes \tilde{m}_2(e, a_{i+1}) \otimes [a_{i+2}| \dots |a_k],$$

and these two terms cancel.

To summarize, we have proved the following theorem.

THEOREM 3.7. *If A is an A_∞-algebra, the Hochschild complex* $(\mathbf{C}(A, A^+), b)$ *may be made into a dg-Λ-module, by means of the operator B.*

4. The cyclic bar complex of a commutative dga

In this section, we will discuss the cyclic bar complex in the special case where A is graded commutative; the example that motivates us is the DGA of differential forms on a smooth manifold, but we will not use any special features of this DGA. Our goal is to prove that underlying the dg-Λ-structure on $\mathbf{C}(A, A^+)$ described above, there is a rich A_∞-algebra structure on the $K[u]$-module $\mathbf{C}(A, A^+)[\![u]\!]$, which comes from a sequence of multilinear operators that generalize Connes's B-operator. The formulas that we give should be thought of as a more precise version of the results of Hood and Jones [3], who only construct the product structure on the cyclic homology spaces $\mathrm{HC}(A; W)$.

What makes the case of a commutative DGA special is that its bar complex $\mathbf{B}(A)$ has a commutative product (the shuffle product) compatible with the coproduct and the differential; in other words, $\mathbf{B}(A)$ is a differential graded Hopf algebra. Let us recall the definition of the shuffle product.

If $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_q)$ are two ordered sets, then a **shuffle** χ of $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_q)$ is a permutation of the ordered set $(a_1, \ldots, a_p, b_1, \ldots, b_q)$ with the property that $\chi(a_i)$ occurs before $\chi(a_j)$, and $\chi(b_i)$ occurs before $\chi(b_j)$, if $i < j$. The shuffle product on $\mathbf{B}(A)$ is defined as the sum over all shuffles on the ordered sets $\left(\binom{1}{1}, \ldots, \binom{i}{1}\right)$ and $\left(\binom{1}{2}, \ldots, \binom{j}{2}\right)$:

$$\left(a_1^1, \ldots, a_1^i\right) * \left(a_2^1, \ldots, a_2^j\right) = \sum_\chi S_\chi\left(sa_1^1 \otimes \cdots \otimes sa_1^i \otimes sa_2^1 \otimes \cdots \otimes sa_2^j\right),$$

where $S_\chi \colon sA^{\otimes I+j} \to sA^{\otimes i+j}$ is the transposition operator which inserts the correct signs. The following proposition summarizes the properties of this product.

PROPOSITION 4.1. *The shuffle product on $\mathbf{B}(A)$ is associative and commutative with identity* [ ] *and it defines a Hopf algebra structure on $\mathbf{B}(A)$. If A is a commutative DGA, the boundary b on $\mathbf{B}(A)$ satisfies Leibniz's rule with respect to the shuffle product, so that $\mathbf{B}(A)$ is made into a commutative DGA.*

If M is a differential graded module over the commutative DGA A, then the space

$$\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)$$

may be made into a differential graded Hopf module over $\mathbf{B}(A)$; since we already know the comodule structure and the differential, we have only to define the module structure, which is done by using the diagram

$$\begin{aligned}
&\mathbf{B}(A) \otimes (\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)) \\
&\xrightarrow{\Delta \otimes 1} \mathbf{B}(A) \otimes \mathbf{B}(A) \otimes (\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)) \\
&\xrightarrow{S_{13425}} (\mathbf{B}(A) \otimes \mathbf{B}(A)) \otimes M(\mathbf{B}(A) \otimes \mathbf{B}(A)) \\
&\xrightarrow{* \otimes 1 \otimes *} \mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)
\end{aligned}$$

Similarly, if M and N are differential graded modules over the commutative DGA A, there is a pairing

$$(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)) \otimes (\mathbf{B}(A) \otimes N \otimes \mathbf{B}(A)) \to \mathbf{B}(A) \otimes (M \otimes N) \otimes \mathbf{B}(A)$$

defined by

$$\begin{aligned}
&(\alpha_1 \otimes x \otimes \beta_1) * (\alpha_2 \otimes y \otimes \beta_2) \\
&\quad = (-1)^{|x||\alpha_2| + |\beta_1||\alpha_2| + |\beta_2||y|} (\alpha_1 * \alpha_2) \otimes (x \otimes y) \otimes (\beta_1 * \beta_2).
\end{aligned}$$

If we restrict this pairing to $\Phi(\mathbf{B}(A) \otimes M \otimes \mathbf{B}(A)) \otimes \Phi(\mathbf{B}(A) \otimes N \otimes \mathbf{B}(A))$, we see that it maps into $\Phi(\mathbf{B}(A) \otimes (M \otimes N) \otimes \mathbf{B}(A))$, and so defines a pairing

$$\mathbf{C}(A, M) \otimes \mathbf{C}(A, N) \to \mathbf{C}(A, M \otimes N),$$

which is given by the sum over all shuffles on the sets $\left(\binom{1}{1}, \ldots, \binom{i}{1}\right)$ and $\left(\binom{1}{2}, \ldots, \binom{j}{2}\right)$:

$$\begin{aligned}
&(x, a_1^1, \ldots, a_1^i) * (y, a_2^1, \ldots, a_2^j) \\
&\quad = \sum_{\chi} (x \otimes y) \otimes S_\chi(sa_1^1 \otimes \cdots \otimes sa_1^i \otimes sa_2^1 \otimes \cdots \otimes sa_2^j).
\end{aligned}$$

As before, the Hochschild boundary b satifies Leibniz's rule with respect to this pairing.

Finally, the space $\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)$ is made into a commutative DGA by the shuffle product, by composing the pairing

$$(\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)) \otimes (\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A)) \to \mathbf{B}(A) \otimes (A^+ \otimes A^+) \otimes \mathbf{B}(A)$$

with the commutative product $A^+ \otimes A^+ \to A^+$. This product restricts to a product on $\mathbf{C}(A, A^+) = \Phi(\mathbf{B}(A) \otimes A^+ \otimes \mathbf{B}(A))$, with the following properties.

PROPOSITION 4.2. (1) *The shuffle product on* $\mathbf{C}(A, A^+)$ *is associative with identity* (e). *If* A *is a commutative DGA, the product is commutative and the Hochschild boundary* b *on* $\mathbf{C}(A, A^+)$ *satisfies Leibniz's rule with respect to the shuffle product, so that* $\mathbf{C}(A, A^+)$ *is a commutative DGA.*

(2) *If* M *is a bimodule for the DGA* A, *the shuffle product action of* $\mathbf{C}(A, A^+)$ *on* $\mathbf{C}(A, M)$ *is associative. If* A *is a commutative DGA, the action satisfies Leibniz's rule with respect to the Hochschild boundaries* b *on* $\mathbf{C}(A, A^+)$ *and* $\mathbf{C}(A, M)$.

It follows from this proposition that the Hochschild homology $\mathrm{HH}(A, A^+)$ of a commutative DGA is a graded commutative algebra, and that $\mathrm{HH}(A, M)$ is always a module for this algebra.

If A is the algebra of smooth functions on manifold M, it was proved by Hochschild, Kostant and Rosenberg that its Hochschild homology $\mathrm{HH}(C^\infty(M))$ is isomorphic to the space of differential forms on M; this isomorphism is realized by the map which sends the chain

$$(f_0, f_1, \ldots, f_n) \in \mathbf{C}\big(C^\infty(M), C^\infty(M)^+\big)$$

to

$$\frac{1}{n!} f_0\, df_1 \ldots df_n \in \Omega(M),$$

and the chain

$$(e, f_1, \ldots, f_n) \in \mathbf{C}\big(C^\infty(M), C^\infty(M)^+\big)$$

to

$$\frac{1}{n!} df_1 \ldots df_n \in \Omega(M).$$

It is fairly easy to see that the product induced $\mathrm{HH}(C^\infty(M))$ by the shuffle product is the exterior product. It is important to observe that the B-operator on $\mathbf{C}(C^\infty(M), C^\infty(M)^+)$ induces a coboundary on $\mathrm{HH}_n(C^\infty(M))$ equal to the exterior differential, as can be seen from the diagram

$$\begin{array}{ccc}
(f_0, f_1, \ldots, f_n) & \xrightarrow{B} & \sum_{i=0}^n (-1)^{in}(e, f_i, \ldots, f_n, f_0, \ldots, f_{i-1}) \\
\downarrow & & \downarrow \\
\frac{1}{n!} f_0\, df_1 \ldots df_n & \xrightarrow{d} & \frac{1}{(n+1)!} df_0\, df_1 \ldots df_n
\end{array}$$

This simple fact is one of the most important reasons for introducing the B-operator.

The differential $b + uB$ is not a derivation with respect to the shuffle product; if we wish to induce a product on the cyclic homology spaces $\mathrm{HC}(A; W)$, we must find a product on $\mathbf{C}(A, A^+)[\![u]\!]$ for which it is. Later in this section, we will define a sequence on maps B_k, $k > 0$, with the following properties.

LEMMA 4.3. *The operator B_k is a multilinear map $\mathbf{C}(A, A^+)^{\otimes k} \to \mathbf{C}_0(A, A^+)$ of degree k such that B_1 is the operator B of Connes, such that $B_k(\alpha_1, \ldots, \alpha_k)$ vanishes if $\alpha_i \in \mathbf{C}_0(A, A^+)$ for any $1 \leq i \leq k$, and satisfying the cocycle property*

$$\begin{aligned} -bB_k(\alpha_1, \ldots, \alpha_k) = & \sum_{i=1}^{k} (-1)^{\varepsilon_{i-1}} B_k(\alpha_1, \ldots, b\alpha_i, \ldots, \alpha_k) \\ & + \alpha_1 * B_{k-1}(\alpha_2, \ldots, \alpha_k) \\ & + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} B_{k-1}(\alpha_1, \ldots, \alpha_i * \alpha_{i+1}, \ldots, \alpha_k) \\ & - (-1)^{\varepsilon_{k-1}} B_{k-1}(\alpha_1, \ldots, \alpha_{k-1}) * \alpha_k. \end{aligned}$$

Using the maps B_k, we define a sequence of multilinear products $\tilde{m}_k$, $k \geq 0$, on $\mathbf{C}(A, A^+)[u]$ as follows:

$$\tilde{m}_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} b\alpha_1 + uB\alpha_1, & k = 1, \\ -(-1)^{|\alpha_1|}\alpha_1 * \alpha_2 + uB_2(\alpha_1, \alpha_2), & k = 2, \\ uB_k(\alpha_1, \ldots, \alpha_k), & \text{otherwise.} \end{cases}$$

From these, we define maps $m_k(\alpha_1, \ldots, \alpha_k)$ as in Section 1 by a sign change

$$\begin{aligned} & m_k(\alpha_1, \ldots, \alpha_k) \\ & \quad = (-1)^{(k-1)|\alpha_1| + (k-2)|\alpha_2| + \cdots + 2|\alpha_{k-2}| + |\alpha_{k-1}| + k(k-1)/2} \tilde{m}_k(\alpha_1, \ldots, \alpha_k). \end{aligned}$$

THEOREM 4.4. *The graded K-module $\mathbf{C}(A, A^+)[\![u]\!]$ with the above multilinear maps m_k is a standard A_∞-algebra.*

Proof. The definition of an A_∞-structure in Section 1 amounts to the following collection of formulas:

$$\sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^{\varepsilon_{i-1}} \tilde{m}_k(\alpha_1, \ldots, \alpha_{i-1}, \tilde{m}_l(\alpha_i, \ldots, \alpha_{i+l-1}), \ldots, \alpha_n) = 0.$$

This formula may be decomposed into three pieces which must vanish separately, corresponding to terms accompanied by no power of u, by the coefficient u, and by the coefficient u^2. The first of these vanishes simply because b satisfies Leibniz's rule with respect to the shuffle product, while the third vanishes because B_k equals zero if any of its arguments lies in $\mathbf{C}_0(A, A^+)$, or in particular is B_l of something. Bearing this in mind, we have only to show that the coefficient of u vanishes; this turns out to be the cocycle formula of 4.3. ■

In order to define the higher maps B_n, we need a little combinatorial machinery. Given numbers $\mu(i)$, $1 \le i \le k$, let $C(\mu(1), \dots, \mu(k))$ be the set

$$\left\{\binom{0}{1}, \dots, \binom{\mu(1)}{1}, \dots, \binom{0}{k}, \dots, \binom{\mu(k)}{k}\right\},$$

ordered lexicographically, that is

$$\binom{j}{i} < \binom{l}{k} \text{ if and only if } i < k \text{ or } i = k \text{ and } j < l.$$

A **cyclic shuffle** σ is a permutation of the set $C(\mu(1), \dots, \mu(k))$ which satisfies the following two conditions:

(1) $\sigma\binom{0}{i_1} < \sigma\binom{0}{i_2}$ if $i_1 < i_2$, and
(2) for each $1 \le i \le k$, there is a number $0 \le j_i \le \mu(i)$ such that

$$\sigma\binom{j_i}{i} < \cdots < \sigma\binom{\mu(i)}{i} < \sigma\binom{0}{i} < \cdots < \sigma\binom{j_i - 1}{i}.$$

We will denote the set of cyclic shuffles by $S(\mu(1), \dots, \mu(k))$.

If we imagine the set $C(\mu(1), \dots, \mu(k))$ arranged as a grid in the plane, so the i-th column is made up of the points $\binom{0}{i}, \binom{1}{i}, \dots, \binom{\mu(i)}{i}$, then a cyclic shuffle is given by first applying a cyclic permutation to each column and then shuffling the columns together in such a way that $\binom{0}{i_1}$ occurs before $\binom{0}{i_2}$ if $i_1 < i_2$.

To each cyclic shuffle σ, there is a corresponding isomorphism of graded K-modules

$$S_\sigma\colon (sA)^{\otimes(i_1 + \cdots + i_k + k)} \to (sA)^{\otimes(i_1 + \cdots + i_k + k)}.$$

If $\alpha_i = (a_i^0, \dots, a_i^{\mu(i)})$, $1 \le i \le k$, is a set of elements of $\mathbf{C}(A, A^+)$, we define

B_k by the formula

$$\begin{aligned}B_k\big((a_1^0, a_1^1, \ldots, a_1^{\mu(1)}), \ldots, (a_k^0, a_k^1, \ldots, a_k^{\mu(k)})\big) \\ = \sum_{\sigma \in \Sigma(\mu(1), \ldots, \mu(k))} e \otimes S_\sigma\big(sa_1^0 \otimes sa_1^1 \otimes \cdots \otimes sa_1^{\mu(1)} \otimes \cdots \otimes sa_k^0 \\ \otimes sa_k^1 \otimes \cdots \otimes sa_k^{\mu(k)}\big).\end{aligned}$$

From this formula, it is immediately clear that $B_k(\alpha_1, \ldots, \alpha_k) \in \mathbf{C}_0(A, A^+)$, and that it vanishes if any of the α_i lie in $\mathbf{C}_0(A, A^+)$. Let us now prove the cocycle formula contained in 4.3, which will complete the construction of the A_∞-structure on $\mathbf{C}(A, A^+)$.

Consider the result of applying $-b$ to a typical term C of the chain $B_k(\alpha_1, \ldots, \alpha_k)$. First of all, note that those terms in bC where the differential d on A hits an entry a_i^j correspond precisely to those terms in $\sum_i (-1)^{\epsilon_{i-1}} B_k(\alpha_1, \ldots, b\alpha_i, \ldots, \alpha_k)$ in which the differential d occurs. The remaining terms involve the product $a_i^j a_k^l$ of two consecutive entries; it is easiest to understand the result by classifying these according to the indices $\binom{j}{i}$ and $\binom{l}{k}$.

(1) The first possibility is that we are forming the product of e and a_1^0 and $C = (e, a_1^0, \ldots)$. The collection of terms coming from this possibility conspire to produce the chain $\alpha_1 * B_{k-1}(\alpha_2, \ldots, \alpha_k)$.

(2) Similarly, we may have the product of a_k^0 and e in $C = (e, \ldots, a_k^0)$. The terms coming from this situation give the chain

$$-(-1)^{\epsilon_{k-1}} B_{k-1}(\alpha_2, \ldots, \alpha_{k-1}) * \alpha_k.$$

(3) The remaining terms involving e also involve an entry a_i^j, $1 \le j \le \mu(i)$, and these terms cancel pairwise, since the entry a_i^j can occur in either the left or the right, and one may check that the signs in these two cases are opposing.

(4) We now come to the products that are "internal" to the chain bC, that is, do not involve its zeroth entry e. If such a product is of the form $a_i^0 a_j^0$, then the nature of a cyclic shuffle shows that $j = i + 1$. This collection of terms produces the sum

$$\sum_{i=1}^{k-1} (-1)^{\epsilon_i} B_{k-1}(\alpha_1, \ldots, \alpha_i * \alpha_{i+1}, \ldots, \alpha_k).$$

(5) Next, we may consider the terms in which we have a product $a_i^j a_i^l$. Since the cyclic shuffle has the effect of cycling the indices $\left(\binom{1}{i}, \ldots, \binom{\mu(i)}{i}\right)$, it is clear that either $l = j + 1$ or $l = 0$ and $j = \mu(i)$, and we find precisely those

terms in $(-1)^{\varepsilon_{i-1}}B_k(\alpha_1,\ldots,b\alpha_i,\ldots,\alpha_k)$ which come from products of elements in α_i, as against differentials.

(6) Finally, we have the products $a_i^j a_k^l$ in which $i \neq k$ and $j+l>0$. These terms cancel from the sum, since $(-1)^{|a_i^j||a_k^l|}a_i^j a_k^l$ occurs with the opposite sign, and the algebra A is (graded) commutative. (This is the only point at which we use the hypothesis that A is graded commutative.)

5. The Chen normalization of the cyclic bar complex

In this section, it will be convenient to change the grading of the complexes of the preceding sections. Our A_∞-algebras will now have a grading for which the multilinear products m_n have degree $2-n$, and the differential b will raise degree on $\mathbf{B}(A)$ by one, while B lowers it. We will assume that our A_∞-algebras A and bimodules M are concentrated in positive degrees, which under the old grading would have corresponded to negative degrees. The original example which motivated the following results is that in which $A=\Omega(X)$ is the DGA of differential forms on a manifold.

Let us introduce a normalization of the cyclic bar complex due to Chen [1]; its purpose is to get rid of chains of negative degree in the cyclic bar complex $\mathbf{C}(A,M)$. If f is an element of A_0, we define operators $S_i(f)$ on $\mathbf{C}(A,M)$ by the formula

$$S_i(f)(x,a_1,\ldots,a_n)=(x,a_1,\ldots,a_{i-1},f,a_i,\ldots,a_n).$$

We now define $\mathbf{D}(A,M)$ to be the subspace of $\mathbf{C}(A,M)$ generated by the images of the operators $S_i(f)$ and $R_i(f)=[b,S_i(f)]$, and the Chen normalised chain complex to be the quotient complex $\mathbf{N}(A,M)=\mathbf{C}(A,M)/\mathbf{D}(A,M)$. The following result shows that the Chen normalization leaves many of the structures that we have considered on $\mathbf{C}(A,M)$ in place.

PROPOSITION 5.1. (1) *The differential b maps $\mathbf{D}(A,M)$ to itself.*

(2) *If $M=A^+$, the shuffle product and the multilinear maps B_k take values in $\mathbf{D}(A,A^+)$ if any of their arguments are in $\mathbf{D}(A,A^+)$.*

It follows that:

(3) *$(\mathbf{D}(A,M),b)$ is a sub-complex of $(\mathbf{C}(A,M),b)$.*

(4) *$(\mathbf{D}(A,A^+),b,B)$ is a sub-dg-Λ-module of $(\mathbf{C}(A,A^+),b,B)$.*

(5) *If A is graded commutative, $\mathbf{D}(A,A^+)[\![u]\!]$ is an A_∞-ideal of $\mathbf{C}(A,A^+)[\![u]\!]$, so that $\mathbf{N}(A,A^+)$ inherits an A_∞-structure from $\mathbf{C}(A,A^+)$.*

Proof. The first part is true by construction. To prove the second part, observe that if in taking the shuffle product of two elements of $\mathbf{C}(A,A^+)$, one of them lies in $\bigcup_i S_i(f)\mathbf{C}(A,A^+)$, then the shuffle product also lies in $\bigcup_i S_i(f)\mathbf{C}(A,A^+)$. Using the fact that b is a derivation with respect to the shuffle product, it follows that $\mathbf{D}(A, A^+)$ is an ideal with respect to the shuffle product:

$$\begin{aligned}(R_i(f)\alpha_1)\alpha_2 &= ([b, S_i(f)]\alpha_1)\alpha_2 \\ &= b((S_i(f)\alpha_1)\alpha_2) + (-1)^{|\alpha_1|}(S_i(f)\alpha_1)(b\alpha_2) \\ &\quad + (S_i(f)b\alpha_1)\alpha_2,\end{aligned}$$

and we have already shown that all three of these terms belong to $\mathbf{D}(A, A^+)$.

The case of $B_k(\alpha_1, \ldots, \alpha_k)$ is similar: if $\alpha_i \in \bigcup_j S_j(f)\mathbf{C}(A, A^+)$, then so is $B_k(\alpha_1, \ldots, \alpha_k)$, so that

$$B_k(\alpha_1, \ldots, R_j(f)\alpha_i, \ldots, \alpha_k) - B_k(\alpha_1, \ldots, bS_j(f)\alpha_i, \ldots, \alpha_k) \in \mathbf{D}(A, A^+).$$

Using the cocycle property of the maps B_k and induction in k, it is easily shown that $B_k(\alpha_1, \ldots, bS_j(f)\alpha_i, \ldots, \alpha_k)$ lies in $\mathbf{D}(A, A^+)$, and hence that $B_k(\alpha_1, \ldots, R_i(f)\alpha_i, \ldots, \alpha_k)$ does too. ■

The following theorem is our transcription to the above setting of a result of Chen [1].

THEOREM 5.2. *If A is connected, that is, $H^0(A) = 0$, then the complex $(\mathbf{D}(A, M), b)$ is acyclic.*

Proof. Let V be a complement of $d[A^0] \subset A^1$, and define a subalgebra $\bar{A} \subset A$ as follows:

$$\bar{A}^k = \begin{cases} 0, & k = 0, \\ V, & k = 1, \\ A^k, & k > 1. \end{cases}$$

By construction, $\bar{A}$ is a subalgebra of A having the same cohomology. We will compare the cyclic bar complexes of A and $\bar{A}$ with coefficients in the bimodule M, under the inclusion

$$\mathbf{C}(\bar{A}, M) \hookrightarrow \mathbf{C}(A, M).$$

Note that $\mathbf{D}(\bar{A}, M)$ is the intersection of $\mathbf{C}(\bar{A}, M)$ and $\mathbf{D}(A, M)$; however, since $\bar{A}^0$ vanishes, it follows that $\mathbf{D}(\bar{A}, M) = 0$. Thus, it suffices to prove that the above inclusion $\mathbf{C}(\bar{A}, M)$ in $\mathbf{C}(A, M)$ induces an isomorphism in cohomology. To do this, we apply the following lemma, which is the generalization to our context of a fundamental lemma of Moore. ■

LEMMA 5.3. *Let M_i be a bimodule over the standard A_∞-algebra A_i for $i \in \{1,2\}$, and let $A_1 \to A_2$ be a homomorphism, such that $M_1 \otimes_{A_1} A_2 \to M_2$ is* *a homomorphism of bimodules over A_2. If these homomorphisms induce isomorphisms in cohomology, then the induced homomorphism*

$$\mathbf{C}(A_1, M_1) \to \mathbf{C}(A_2, M_2)$$

induces an isomorphism in cohomology.

Proof. If A is a standard A_∞-algebra with M is a bimodule over A, consider the bar filtration on $\mathbf{C}(A, M)$

$$F^{-k}\mathbf{C}(A, M) = \operatorname{span}\{(m, a_1, \ldots, a_i) | i \le k\}.$$

The E_1-term of the spectral sequence associated to this filtration is easily seen to be isomorphic to $\mathbf{C}(H(A), H(M))$; more precisely, $E_1^{-p,q}$ is equal to the span of the collection of chains in $\mathbf{C}(H(A), H(M))$ of the form

$$([x], [a_1], \ldots, [a_p])$$

where

$$[x] \in H(M), [a_i] \in H(A) \text{ and } |x| + \sum |a_i| = q.$$

From this, it is easy to see that the spectral sequence converges, from which the lemma follows easily by a comparison theorem. ■

Note that the definition of a connected A_∞-algebra which we use in this theorem, that is $H^0(A) = 0$, is the correct one in the category of A_∞-algebras without identity. If we were to state the corresponding result for A_∞-algebras with identity (which we leave it for the reader to do), we would demand that $H^0(A) = K$.

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