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SPECIAL VALUES OF HYPERLOGARITHMS AND LINEAR DIFFERENCE SCHEMES

BY

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To the memory of Prof. K. T. Chen

1. Introduction

The values of iterated integrals in the sense of K. T. Chen,

$$(1.1)\qquad L_{r,s} = \int_0^1 (d\log x)^r \cdot (d\log(1-x))^s, \quad r, s \geq 1,$$

are equal to

$$(1.2)\qquad L_{r,s} = \frac{(-1)^r}{r!s!} \frac{\partial^{r+s}}{\partial\alpha^r \partial\beta^s}\left[\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}\right]_{\alpha=\beta=0}.$$

By using these formulae, one can prove that $L_{r,s}$ $(r, s \geq 1)$ are linearly dependent over

$$\mathbf{Q}(L_{j,1}, 1 \leq j \leq r+s-1),$$

where $L_{j,1}$ are also the values at $z = 1$ of the polylogarithms

$$L_{j+1}(z) = \int_0^z (d\log x)^j \cdot d\log(1-x).$$

In this sense from transcendental point of view, $L_{r,s}$ do not give any essentially new features than the values $L_{j,1}$ themselves. Due to R. Ree (see [24]), it is known that the dimension $N_{r,s}$ of basic Lie elements of degree r and s of a free Lie algebra of two generators is generally greater than 1. Hence more complicated iterated integrals of type $(r_1, r_2, r_3, \ldots, r_n)$ of degree r, $r = r_1 + r_2 + r_3 + \cdots + r_n$, hyperlogarithms in the sense of Poincaré, Lappo-

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Danilevskii,

$$(1.3)\quad L_{r_1,r_2,r_3,\cdots} = \int_0^1 (d\log x)^{r_1}\cdot(d\log(1-x))^{r_2}\cdot(d\log x)^{r_3}\cdots,$$

for $r_1, r_2, r_3, \cdots \geq 1$ are expected to be linearly independent over $\mathbf{Q}(L_{j,1}, 1 \leq j \leq r-1)$. Here the integrals must take their regular parts when they diverge (see the text for details). A few special cases of $L_{r_1,r_2,r_3,\ldots}$, in particular $L_{r_1,1,r_2}$, have been investigated by J. L. Dupont from an algebraic view point (see [11]).

The purpose of this note is to generalize the above formulae (1.2) to the cases $L_{r_1,r_2,r_3,\ldots}$, i.e., to obtain equalities between these and coefficients of Taylor expansions of certain analytic functions $F_n^*(z_1, z_2, z_3, \ldots | x, y)$ at the origin $z_1 = z_2 = z_3 = \cdots = 0$ and at $(x, y) = (1,0)$.

$$(1.4)\quad \int_0^1 \omega_1^{r_1}\omega_2^{r_2}\omega_1^{r_3}\cdots = \frac{1}{r_1!\,r_2!\,r_3!\cdots} \times \frac{\partial^{r_1+r_2+r_3+\cdots}}{\partial z_1^{r_1}\,\partial z_2^{r_2}\,\partial z_3^{r_3}\cdots} F_n^*(0,0,0,\cdots|1,0)$$

for $\omega_1 = d\log x$ and $\omega_2 = d\log(1-x)$, where $F_n^*(z_1, z_2, z_3, \ldots|1,0)$ is meromorphic in $(z_1, z_2, z_3, \ldots) \in \mathbf{C}^n$ and satisfies *integrable* (*i.e.*, *holonomic*) *linear difference equations* in $z_1, z_2, z_3, \ldots$ which characterize it uniquely. $F_n^*(z_1, z_2, z_3, \ldots | x, y)$ belong to one of wider classes of functions which are defined by integrals of certain multiplicative functions (multiplicative functions are defined generally on Kähler varieties but here we consider only product of powers of linear functions on affine spaces). For this see [1] and [3].

Recently I. M. Gelfand and his collabolators have presented this kind of integrals called "hypergeometric integrals" in the framework of Grassmannian geometry ([12] and [13]). We consider separately the following four types of iterated integrals:

$$(1.5)\quad L_{r_1,r_2,\ldots,r_{2m}} = \int_0^1 \omega_1^{r_1}\omega_2^{r_2}\cdots\omega_2^{r_{2m}},$$

$$(1.6)\quad L_{r_1,r_2,\ldots,r_{2m-1}} = \operatorname{reg}\int_0^1 \omega_1^{r_1}\omega_2^{r_2}\cdots\omega_1^{r_{2m-1}},$$

$$(1.7)\quad L'_{r_1,r_2,\ldots,r_{2m}} = \operatorname{reg}\int_0^1 \omega_2^{r_1}\omega_1^{r_2}\cdots\omega_1^{r_{2m}},$$

$$(1.8)\quad L'_{r_1,r_2,\ldots,r_{2m-1}} = \operatorname{reg}\int_0^1 \omega_2^{r_1}\omega_1^{r_2}\cdots\omega_2^{r_{2m-1}},$$

for $r_1, r_2, r_3, \cdots \geq 1$, where reg denotes the regularization of divergent integrals, since (1.4) is generally divergent. An iterated integral in the sense of

K. T. Chen,

$$\int_y^x \theta_1\theta_2 \cdots \theta_n = \int_y^x \theta_1(x_1, dx_1) \int_y^{x_1} \theta_2(x_2, dx_2) \cdots \int_y^{x_{n-1}} \theta_n(x_n, dx_n), \tag{1.9}$$

is defined in a standard way along a suitable path in a space where 1-forms $\theta_1, \theta_2, \ldots, \theta_n$ are defined. To avoid confusion, the product $\theta_1\theta_2 \cdots \theta_n$ in the Chen algebra (reduced bar construction) will also be denoted by $\theta_1|\theta_2| \cdots |\theta_n$. The regularization of integrals is the same thing as finite parts in the sense of Hadamard-Leray (see [14], [20] or [23]).

We want to raise the following interesting question:

QUESTION. *Are $L_{r_1,\ldots,r_n}$ (or $L'_{r_1,\ldots,r_n}$) for $r_1, \ldots, r_n \geq 1$ linearly independent over rational numbers?*

2. Basic relation in twisted de Rham cohomology

Let $\lambda_{r,\alpha} \in \mathbf{C}$, $1 \leq r \leq n$, $1 \leq \alpha \leq m$ and $\lambda'_{r,r+1}$, $1 \leq r \leq n-1$, be arbitrarily given such that $\operatorname{Re}\lambda_{r,\alpha} > 0$ and $\operatorname{Re}\lambda'_{r,r+1} > 0$. We denote by λ_r, each m-dimensional vector

$$(\lambda_{r,1}, \lambda_{r,2}, \ldots, \lambda_{r,m}) \in \mathbf{C}^m,$$

and by $\lambda \in \mathbf{C}^{mn+n-1}$ the vector whose components are $\lambda_{r,\alpha}$ and $\lambda'_{r,r+1}$.

Suppose further that a sequence of complex numbers $\{a_\alpha, 1 \leq \alpha \leq m\}$ are given such that $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$.

Let $\Phi_m(\lambda|a_1, \ldots, a_m) = \Phi_m(\lambda_1, \ldots, \lambda_n; \lambda'_{1,2}, \ldots, \lambda'_{n-1,n}|a_1, \ldots, a_m)$ be the multiplicative function

$$\Phi_m = \prod_{r=1}^{n} \prod_{\alpha=1}^{m} (x_r - a_\alpha)^{\lambda_{r,\alpha}} \prod_{r=1}^{n-1} (x_r - x_{r+1})^{\lambda'_{r,r+1}} \tag{2.1}$$

and $\omega = d\log\Phi_m$ be the associated logarithmic 1-form on the space $X = X_n$, where X_n denotes the n-dimensional affine variety:

$$X_n = \mathbf{C}^n - \bigcup_{\substack{1\leq r\leq n \\ 1\leq\alpha\leq m}} (x_r = a_\alpha) \bigcup_{1\leq r\leq n-1} (x_r = x_{r+1}). \tag{2.2}$$

We denote by $H^n(X, \nabla_\omega)$ the twisted de Rham n-cohomology on X associated with the covariant differentiation ∇_ω defined by ω.

Notation. In the sequel, for simplicity, we let

$$(j,k) = a_j - a_k \quad \text{for } 1 \le j, k \le m \text{ and } j \ne k,$$
$$(m+j,k) = -(k,m+j) = x_j - a_k \quad \text{for } 1 \le j \le n \text{ and } 1 \le k \le m,$$
$$(m+j, m+j+1) = -(m+j+1, m+j) = x_j - x_{j+1} \quad \text{for } 1 \le j \le n-1.$$

We define the analytic function

$$\tilde{\Phi}_m(\lambda | a_1, \ldots, a_m) = \tilde{\Phi}_m(\lambda_1, \ldots, \lambda_n; \lambda'_{1,2}, \ldots, \lambda'_{n-1,n} | a_1, \ldots, a_m)$$

by

$$\tilde{\Phi}_m = \int \Phi_m \, dx_1 \wedge \cdots \wedge dx_n. \tag{2.3}$$

The integral is done over a suitable twisted cycle defined in the affine variety X. This is meromorphic in λ but generally many valued in $a_1, \ldots, a_m$. It satisfies holonomic differential equations in a_α and holonomic difference equations (sometimes called contiguity relations) in λ. First we want to write them down. For this we define the difference operators $T_{\pm e_{r,\alpha}}$ and $T_{\pm e'_{r,r+1}}$ for meromorphic functions on a complex affine space as follows:

DEFINITION 1. We denote by $e_{r,\alpha}$ (or $e'_{r,r+1}$) the unit vector with elements $\lambda_{s,\beta} = 0$, $\lambda'_{s,s+1} = 0$ except for the $\lambda_{r,\alpha}$ (or $\lambda'_{r,r+1}$)-component which is equal to 1 respectively. These span the $(mn+n-1)$-dimensional linear space $\mathbf{C}^{mn+n-1}$. Then

$$T_{\pm e_{r,\alpha}} f(\lambda) = f(\lambda \pm e_{r,\alpha}), \quad T_{\pm e'_{r,r+1}} f(\lambda) = f(\lambda \pm e'_{r,r+1}). \tag{2.4}$$

For a sequence I of n arguments i_r, $1 \le r \le n$, such that $1 \le i_r \le m$ or $i_r = r - 1 + m$, we shall denote by $\langle I \rangle = \langle i_1, \ldots, i_n \rangle$ the logarithmic n-form

$$d\log(m+1, i_1) \wedge \cdots \wedge d\log(m+n, i_n). \tag{2.5}$$

Remark that $i_r < r + m$. Then obviously we have

$$\int \Phi_m \langle I \rangle = \tilde{\Phi}_m(\lambda - e_{1,i_1} - \cdots - e_{n,i_n} | a_1, \ldots, a_m). \tag{2.6}$$

The left hand side will also be denoted by $\langle \tilde{I} \rangle$.

LEMMA 2.1. *The twisted de Rham cohomology $H^n(X, \nabla_\omega)$ is isomorphic to the space of logarithmic n-forms $\Omega^n_{\log}(X)$ generated by*

$$d\log(m+r, \alpha), \quad 1 \le r \le n, 1 \le \alpha \le m$$

and

$$d\log(m+r,\ m+r+1),\quad 1 \le r \le n-1$$

as follows:

$$H^n(X,\nabla_\omega) \simeq \Omega^n_{\log}(X)/\omega \wedge \Omega^{n-1}_{\log}(X) \tag{2.7}$$

where $\Omega^{n-1}_{\log}(X)$ *denotes the space of logarithmic* $(n-1)$*-forms generated by*

$$d\log(m+r,\alpha) \quad \text{and} \quad d\log(m+r,m+r+1).$$

The dimension of $H^n(X,\nabla_\omega)$ *coincides with the absolute value of the Euler number of* X *which is equal to* $(m-1)m^{n-1}$.

Proof. See [1] and also [3].

Let $V_{n,m}$ be the space spanned by exterior products of degree n generated by $d\log(m+r,\alpha)$, $1 \le r \le n$ and $1 \le \alpha \le m$, i.e., spanned by $\langle I\rangle$ such that all $i_r \le m$. We shall call these $\langle I\rangle$ and also their integrals $\langle \tilde{I}\rangle$ "admissible ones". $H^n(X,\nabla_\omega)$ is spanned by $V_{n,m}$, i.e., the following exact sequence holds:

$$0 \to W_{n,m} \to V_{n,m} \to H^n(X,\nabla_\omega) \to 0, \tag{2.8}$$

for a subspace $W_{n,m}$ of $V_{n,m}$.

To see the structure of $H^n(X,\nabla_\omega)$ in more details, we want to present two kinds of basic relations among the logarithmic forms $\langle I\rangle$. For that purpose it is useful to define two kinds of endomorphisms $P^r_{s,\alpha}$ and Q^r_s on $V_{n,m}$.

DEFINITION 2. We denote by $P^r_{s,\alpha}$, $1 \le r,\ s \le n$, $1 \le \alpha \le m$, the endomorphisms on $V_{n,m}$ defined by

$$P^r_{s,\alpha}\langle i_1,\dots,i_n\rangle = \langle i_1,\dots,i_{s-1},\alpha,i_s,\dots,i_{r-1},i_{r+1},\dots,i_n\rangle \quad \text{for } 1 \le s < r \le n, \tag{2.9}$$

i.e., delete the r-the element i_r, insert the argument α into the s-th place and shift the arguments $i_s,\dots,i_{r-1}$ to the right.

Also

$$P^r_{r,\alpha}\langle I\rangle = \langle i_1,\dots,i_{r-1},\alpha,i_{r+1},\dots,i_n\rangle \quad \text{for } r = s. \tag{2.10}$$

$$P^r_{s,\alpha}\langle I\rangle = \langle i_1,\dots,i_{r-1},i_{r+1},\dots,i_s,\alpha,i_{s+1},\dots,i_n\rangle \quad \text{for } 1 \le r < s \le n, \tag{2.11}$$

$$Q^r_s\langle I\rangle = \langle i_1,\dots,i_{s-1},i_s,i_s,i_{s+1},\dots,i_{r-1},i_{r+1},\dots,i_n\rangle \quad \text{for } 1 \le s < r \le n, \tag{2.12}$$

i.e., delete the r-th element i and double the s-th element i_s and shift the elements between them to the right.

Similarly

$$Q_s^r\langle I\rangle = \langle i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_{s-1}, i_s, i_s, i_{s+1}, \dots, i_n\rangle \quad \text{for } 1 \le r < s \le n. \tag{2.13}$$

It is obvious that both $P_{s,\alpha}^r\langle I\rangle$ and $Q_s^r\langle I\rangle$ are independent of choice of the argument i_r. In the same way we can define the endomorphisms $\tilde{P}_{s,\alpha}^r$ and $\tilde{Q}_s^r$ for the integrals $\langle \tilde{I}\rangle$. The first basic relations are as follows:

LEMMA 2.2. *For a fixed r, $1 \le r \le n-1$, let $i_1, \dots, i_n$ be arbitrarily given such that $1 \le i_s \le m$. Then we have cohomological relations in $H^n(X, \nabla_\omega)$:*

$$(R_r) \qquad \lambda'_{r,r+1}\langle i_1, \dots, i_r, m+r, i_{r+2}, \dots, i_n\rangle \sim \left\{\sum_{s=1}^{r}\sum_{\alpha=1}^{m} \lambda_{s,\alpha}P_{s,\alpha}^{r+1} + \sum_{s=1}^{r}\lambda'_{s,s+1}Q_s^{r+1}\right\}\langle I\rangle.$$

The right hand side does not depend on i_{r+1}.

Proof. (R_r) can be proved by induction on r for $r \ge 0$. In fact we have the cohomological identity in $H^n(X, \nabla_\omega)$:

$$\begin{aligned}(2.14)\qquad 0 \sim \nabla_\omega\{&(-1)^{r-1} d\log(m+1, i_1) \wedge \cdots \wedge d\log(m+r-1, i_{r-1}) \\ &\wedge d\log(m+r+1, i_r) \wedge d\log(m+r+2, i_{r+2}) \\ &\wedge \cdots \wedge d\log(m+n, i_n)\} \\ = &\sum_{\alpha=1}^{m} \lambda_{r,\alpha}P_{r,\alpha}^{r+1}\langle I\rangle + \lambda'_{r,r+1}Q_r^{r+1}\langle I\rangle \\ &+ \lambda'_{r-1,r}\langle i_1, \dots, i_{r-1}, m+r-1, i_r, i_{r+2}, \dots, i_n\rangle \\ &- \lambda'_{r,r+1}\langle i_1, \dots, i_{r-1}, i_r, m+r, i_{r+2}, \dots, i_n\rangle\end{aligned}$$

since

$$\begin{aligned}(2.15)\qquad d\log(r+m, r+m+1) \wedge d\log(r+m+1, i_r) &= d\log(r+m, i_r) \wedge d\log(r+m+1, i_r) \\ &\quad - d\log(r+m, i_r) \wedge d\log(r+m+1, r+m).\end{aligned}$$

By repeating this procedure we arrive at (R_r).

Hence $\langle i_1, \dots, i_r, m+r, i_{r+2}, \dots, i_n\rangle$ is expressed explicitly as a linear combination of admissible $\langle j_1, \dots, j_n\rangle$.

LEMMA 2.3. *Fix r, $1 \le r \le n$. Then we have*

$$(R'_r) \quad 0 \sim \left\{\sum_{s=1}^{n}\sum_{\beta=1}^{m} \lambda_{s,\beta}P_{s,\beta}^r + \sum_{s=1}^{r-1}\lambda'_{s,s+1}Q_s^r + \sum_{s=r+1}^{n}\lambda'_{s-1,s}Q_s^r\right\}\langle I\rangle,$$

for an arbitrary admissible $\langle I\rangle$. Hence $W_{n,m}$ coincides with the image

$$\left\{\sum_{s=1}^{n}\sum_{\beta=1}^{m}\lambda_{s,\beta}P^r_{s,\beta}+\sum_{s=1}^{r-1}\lambda'_{s,s+1}Q^r_s+\sum_{s=r+1}^{n}\lambda'_{s-1,s}Q^r_s\right\}V_{n,m}.$$

Proof. The identity

$$(2.16)\quad 0\sim\nabla_\omega\{(-1)^{n-1}d\log(m+1,i_1)\wedge\cdots\wedge d\log(m+n-1,i_{n-1})\}$$
$$=\sum_{\alpha=1}^{m}\lambda_{n,\alpha}P^n_{n,\alpha}\langle I\rangle+\lambda'_{n-1,n}\langle i_1,\ldots,i_{n-1},m+n-1\rangle.$$

But (R_{n-1}) shows that the right hand side is equal to

$$(2.17)\qquad \left\{\sum_{r=1}^{n}\sum_{\alpha=1}^{m}\lambda_{r,\alpha}P^n_{r,\alpha}+\sum_{r=1}^{n-1}\lambda'_{r,r+1}Q^n_r\right\}\langle I\rangle,$$

which implies (R'_n). We get (R'_r) from (R'_n) be relabelling the arguments i_s like $i_s\to i_s$ for $s\le r-1$, $i_s\to i_{s+1}$ for $r\le s\le n-1$. Lemma 2.3 has thus been proved.

As a result of (R_r) and (R'_r) we derive the following fundamental difference equations for the integral $\langle\tilde I\rangle$:

$$(\tilde R_r)\quad \lambda'_{r,r+1}\langle i_1,\ldots,i_r,\widetilde{m+r},i_{r+1},\ldots,i_n\rangle$$
$$=\left\{\sum_{s=1}^{r}\sum_{\alpha=1}^{m}\lambda_{s,\alpha}\tilde P^{r+1}_{s,\alpha}+\sum_{s=1}^{r}\lambda'_{s,s+1}\tilde Q^{r+1}_s\right\}\langle\tilde I\rangle,\quad 1\le r\le n-1.$$

$$(\tilde R'_r)\quad 0=\left\{\sum_{s=1}^{n}\sum_{\beta=1}^{m}\lambda_{s,\beta}\tilde P^r_{s,\beta}+\sum_{s=1}^{r-1}\lambda'_{s,s+1}\tilde Q^r_s+\sum_{s=r+1}^{n}\lambda'_{s,s+1}\tilde Q^r_s\right\}\langle\tilde I\rangle,$$
$$1\le r\le n.$$

Note that *the relations $(\tilde R'_r)$ are all equivalent.* Hence the number of linearly independent relations is just equal to m^{n-1}. This shows that the admissible integrals $\langle\tilde I\rangle$ with $i_r=\alpha$ are all expressed by a linear combination of $\langle J\rangle$ for $J=(j_1,\ldots,j_n)$ such that $j_r\ne\alpha$:

$$(2.18)\qquad \langle i_1,\ldots,i_{r-1},\widetilde{\alpha,ir+1},\ldots,i_n\rangle$$
$$=\sum_{j_1,\ldots,j_n}\sum_{j_r\ne\alpha}\langle \widetilde{j_1\cdots j_n}\rangle B_r\begin{pmatrix} j_1,\ldots, & & ,\ldots,j_n\\ i_1,\ldots,i_{r-1}, & \alpha & i_{r+1},,\ldots,i_n\end{pmatrix}$$

where

$$B_r\begin{pmatrix} j_1, \dots, & & , \dots, j_n \\ i_1, \dots, i_{r-1}, & \alpha & i_{r+1}, \dots, i_n \end{pmatrix} = B_r\left(\begin{matrix} j_1, \dots, & & & , \dots, j_n \\ i_1, \dots, & i_{r-1}, & \alpha & i_{r+1}, \dots, i_n \end{matrix}\middle| \lambda\right)$$

denote *rational functions in* λ, *whose denominators are products of linear factors*

$$\sum_{s=r-t_1}^{r+t_2} \lambda_{s,\alpha} + \sum_{s=r-t_1}^{r+t_2-1} \lambda'_{s,s+1} \quad \textit{for } 0 \le t_1 \le r-1, 0 \le t_2 \le n-r.$$

This formula can be proved by induction on the numbers t_1 and t_2 such that $i_{r-t_1} = \cdots = i_{r+t_2} = \alpha$ and that i_{r-t_1-1} and i_{r+t_2+1} are different from α.

In fact we denote by $\tilde{V}^{(r)}_{n,m}(t_1, t_2)$ the linear space spanned by admissible $\langle \tilde{I} \rangle$ such that $i_{r-t_1} = \cdots = i_{r+t_2} = \alpha$ and $i_{r-t_1-1}, i_{r+t_2+1} \neq \alpha$. If $t_1 = t_2 = 1$, then (R'_r) shows that

$$(2.19) \quad -\lambda_{r,\alpha}\langle i_1, \dots, i_{r-1}, \widetilde{\alpha, ir}+1, \dots, i_n\rangle$$
$$= \left\{ \sum_{\substack{s=1 \\ s\neq r}}^{n} \sum_{\beta=1}^{m} \lambda_{s,\beta}\tilde{P}^r_{s,\beta} + \sum_{\beta\neq\alpha} \lambda_{r,\beta}\tilde{P}^r_{r,\beta} + \sum_{s=1}^{r-1} \lambda'_{s,s+1}\tilde{Q}^r_s + \sum_{s=r+1}^{n} \lambda'_{s-1,s}Q^r_s \right\} \langle i_1, \dots, i_{r-1}, \widetilde{\alpha, ir}+1, \dots, i_n\rangle$$
$$\equiv 0 \mod \tilde{V}^{(r)}_{n,m}(0,0).$$

Let $\langle \widetilde{I_{t_1,t_2}} \rangle$ be an arbitrary $\langle i_1, \dots, i_{r-t_1}, \widetilde{\alpha \cdots \alpha}, i_{r+t_2}, \dots, i_n\rangle \in \tilde{V}^{(r)}_{n,m}(t_1, t_2)$. Since

$$(2.20)$$
$$-\left(\sum_{s=r-t_1}^{r+t_2} \lambda_{s,\alpha} + \sum_{s=r-t_1}^{r+t_2-1} \lambda'_{s,s+1}\right)\langle \widetilde{I_{t_1,t_2}} \rangle$$
$$= \left\{ \left(\sum_{s=1}^{r-t_1} + \sum_{s=r+t_2+1}^{n}\right)\left(\sum_{\beta=1}^{m} \lambda_{s,\beta}\tilde{P}^r_{s,\beta}\right) + \sum_{s=r-t_1}^{r+t_2} \sum_{\beta\neq\alpha} \lambda_{s,\beta}\tilde{P}^r_{s,\beta} + \sum_{s=1}^{r-t_1} \lambda'_{s,s+1}\tilde{Q}^r_s + \sum_{s=r+t_2}^{n} \lambda'_{s-1,s}\tilde{Q}^r_s \right\}\langle \widetilde{I_{t_1,t_2}} \rangle$$
$$\equiv 0 \mod_{\substack{0\le t'_1\le t_1, 0\le t'_2\le t_2 \\ t'_1+t'_2<t_1+t_2}} \tilde{V}^{(r)}_{n,m}(t'_1, t'_2).$$

Hence by induction hypotheses we have proved the lemma.

In the same manner one can prove:

LEMMA 2.4. *For fixed r and* α,

$$(2.21)\quad \langle i_1,\ldots,i_{r-1},\alpha,\widetilde{\alpha,i_{r+2}},\ldots,i_n\rangle = \sum B'_{r,r+1}\begin{pmatrix} j_1,\ldots,j_{r-1}, & j_r, & j_{r+1}, & j_{r+2},\ldots,j_n \\ i_1,\ldots,i_{r-1}, & \alpha, & \alpha, & i_{r+2},\ldots,i_n \end{pmatrix}\langle\tilde{J}\rangle,$$

where $J = \{j_1,\ldots,j_n\}$ *run over admissible sequences such that* $j_r \neq j_{r+1}$. *Here,* $B'_{r,r+1} = B'_{r,r+1}\left(\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\middle|\lambda\right)$ *denote rational functions in* λ *whose denominators are products of linear factors*

$$\sum_{s=r-t_1}^{r+t_2}\lambda_{s,\alpha} + \sum_{s=r-t_1}^{r+t_2-1}\lambda'_{s,s+1}$$

for $0 \le t_1 \le r-1$ *and* $1 \le t_2 \le n$.

The symmetry properties below for B_r and $B'_{r,r+1}$ follow immediately from the definitions:

Let σ be a permutation of m arguments $1,2,\ldots,m$. Then σ entails the transformation for each λ_r: $\lambda_r \to \sigma\lambda_r$ where $(\sigma\lambda)_\alpha = \lambda_{r,\sigma(\alpha)}$. Then:

LEMMA 2.5.

$$(2.22)\quad B_r\left(\begin{matrix}\sigma(i_1),\ldots,\sigma(i_n)\\ \sigma(j_1),\ldots,\sigma(j_n)\end{matrix}\middle|\sigma\lambda_1,\ldots,\sigma\lambda_n;\lambda'_{1,2},\ldots,\lambda'_{n-1,n}\right) = B_r\left(\begin{matrix}i_1,\ldots,i_n\\ j_1,\ldots,j_n\end{matrix}\middle|\lambda_1,\ldots,\lambda_n;\lambda'_{1,2},\ldots,\lambda'_{n-1,n}\right)$$

$$(2.23)\quad B_r\left(\begin{matrix}i_n, & i_{n-1},\ldots,i_1\\ j_n, & j_{n-1},\ldots,j_1\end{matrix}\middle|\lambda_n,\lambda_{n-1},\ldots,\lambda_1;\lambda'_{n-1,n},\ldots,\lambda'_{1,2}\right) = B_r\left(\begin{matrix}i_1,\ldots,i_n\\ j_1,\ldots,j_n\end{matrix}\middle|\lambda_1,\ldots,\lambda_n;\lambda'_{1,2},\ldots,\lambda'_{n-1,n}\right)$$

$$(2.24)\quad B'_{r,r+1}\left(\begin{matrix}\sigma(i_1),\ldots,\sigma(i_n)\\ \sigma(j_1),\ldots,\sigma(j_n)\end{matrix}\middle|\sigma\lambda_1,\ldots,\sigma\lambda_n;\lambda'_{1,2},\ldots,\lambda'_{n-1,n}\right) = B'_{r,r+1}\left(\begin{matrix}i_1,\ldots,i_n\\ j_1,\ldots,j_n\end{matrix}\middle|\lambda_1,\ldots,\lambda_n;\lambda'_{1,2},\ldots,\lambda'_{n-1,n}\right)$$

$$(2.25)\quad B'_{n-r-1,n-r}\left(\begin{matrix}i_n, & i_{n-1},\ldots,i_1\\ j_n, & j_{n-1},\ldots,j_1\end{matrix}\middle|\lambda_n,\ldots,\lambda_1;\lambda'_{n-1,n},\ldots,\lambda'_{1,2}\right) = B'_{r,r+1}\left(\begin{matrix}i_1,\ldots,i_n\\ j_1,\ldots,j_n\end{matrix}\middle|\lambda_1,\ldots,\lambda_n;\lambda'_{1,2},\ldots,\lambda'_{n-1,n}\right)$$

Suppose that a_α are all real and $a_1 > a_2 > \cdots > a_m$. Then the dual space $H^n(X, \nabla_\omega)^*$ of $H^n(X, \nabla_\omega)$ has a basis of twisted cycles defined by the following inequalities. For arbitrary $\alpha_1, \alpha_2, \ldots, \alpha_n$, $1 \le \alpha_r \le m$, we define the relatively compact domains $\Delta(\alpha_1, \ldots, \alpha_n)$ in $\mathbf{R}^n$:

$$\begin{aligned} &a_{\alpha_r} > x_r > a_{\alpha_r+1}, \\ &x_r - x_{r+1} > 0 \text{ or } < 0, \quad 1 \le r \le n, \end{aligned} \tag{2.26}$$

where $\alpha_r > \alpha_{r+1} + 1$ or $\alpha_r + 1 < \alpha_{r+1}$ according as $x_r - x_{r+1} > 0$ or < 0. The number of such domains is just equal to $m^{n-1}(m-1)$.

Proof. See [1].

3. Complete systems of difference and differential equations for $\tilde{\Phi}_m$

The relations (R'_r) are rewritten in terms of $\tilde{\Phi}_m$ as follows:

(3.1)
$$\begin{aligned} &\sum_{s=1}^{r} \sum_{\beta=1}^{m} \lambda_{s,\beta} \tilde{\Phi}_m \left(\lambda - \sum_{t=1}^{s-1} e_{t,i_t} - e_{s,\beta} - \sum_{t=s+1}^{r} e_{t,i_{t-1}} - \sum_{t=r+1}^{n} e_{t,i_t} \right) \\ &\quad + \sum_{s=r+1}^{n} \sum_{\beta=1}^{m} \lambda_{s,\beta} \tilde{\Phi}_m \left(\lambda - \sum_{t=1}^{r-1} e_{t,i_t} - \sum_{t=r}^{s-1} e_{t,i_{t+1}} - e_{s,\beta} - \sum_{t=s+1}^{n} e_{t,i_t} \right) \\ &\quad + \sum_{s=1}^{r-1} \lambda'_{s,s+1} \tilde{\Phi}_m \left(\lambda - \sum_{t=1}^{s-1} e_{t,i_t} - e_{s,i_s} - \sum_{t=s+1}^{r} e_{t,it-1} - \sum_{t=r+1}^{n} e_{t,i_t} \right) \\ &\quad + \sum_{s=r+1}^{n} \lambda'_{s-1,s} \tilde{\Phi}_m \left(\lambda - \sum_{t=1}^{r-1} e_{t,i_t} - \sum_{t=r}^{s-1} e_{t,i_{t+1}} - e_{s,i_s} - \sum_{t=s+1}^{n} e_{t,i_t} \right) \\ &\quad = 0. \end{aligned}$$

In addition to this, the partial fractions

$$\frac{1}{(m+r,\alpha)(m+r,\beta)} = \frac{1}{(\alpha,\beta)} \left\{ \frac{1}{(m+r,\alpha)} - \frac{1}{(m+r,\beta)} \right\}, \tag{3.2}$$

$$\begin{aligned} &\frac{1}{(m+r,\alpha)(m+r+1,\beta)(m+r+1,m+r)} \\ &= \frac{1}{(\alpha,\beta)} \left\{ \frac{1}{(m+r,\alpha)(m+r+1,m+r)} \right. \\ &\qquad + \frac{1}{(m+r,\beta)(m+r+1,\beta)} \\ &\qquad \left. - \frac{1}{(m+r,\beta)(m+r+1,m+r)} - \frac{1}{(m+r,\alpha)(m+r+1,\beta)} \right\} \end{aligned} \tag{3.3}$$

imply the following identities:

$$(R''_{r,\alpha,\beta})\quad \tilde{\Phi}_m(\lambda - e_{r,\alpha} - e_{r,\beta}) = \frac{1}{(\alpha,\beta)}\left\{\tilde{\Phi}_m(\lambda - e_{r,\alpha}) - \tilde{\Phi}_m(\lambda - e_{r,\beta})\right\},$$

for $1 \le r \le n$, $1 \le \alpha \ne \beta \le m$ and

$$\begin{aligned}&(R'''_{r,\alpha,\beta})\\ &\quad\tilde{\Phi}_m(\lambda - e_{r,\alpha} - e_{r+1,\beta} - e'_{r,r+1})\\ &\qquad = \frac{1}{(\alpha,\beta)}\left\{\tilde{\Phi}_m(\lambda - e_{r,\alpha} - e'_{r,r+1}) + \tilde{\Phi}_m(\lambda - e_{r,\beta} - e_{r+1,\beta})\right.\\ &\qquad\qquad \left. - \tilde{\Phi}_m(\lambda - e_{r,\beta} - e'_{r,r+1}) - \tilde{\Phi}_m(\lambda - e_{r,\alpha} - e_{r+1,\beta})\right\}.\end{aligned}$$

We can now get explicit formulae for difference operators $T_{-e_{r,\alpha}}$ and $T_{-e'_{r,r+1}}$ as follows:

PROPOSITION 1. (i) *If* $i_r \ne \alpha$,

$$T_{-e_{r,\alpha}}\langle \tilde{I}\rangle = \frac{1}{(i_r,\alpha)}(1 - P^r_{r,\alpha})\langle \tilde{I}\rangle, \tag{3.4}$$

(ii) *If* $i_r = \alpha$,

$$T_{-e_{r,\alpha}}\langle \tilde{I}\rangle = \sum_J \frac{B_r\left(\begin{matrix}J\\ I\end{matrix}\middle|\lambda - e_{r,\alpha}\right)}{(j_r,\alpha)}(1 - P^r_{r,\alpha})\langle \tilde{J}\rangle \tag{3.5}$$

where J *moves over admissible sequences such that* $j_r \ne \alpha$.

Proof. (3.4) follows from $(R''_{r,\alpha,\beta})$. (3.5) follows from (2.18) and (3.4).

PROPOSITION 2. (i) *If* $i_r \ne i_{r+1}$,

$$\begin{aligned}(i_r, i_{r+1})T_{-e'_{r,r+1}}\langle \tilde{I}\rangle &= (1 - \tilde{Q}^r_{r+1})\langle \tilde{I}\rangle \\ &\quad - \left(\sum_{s=1}^{r+1}\sum_{\beta=1}^{m}\lambda_{s,\beta}\tilde{P}^{r+1}_{s,\beta} + \sum_{s=1}^{r}\lambda'_{s,s+1}\tilde{Q}^{r+1}_s\right)\\ &\quad \times(1 - \tilde{Q}^r_{r+1})\langle \tilde{I}\rangle.\end{aligned} \tag{3.6}$$

(ii) *If* $i_r = i_{r+1}$,

(3.7)

$$T_{-e'_{r,r+1}}\langle\tilde{I}\rangle = \sum_J \frac{B_r\left(\begin{matrix}J\\I\end{matrix}\middle|\lambda - e'_{r,r+1}\right)}{(j_r, j_{r+1})}\left\{(1 - \tilde{Q}^r_{r,r+1})\langle\tilde{J}\rangle\right.$$
$$\left. - \left(\sum_{s=1}^{r+1}\sum_{\beta=1}^{m}\lambda_{s,\beta}\tilde{P}^{r+1}_{s,\beta} + \sum_{s=1}^{r}\lambda'_{s,s+1}\tilde{Q}^{r+1}_s\right)(1 - \tilde{Q}^r_{r+1})\langle\tilde{J}\rangle\right\}.$$

where J *moves over admissible sequences such that* $j_r \neq j_{r+1}$.

Proof. (3.6) follows from $(R'''_{r,\alpha,\beta})$ and $(\tilde{R}_r)$. (3.7) follows from (2.21) and (3.6). Summing up these two propositions we have the following result.

THEOREM 1. *The equations* (R'_r), $(R''_{r,\alpha,\beta})$ *and* $(R'''_{r,\alpha,\beta})$ *or equivalently* (3.4)–(3.7) *define holonomic linear difference equations for the meromorphic function* Φ_m *in* λ. *It is completely determined by the asymptotic behaviour for* $l \to \infty$, *for*

$$\lambda_{r,\alpha} = \lambda^{(0)}_{r,\alpha}l + \lambda^{(1)}_{r,\alpha} \quad \textit{and} \quad \lambda'_{r,r+1} = \lambda'^{(0)}_{r,r+1}l + \lambda'^{(1)}_{r,r+1}$$

where $\lambda^{(0)}_{r,\alpha}, \lambda'^{(0)}_{r,r+1} \in \mathbf{R}^+$ *and* $\lambda^{(1)}_{r,\alpha}, \lambda'^{(1)}_{r,r+1} \in \mathbf{C}$:

$$\tilde{\Phi}_m(\lambda) \sim \Phi_m(c)\frac{(l/2)^{-n/2}}{\sqrt{\operatorname{Hess}\Phi_m(c)}}\left\{1 + O\left(\frac{1}{l}\right)\right\}. \tag{3.8}$$

$c = (c_1, \ldots, c_n) \in \mathbf{R}^n$ *is, by saddle point method, uniquely determined as the solution of the equations*

(3.9)

$$\sum_{r=1}^{n}\sum_{\alpha=1}^{m}\lambda^{(0)}_{r,\alpha}d\log(m + r, \alpha) + \sum_{r=1}^{n-1}\lambda'^{(0)}_{r,r+1}d\log(m + r, m + r + 1) = 0$$

at $x = c$ *such that* $0 \leq c_r \leq 1$.

Proof. The computation of asymptotic behaviours is standard. We have only to find the critical points of the function $\operatorname{Re}\log\tilde{\Phi}_m$ for $\lambda_{r,\alpha} = \lambda^{(0)}_{r,\alpha}$ and $\lambda'_{r,r+1} = \lambda'^{(0)}_{r,r+1}$ respectively. See [1] for details.

Now let us compute integrable differential equations for $\tilde{\Phi}_m$ in the variables $a_1, \ldots, a_m$ for $a_\alpha \neq a_\beta$, $\alpha \neq \beta$. Since $\tilde{\Phi}_m$ is a special case of the integral (0.1) in [3], we can apply the formula (0.3) loc. cit. to the $\langle\tilde{I}\rangle = \tilde{\Phi}_m(\lambda - e_{1,i_1}$

$- \cdots - e_{n,i_n})$, use the relations (R_r) and get the following *differential system* (*Gauss-Manin connection*):

$$(\mathscr{E})\quad d\langle\tilde{I}\rangle = \sum_{r=1}^{n}\sum_{\substack{\alpha=1\\ \alpha\neq i_r}}^{m}\lambda_{r,\alpha}d\log(i_r,\alpha)\left(1-\tilde{P}^r_{r,\alpha}\right)\langle\tilde{I}\rangle$$
$$+\sum_{r=1}^{n-1}d\log(i_r,i_{r+1})\left\{\lambda'_{r,r+1}\left(\tilde{Q}^r_r-\tilde{Q}^r_{r+1}\right)\right.$$
$$\left.+\sum_{s=1}^{r}\sum_{\alpha=1}^{m}\lambda_{s,\alpha}\left(\tilde{P}^s_{r,\alpha}-\tilde{P}^s_{r+1,\alpha}\right)\right\}\langle\tilde{I}\rangle$$

where the terms $d\log(i_r, i_{r+1})\{\cdots\}$ *vanish for* $i_r = i_{r+1}$ *in the right hand side.*

4. Case where $m = 2$

In this part we put $a_1 = 0$, and $a_2 = 1$ and consider a slightly modified form of the integral (2.3) as follows:

$$(4.1)\qquad \tilde{\Phi}_2^{(0)}(\lambda_1,\ldots,\lambda_n|x,y) = \int_{x\geq x_1\geq x_2\geq\cdots\geq x_n\geq y}\Phi_2^{(0)}(\lambda|1,0)\,dx_1\wedge\cdots\wedge dx_n$$

for $1 > x > y > 0$ where $\Phi_2^{(0)}$ denotes the function Φ_2 restricted to the subspace $\lambda'_{1,2} = \cdots = \lambda'_{n-1,n} = 0$. We abbreviate $\tilde{\Phi}_2^{(0)}(\lambda|1,0)$ simply by $\tilde{\Phi}_2^{(0)}(\lambda_1,\ldots,\lambda_n)$ or $\tilde{\Phi}_2^{(0)}(\lambda)$. $\partial_r\langle I\rangle$ denotes the deletion of the r-th element i_r: $\partial_r\langle I\rangle = \langle i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_n\rangle$. Then $\tilde{\Phi}_2^{(0)}(\lambda)$ satisfies the difference equations in $\lambda_1,\lambda_2,\ldots,\lambda_n$ as follows.

LEMMA 4.1.

$$(4.2)\qquad \left[(-1)^r\partial_{r+1}\langle\tilde{I}\rangle\right]_{x_r=x_{r+1}} = \sum_{s=1}^{r+1}\sum_{\alpha=1}^{2}\lambda_{s,\alpha}P^{r+1}_{s,\alpha}\langle\tilde{I}\rangle,$$

$$(4.3)\qquad \sum_{s=1}^{n}\sum_{\beta=1}^{2}\lambda_{s,\beta}P^r_{s,\beta}\langle\tilde{I}\rangle = 0.$$

Proof. Both are immediate consequences of (R_r) and (R'_r) in §2 respectively, taking the limit $\lambda'_{s,s+1} = 0$ for all s. The left hand side of (4.2) comes from the residue in (R_r) along the subspace $x_r = x_{r+1}$.

COROLLARY. *We can choose as a basis of* $H^n(X,\nabla_\omega)$ *admissible* $\langle I\rangle$ *such that* $i_n = 1$. *Hence* $\dim H^n(X,\nabla_\omega) = 2^{n-1}$.

The equations ($\mathscr{E}$) become

(4.4)

$$d\langle\tilde{I}\rangle = \sum_{r=1}^{n} \sum_{\substack{\alpha=1 \\ \alpha\neq i_r}}^{2} \lambda_{r,\alpha} d\log(i_r, \alpha)\left(1 - \tilde{P}^r_{r,\alpha}\right)\langle\tilde{I}\rangle + d\log(x - a_{i_1})\left[\partial_1\langle\tilde{I}\rangle\right]_{x_1=x}$$
$$+ \sum_{r=1}^{n-1} d\log(i_r, i_{r+1})\left[(\partial_{r+1} - \partial_r)\langle\tilde{I}\rangle\right]_{x_r=x_{r+1}}$$
$$+ d\log(a_{i_n} - y)\left[\partial_n\langle\tilde{I}\rangle\right]_{x_n=y},$$

where the terms $d\log(i_r, i_{r+1})\{\cdots\}$ vanish for $i_r = i_{r+1}$.

5. Iterated integrals of logarithmic 1-forms

We put again $\omega_r = d\log(x - a_r)$ and assume $a_1 = 0$, $a_2 = 1$ or $a_1 = 1$, $a_2 = 0$.

We are specially interested in the following iterated integrals (hyperlogarithms of Gaussian type in [19]):

DEFINITION 3. For $\omega_1 = d\log x$ and $\omega_2 = d\log(x-1)$, we put

$$L_1^2(r_1, \ldots, r_{2m}|x, y) = \int_y^x \omega_1^{r_1}|\omega_2^{r_2}|\cdots|\omega_2^{r_{2m}}, \tag{5.1}$$

$$L_1^1(r_1, \ldots, r_{2m-1}|x, y) = \int_y^x \omega_1^{r_1}|\omega_2^{r_2}|\cdots|\omega_1^{r_{2m-1}}, \tag{5.2}$$

$$L_2^1(r_1, \ldots, r_{2m}|x, y) = \int_y^x \omega_2^{r_1}|\omega_1^{r_2}|\cdots|\omega_1^{r_{2m}}, \tag{5.3}$$

$$L_2^2(r_1, \ldots, r_{2m-1}|x, y) = \int_y^x \omega_2^{r_1}|\omega_1^{r_2}|\cdots|\omega_2^{r_{2m-1}}, \tag{5.4}$$

respectively.

For $(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$ consider the following generating functions

$$F_n(z_1, \ldots, z_n|a_1, \ldots, a_n; x, y) = \sum_{r_1,\ldots,r_n\geq 0} z_1^{r_1}z_2^{r_2}\cdots z_n^{r_n}\int_y^x \omega_1^{r_1}|\cdots|\omega_n^{r_n}, \tag{5.5}$$

$$F_n^*(z_1, \ldots, z_n|a_1, \ldots, a_n; x, y) = \sum_{r_1,\ldots,r_n\geq 1} z_1^{r_1}z_2^{r_2}\cdots z_n^{r_n}\int_y^x \omega_1^{r_1}|\cdots|\omega_n^{r_n}, \tag{5.6}$$

where, we assume, x, y and $a_1, \ldots, a_n$ are all different from each other. We put $F_0^* = 1$. The F_n are meromorphic functions in $z_1, \ldots, z_n$ and many valued analytic functions in $x, y, a_1, \ldots, a_n$ on the affine variety $X_{n+2} = \{(x, y, a_1, \ldots, a_n) \in \mathbf{C}^{n+2};\ x, y, a_1, \ldots, a_n$ are different from each other$\}$. In particular we have

$$(5.7) \qquad F_1(z_j|a_j;\, x, y) = \sum z_j^r \int_y^x \omega_j^r = \exp\left(z_j \int_y^x \omega_j \right) = \left(\frac{x - a_j}{y - a_j} \right)^{z_j}.$$

We abbreviate this function simply by $E_j(z_j)$.

By definition, $F_n(z_1, \ldots, z_n|x, y)$ is equal to the sum

$$(5.8) \qquad \sum_{r=0}^{\infty} \sum_{1 \le \nu_1 < \cdots < \nu_r \le n} F_r^*(z_{\nu_1}, \ldots, z_{\nu_r}|a_{\nu_1}, \ldots, a_{\nu_r};\, x, y)$$

To get an integral representation of F_n, we first derive a recurrent system of differential relations for F_n.

LEMMA 5.1.

$$(5.9) \qquad F_{n-j+1}(z_j, \ldots, z_n|a_j, \ldots, a_n;\, x, y) = 1 + E_j(z_j) \int_y^x E_j^{-1}(z_j)\, dF_{n-j}(z_{j+1}, \ldots, z_n|a_{j+1}, \ldots, a_n;\, x, y)$$

Proof. In fact by Chen's formula F_{n-j+1} satisfies the equation

$$(5.10) \qquad \frac{dF_{n-j+1}}{dx} = \frac{z_j}{x - a_j} F_{n-j+1} + \frac{dF_{n-j}}{dx}.$$

with the initial condition $[F_{n-j+1}]_{x=y} = 1$. This can be solved uniquely as in (5.9).

Lemma 5.1 shows

$$(5.11) \qquad dF_{n-j+1} = dE_j \int_y^x E_j^{-1}\, dF_{n-j} + dF_{n-j},$$

or equivalently

$$dF_{n-j+1} = dE_j|E_j^{-1}\, dF_{n-j} + dF_{n-j}. \tag{5.12}$$

By repeating this procedure we can prove the following identity in the Chen algebra of differential 1-forms.

LEMMA 5.2.

$$\begin{aligned} &dF_n(z_1,\ldots,z_n|a_1,\ldots,a_n;x,y) \\ &= \sum_{r=1}^{n} \sum_{1\le\nu_1<\cdots<\nu_r\le n} dE_{\nu_1}|E_{\nu_1}^{-1}\, dE_{\nu_2}|\cdots|E_{\nu_{r-1}}^{-1}\, dE_{\nu_r}. \end{aligned} \tag{5.13}$$

Hence we have:

PROPOSITION 3.

$$\begin{aligned} &F_n(z_1,\ldots,z_n|a_1,\ldots,a_n;x,y) \\ &= 1 + \sum_{r=1}^{n} \sum_{1\le\nu_1<\cdots<\nu_r\le n} \int_y^x dE_{\nu_1}(z_{\nu_1})|E_{\nu_1}^{-1}(z_{\nu_1})\, dE_{\nu_2}(z_{\nu_2}) \\ &\quad |\cdots|E_{\nu_{r-1}}^{-1}(z_{\nu_{r-1}})\, dE_{\nu_r}(z_{\nu_r}), \end{aligned} \tag{5.14}$$

$$\begin{aligned} &F_n^*(z_1,\ldots,z_n|a_1,\ldots,a_n;x,y) \\ &= \int_y^x dE_1(z_1)|E_1^{-1}(z_1)\, dE_2(z_2)|\cdots|E_{n-1}^{-1}(z_{n-1})\, dE_n(z_n) \\ &= \sum_{r=1}^{n}(-1)^{r-1}E_r(z_r)\int_y^x E_r^{-1}(z_r)\, dE_{r+1}(z_{r+1})|\cdots \\ &\quad |E_{n-1}^{-1}(z_{n-1})\, dE_n(z_n) + (-1)^n. \end{aligned} \tag{5.15}$$

The last equality follows by partial integration or Chen's product formula.

We denote by

$$F_{2,1}^{*2}(z_1,\ldots,z_{2m}|x,y),\quad F_{2,1}^{*1}(z_1,\ldots,z_{2m-1}|x,y),$$
$$F_{2,2}^{*1}(z_1,\ldots,z_{2m}|x,y),\quad F_{2,2}^{*2}(z_1,\ldots,z_{2m-1}|x,y),$$

the generating functions for the values

$$L_1^2(r_1,\ldots,r_{2m}|x,y),\quad L_1^1(r_1,\ldots,r_{2m-1}|x,y),$$
$$L_2^1(r_1,\ldots,r_{2m}|x,y),\quad L_2^2(r_1,\ldots,r_{2m-1}|x,y),$$

respectively. These are special cases of $F_n^*(z_1,\ldots,z_n|x,y)$, where $a_1 = a_3 = \cdots = 0$ or 1 and $a_2 = a_4 = \cdots = 1$ or 0.

In this circumstance it is also convenient to give the following:

DEFINITION 4. The functions

$$\psi_1^2(z_1,\dots,z_{2m}|x,y),\quad \psi_1^1(z_1,\dots,z_{2m-1}|x,y),$$
$$\psi_2^1(z_1,\dots,z_{2m}|x,y)\quad\text{and}\quad \psi_1^2(z_1,\dots,z_{2m-1}|x,y)$$

will mean the integrals of the right hand side of (5.14). $a_1 = a_3 = \cdots = 0$, $a_2 = a_4 = \cdots = 1$ correspond to ψ_1^2 and ψ_1^1, while $a_1 = a_3 = \cdots = 1$, $a_2 = a_4 = \cdots = 0$ correspond to ψ_2^1 and ψ_2^2. We also denote by $\psi_j^i(z_1,\dots,z_n)$ the values of $\psi_j^i(z_1,\dots,z_n|1,0)$ at $(x,y)=(1,0)$ respectively.

Then from (4.1) and Proposition 3, by means of partial integration we have:

LEMMA 5.3.

$$(5.16)\quad \psi_1^2(z_1,\dots,z_{2m}|x,y)$$
$$= (-1)^m z_1\cdots z_{2m}\tilde{\Phi}_2^{(0)}\left(\begin{matrix} z_1-1, & -z_1,\dots,-z_{2m-1}\\ 0, & z_2-1,\dots,z_{2m}-1\end{matrix}\middle| x,y\right)$$
$$= (1-y)^{-z_{2m}}\Bigg\{\sum_{r=1}^{m}(-1)^{m-r+1}x^{z_{2r-1}}z_{2r}z_{2r+1}\cdots z_{2m}$$
$$\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix} -z_{2r-1}, & z_{2r+1}-1,\dots,-z_{2m-1}\\ z_{2r}-1, & -z_{2r},\dots,z_{2m}-1\end{matrix}\middle| x,y\right)$$
$$+\sum_{r=1}^{m}(-1)^{m-r+1}(1-x)^{z_{2r}}z_{2r+1}\cdots z_{2m}$$
$$\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix} z_{2r+1}-1, & -z_{2r+1},\dots,-z_{2m-1}\\ -z_{2r}, & z_{2r+2}-1,\dots,z_{2m}-1\end{matrix}\middle| x,y\right)\Bigg\}+1.$$

$$(5.17)\quad \psi_1^1(z_1,\dots,z_{2m-1}|x,y)$$
$$= (-1)^{m-1}z_1\cdots z_{2m-1}$$
$$\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix} z_1-1, & -z_1,\dots,z_{2m-1}-1\\ 0, & z_2-1,\dots,-z_{2m-2}\end{matrix}\middle| x,y\right)$$
$$= y^{-\lambda_{2m-1}}\Bigg\{\sum_{r=1}^{m}(-1)^{m-r}x^{z_{2r-1}}z_{2r}\cdots z_{2m-1}$$
$$\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix} -z_{2r-1},\dots,z_{2m-1}-1\\ z_{2r}-1,\dots,-z_{2m-2}\end{matrix}\middle| x,y\right)$$
$$+\sum_{r=1}^{m-1}(-1)^{m-r}(1-x)^{z_{2r}}z_{2r+1}\cdots z_{2m-1}$$
$$\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix} z_{2r+1}-1,\dots,z_{2m-1}-1\\ -z_{2r},\dots,-z_{2m-2}\end{matrix}\middle| x,y\right)\Bigg\}-1.$$

(5.18)

$$\begin{aligned}
&\psi_2^1(z_1,\dots,z_{2m}|x,y)\\
&= (-1)^m z_1\cdots z_{2m}\tilde{\Phi}_2^{(0)}\left(\begin{matrix}0 & z_2-1,\dots,z_{2m}-1\\ z_1-1, & -z_1,\dots,-z_{2m-1}\end{matrix}\middle| x,y\right)\\
&= y^{-z_{2m}}\left\{\sum_{r=1}^{m}(-1)^{m-r}(1-x)^{z_{2r-1}}z_{2r}\cdots z_{2m}\right.\\
&\qquad\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix}z_{2r}-1,\dots,z_{2m}-1\\ -z_{2r-1},\dots,-z_{2m-2}\end{matrix}\middle| x,y\right)\\
&\qquad+\sum_{r=1}^{m}(-1)^{m-r}x^{z_{2r}}z_{2r+1}\cdots z_{2m}\\
&\qquad\left.\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix}-z_{2r},\dots,z_{2m}-1\\ z_{2r+1}-1,\dots,-z_{2m-1}\end{matrix}\middle| x,y\right)\right\}+1.
\end{aligned}$$

(5.19)

$$\begin{aligned}
&\psi_2^2(z_1,\dots,z_{2m-1}|x,y)\\
&= (-1)^{m-1}z_1\cdots z_{2m-1}\\
&\qquad\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix}0, & z_2-1,\dots,-z_{2m-2}\\ z_1-1, & -z_1,\dots,z_{2m-1}-1\end{matrix}\middle| x,y\right)\\
&= (1-y)^{-z_{2m-1}}\left\{\sum_{r=1}^{m}(-1)^{m-r}(1-x)^{z_{2r-1}}z_{2r}\cdots z_{2m-1}\right.\\
&\qquad\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix}z_{2r}-1,\dots,-z_{2m-2}\\ -z_{2r-1},\dots,z_{2m-1}-1\end{matrix}\middle| x,y\right)\\
&\qquad+\sum_{r=1}^{m-1}(-1)^{m-r}x^{z_{2r}}z_{2r+1}\cdots z_{2m-1}\\
&\qquad\left.\cdot\tilde{\Phi}_2^{(0)}\left(\begin{matrix}-z_{2r},\dots,-z_{2m-2}\\ z_{2r+1}-1,\dots,z_{2m-1}-1\end{matrix}\middle| x,y\right)\right\}-1.
\end{aligned}$$

The following lemma is an elementary consequence of integrals $\tilde{\Phi}_2^{(0)}(\lambda_1,\dots,\lambda_n|x,y)$. So we omit a proof.

LEMMA 5.4. *The functions $\psi_j^i(z_1,\dots,z_n)$ are all holomorphic at $z_1=\cdots=z_n=0$.*

But there appear singularities of L^i_j at $(x, y) = (1,0)$. These are described as simple logarithmic expansions:

LEMMA 5.5. *For* $r_1, \dots, r_n \geq 1$, $L^2_1(r_1, \dots, r_{2m}|x, y)$ *is holomorphic at* $(x, y) = (1,0)$; *i.e., if we put* $L^2_1(r_1, \dots, r_{2m}|x, y) = A^{(0,0)}(x, y)$, *then*

$$A^{(0,0)}(1,0) = \int_0^1 \omega_1^{r_1}| \cdots |\omega_{2m}^{r_{2m}}. \tag{5.20}$$

The other L^i_j *have expansions at* $(x, y) = (1,0)$ *as follows*:

$$L^1_1 = A^{(0,0)}(x, y) + \sum_{k=0}^{r} A^{(0,l)}(x, y)(\log y)^l, \tag{5.21}$$

$$L^1_2 = \sum_{k=l=0}^{r} A^{(k,l)}(x, y)(\log(1-x))^k(\log y)^l, \tag{5.22}$$

$$L^2_2 = \sum_{k=0}^{r} A^{(k,0)}(x, y)(\log(1-x))^k, \tag{5.23}$$

where the functions $A^{(k,l)}$ *are all holomorphic at* $(1,0)$.

In all the 4 cases $A^{(0,0)}((1,0))$ has definite values. These values are defined to be *regularized ones* of L^i_j and are written as reg $\int_0^1 \omega_1^{r_1}| \cdots |\omega_2^{r_{2m}}$ which is equal to

$$\int_0^1 \omega_1^{r_1}| \cdots |\omega_2^{r_{2m}}, \quad \text{reg} \int_0^1 \omega_1^{r_1}| \cdots |\omega_1^{r_{2m-1}},$$

$$\text{reg} \int_0^1 \omega_2^{r_1}| \cdots |\omega_1^{r_{2m}} \quad \text{and} \quad \text{reg} \int_0^1 \omega_2^{r_1}| \cdots |\omega_2^{r_{2m-1}}$$

respectively. These are nothing else than the finite part of divergent integrals in the sense of Hadamard-Leray.

LEMMA 5.6 . *The functions* ψ^i_j *have the unique expansions in* $z_1, \dots, z_n$ *and* $\log(1-x)$, $\log y$ *as follows*:

$$\psi^i_j = \sum_{r_1, r_2, \dots, r_{2m} \geq 1} \sum_{\substack{k, l=0 \\ 0 \leq k+l \leq r_1 + \cdots + r_n}} A^{(k,l)}_{r_1, \dots, r_n}(x, y) \times z_1^{r_1} \cdots z_n^{r_n}(\log(1-x))^k \cdot (\log y)^l \tag{5.24}$$

where $A^{(k,l)}_{r_1, \dots, r_n}(x, y)$ *are holomorphic at* $(x, y) = (1,0)$.

Proof. This follows from the series expansions in $z_1, \ldots, z_n$ of the integral representations (5.16)–(5.17) by using the Taylor expansions of $(1-x)^s$ and y^s in s.

Hence:

LEMMA 5.7.

$$\psi_j^i(z_1, \ldots, z_n) = \sum_{r_1, \ldots, r_n \geq 1} z_1^{r_1} \cdots z_n^{r_n} A^{(0,0)}_{r_1, \ldots, r_n}(1,0). \tag{5.25}$$

Summing up the above, we have proved the basic formulae:

PROPOSITION 4.

$$\psi_1^2(z_1, \ldots, z_n) = \int_0^1 d(x^{z_1})|x^{-z_1} d(1-x)^{z_2}| \cdots |x^{-z_{2m-1}} d(1-x)^{z_{2m}}, \tag{5.26}$$

$$\psi_1^1(z_1, \ldots, z_n) = \int_0^1 d(x^{z_1})|x^{-z_1} d(1-x)^{z_2}| \cdots |(1-x)^{-z_{2m-2}} d(x^{z_{2m-1}}), \tag{5.27}$$

$$\psi_2^1(z_1, \ldots, z_n) = \int_0^1 d(1-x)^{z_1}|(1-x)^{-z_1} d(x^{z_2})| \cdots |(1-x)^{-z_{2m-1}} d(x^{z_{2m}}), \tag{5.28}$$

$$\psi_2^2(z_1, \ldots, z_n) = \int_0^1 d(1-x)^{z_1}|(1-x)^{-z_1} d(x^{z_2})| \cdots |x^{-z_{2m-2}} d(1-x)^{z_{2m-1}}. \tag{5.29}$$

These functions are specializations of $\tilde{\Phi}_m$ satisfying the difference equations (3.4)–(3.7). Finally we can state the main result.

THEOREM 2. *The four functions $\psi_j^i(z_1, \ldots, z_n)$ are all holomorphic at the origin. Special values of hyperlogarithms $L_{r_1, r_2, r_3, \cdots}$ and $L'_{r_1, r_2, r_3, \cdots}$ coincide with* *coefficients of the Taylor expansions of* ψ^i_j *at the origin*:

(5.30)
$$\int_0^1 \omega_1^{r_1}|\omega_2^{r_2}|\cdots|\omega_2^{r_{2m}} = \frac{1}{r_1!\,r_2!\cdots r_{2m}!}\frac{\partial^{r_1+\cdots+r_{2m}}}{(\partial z_1)^{r_1}\cdots(\partial z_{2m})^{r_{2m}}}\psi_1^2(0,\ldots,0),$$

(5.31)
$$\operatorname{reg}\int_0^1 \omega_1^{r_1}|\omega_2^{r_2}|\cdots|\omega_1^{r_{2m-1}} = \frac{1}{r_1!\,r_2!\cdots r_{2m-1}!}\frac{\partial^{r_1+\cdots+r_{2m-1}}}{(\partial z_1)^{r_1}\cdots(\partial z_{2m-1})^{r_{2m-1}}}\psi_1^1(0,\ldots,0),$$

(5.32)
$$\operatorname{reg}\int_0^1 \omega_2^{r_1}|\omega_1^{r_2}|\cdots|\omega_1^{r_{2m}} = \frac{1}{r_1!\,r_2!\cdots r_{2m}!}\frac{\partial^{r_1+\cdots+r_{2m}}}{(\partial z_1)^{r_1}\cdots(\partial z_{2m})^{r_{2m}}}\psi_2^1(0,\ldots,0),$$

(5.33)
$$\operatorname{reg}\int_0^1 \omega_2^{r_1}|\omega_1^{r_2}|\cdots|\omega_2^{r_{2m-1}} = \frac{1}{r_1!\,r_2!\cdots r_{2m-1}!}\frac{\partial^{r_1+\cdots+r_{2m-1}}}{(\partial z_1)^{r_1}\cdots(\partial z_{2m-1})^{r_{2m-1}}}\psi_2^2(0,\ldots,0),$$

for $n = 2m,\ 2m-1,\ 2m$ *and* $2m-1$ *respectively which are exactly the identities* (1.4) *stated in the introduction.*

6. Examples

(1) *Case* $n = 2$. Since

(6.1)
$$\psi_1^2(z_1, z_2) = \int_0^1 d(x^{z_1})|x^{-z_1}d(1-x)^{z_2} = \frac{\Gamma(1-z_1)\Gamma(1+z_2)}{\Gamma(1-z_1+z_2)} - 1.$$

Theorem 2 implies the formula (1.2).

(2) *Case* $n = 3$. $\psi_1^1(z_1, z_2, z_3)$ and $\psi_2^2(z_1, z_2, z_3)$ are equal to

(6.2)
$$- z_2 z_3 \tilde{\Phi}_2^{(0)}\begin{pmatrix} -z_1, & z_3 - 1 \\ z_2 - 1, & -z_2 \end{pmatrix} = -z_2 z_3 \int_{1 \geq x_1 \geq x_2 \geq 0} x_1^{-z_1} x_2^{z_3 - 1}(1 - x_1)^{z_2 - 1}(1 - x_2)^{-z_2}\, dx_1\, dx_2$$

and

$$(6.3) \qquad - z_3 \tilde{\Phi}_2^{(0)}\begin{pmatrix} -z_2 \\ z_3 - 1 \end{pmatrix} = \frac{\Gamma(1 - z_2)\Gamma(1 + z_3)}{\Gamma(1 - z_2 + z_3)} - 1$$

respectively.

$\tilde{\Phi}_2(\lambda_1, \lambda_2; \lambda'_{1,2})$ coincides with the value

$$\mathscr{F}\begin{pmatrix} \lambda_{1,1}, & \lambda_{2,1} & \\ \lambda_{1,2}, & \lambda'_{1,2}, & \lambda_{2,2} \end{pmatrix}$$

in [4] which is also equal to the value at $z = 1$ of *the Goursat-Thomae-Whipple hypergeometric function* ${}_3F_2(z)$ apart from a Γ-factor; more exactly it is equal to

$$(6.4) \quad \frac{\Gamma(\lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2} + 2)\Gamma(\lambda_{2,1} + 1)\Gamma(\lambda_{1,2} + 1)\Gamma(\lambda'_{1,2} + 1)}{\Gamma(\lambda_{1,1} + \lambda_{1,2} + \lambda_{2,1} + \lambda'_{1,2} + 3)\Gamma(\lambda_{2,1} + \lambda'_{1,2} + 2)} \cdot {}_3F_2\begin{pmatrix} \lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2} + 2, & \lambda_{2,1} + 1, & -\lambda_{2,2} \\ \lambda_{1,1} + \lambda_{2,1} + \lambda_{1,2} + \lambda'_{1,2} + 3, & \lambda_{2,1} + \lambda'_{1,2} + 2 & \end{pmatrix}$$

The coefficients

$$B_1\begin{pmatrix} 2, & j_2 \\ 1, & i_2 \end{pmatrix} \quad \text{and} \quad B'_1\begin{pmatrix} j_1, & j_2 \\ 1, & 1 \end{pmatrix}$$

for $\tilde{\Phi}_2^{(0)}$ are given by

$$(6.5) \quad \lambda_{1,1}(\lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2})\langle\widetilde{1,1}\rangle = (\lambda_{2,1}\lambda_{2,2} - \lambda_{1,1}\lambda_{1,2})\langle\widetilde{2,1}\rangle + \lambda_{2,2}(\lambda_{1,2} + \lambda_{2,2} + \lambda'_{1,2})\langle\widetilde{2,2}\rangle,$$

$$(6.6) \quad - \lambda_{1,1}\langle\widetilde{1,2}\rangle = \lambda_{2,1}\langle\widetilde{2,1}\rangle + (\lambda_{1,2} + \lambda_{2,2} + \lambda'_{1,2})\langle\widetilde{2,2}\rangle,$$

$$(6.7) \quad - (\lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2})\langle\widetilde{1,1}\rangle = \lambda_{1,2}\langle\widetilde{2,1}\rangle + \lambda_{2,2}\langle\widetilde{1,2}\rangle,$$

$$(6.8) \quad \lambda'_{1,2}\langle\widetilde{1,3}\rangle = (\lambda_{1,1} + \lambda'_{1,2})\langle\widetilde{1,1}\rangle + \lambda_{1,2}\langle\widetilde{2,1}\rangle,$$

The other coefficients

$$B_r\begin{pmatrix} j_1, & j_2 \\ i_1, & i_2 \end{pmatrix}, \quad r = 1,2$$

and

$$B'_{1,2}\begin{pmatrix} j_1, & j_2 \\ i, & i \end{pmatrix}$$

are evaluated by the symmetry properties (2.22)–(2.25).

We can take as a basis the integrals $\widetilde{\langle 1,1\rangle}$ and $\widetilde{\langle 2,1\rangle}$ (Corollary of Lemma 4.1).

The difference equations with respect to this basis are

$$T_{-e_{1,1}}\widetilde{\langle 2,1\rangle} = \widetilde{\langle 2,1\rangle} - \widetilde{\langle 1,1\rangle}, \tag{6.9}$$

$$\begin{aligned}(\lambda_{1,1} - 1)(\lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2} - 1)T_{-e_{1,1}}\widetilde{\langle 1,1\rangle} &= (\lambda_{2,1}\lambda_{2,2} - \lambda_{1,1}\lambda_{1,2} + \lambda_{1,2})\big(\widetilde{\langle 2,1\rangle} - \widetilde{\langle 1,1\rangle}\big)\\ &\quad + \lambda_{2,2}(\lambda_{1,2} + \lambda_{2,2} + \lambda'_{1,2})\big(\widetilde{\langle 2,2\rangle} - \widetilde{\langle 1,2\rangle}\big),\end{aligned} \tag{6.10}$$

$$T_{-e_{1,2}}\widetilde{\langle 1,1\rangle} = \widetilde{\langle 2,1\rangle} - \widetilde{\langle 1,1\rangle}, \tag{6.11}$$

$$\begin{aligned}&-(\lambda_{1,2} - 1)T_{-e_{1,2}}\widetilde{\langle 2,1\rangle}\\ &\quad = (\lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2})\big(\widetilde{\langle 2,1\rangle} - \widetilde{\langle 1,1\rangle}\big) + \lambda_{2,2}\big(\widetilde{\langle 2,2\rangle} - \widetilde{\langle 1,2\rangle}\big),\end{aligned} \tag{6.12}$$

$$\begin{aligned}(\lambda_{2,1} - 1)(\lambda_{1,1} + \lambda_{2,1} + \lambda'_{1,2} - 1)T_{-e_{2,1}}\widetilde{\langle 1,1\rangle} &= \lambda_{1,2}(\lambda_{1,2} + \lambda_{2,2} + \lambda'_{1,2})\big(\widetilde{\langle 2,2\rangle} - \widetilde{\langle 2,1\rangle}\big)\\ &\quad + (\lambda_{1,2}\lambda_{1,1} - \lambda_{2,1}\lambda_{2,2} + \lambda_{2,2})\big(\widetilde{\langle 1,2\rangle} - \widetilde{\langle 1,1\rangle}\big),\end{aligned} \tag{6.13}$$

$$\begin{aligned}&-(\lambda_{2,1} - 1)T_{-e_{2,1}}\widetilde{\langle 2,1\rangle}\\ &\quad = \lambda_{1,1}\big(\widetilde{\langle 1,2\rangle} - \widetilde{\langle 1,1\rangle}\big) + (\lambda_{1,2} + \lambda_{2,2} + \lambda'_{1,2})\big(\widetilde{\langle 2,2\rangle} - \widetilde{\langle 2,1\rangle}\big).\end{aligned} \tag{6.14}$$

$$T_{-e_{2,2}}\widetilde{\langle 1,1\rangle} = \widetilde{\langle 1,2\rangle} - \widetilde{\langle 1,1\rangle}, \tag{6.15}$$

$$T_{-e_{2,2}}\widetilde{\langle 2,1\rangle} = \widetilde{\langle 2,2\rangle} - \widetilde{\langle 2,1\rangle}, \tag{6.16}$$

Finally, using (3.2) and (3.3), we have

$$(6.17)\qquad \begin{aligned} -(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2}-1)T_{-e'_{1,2}}\widetilde{\langle 1,1\rangle} &= \lambda_{1,2}\big(\widetilde{\langle 2,1\rangle}-\widetilde{\langle 1,1\rangle}-\widetilde{\langle 2,3\rangle}+\widetilde{\langle 1,3\rangle}\big)\\ &\quad+\lambda_{2,2}\big(\widetilde{\langle 1,2\rangle}-\widetilde{\langle 2,2\rangle}-\widetilde{\langle 1,3\rangle}+\widetilde{\langle 2,3\rangle}\big),\end{aligned}$$

$$(6.18)\qquad T_{-e'_{1,2}}\widetilde{\langle 2,1\rangle}=\widetilde{\langle 2,1\rangle}-\widetilde{\langle 1,1\rangle}-\widetilde{\langle 2,3\rangle}-\widetilde{\langle 1,3\rangle}.$$

(2) *Case* $n=4$. ψ_1^2 and ψ_2^1 are expressed by means of $\tilde{\Phi}_2^{(0)}(z_1, z_2, z_3)$ which is no more hypergeometric. We only give the formulae for

$$B_1\begin{pmatrix}2, & j_2, & j_3\\ 1, & i_2, & i_3\end{pmatrix}\quad\text{and}\quad B'_{1,2}\begin{pmatrix}j_1, & j_2, & j_3\\ 1, & 1, & i_3\end{pmatrix}.$$

The other B_r and $B'_{r,r+1}$ are obtained according to (2.22)–(2.25):

$$(6.19)\qquad \begin{aligned}-\lambda_{1,1}\widetilde{\langle 1,2,2\rangle} &= \lambda_{2,1}\widetilde{\langle 2,1,2\rangle}+\lambda_{3,1}\widetilde{\langle 2,2,1\rangle}\\ &\quad+(\lambda_{1,2}+\lambda_{2,2}+\lambda_{3,2}+\lambda'_{1,2}+\lambda'_{2,3})\widetilde{\langle 2,2,2\rangle},\end{aligned}$$

$$(6.20)\qquad \begin{aligned}-\lambda_{1,1}\widetilde{\langle 1,2,1\rangle} &= (\lambda_{2,1}+\lambda_{3,1}+\lambda'_{2,3})\widetilde{\langle 2,1,1\rangle}+\lambda_{3,2}\widetilde{\langle 2,1,2\rangle}\\ &\quad+(\lambda_{1,2}+\lambda_{2,2}+\lambda'_{1,2})\widetilde{\langle 1,1,2\rangle},\end{aligned}$$

(6.21)

$$\begin{aligned}&\lambda_{1,1}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2})\widetilde{\langle 1,1,2\rangle}\\ &\quad=\lambda_{3,1}(\lambda_{2,1}+\lambda_{3,1}+\lambda'_{2,3})\widetilde{\langle 2,1,1\rangle}\\ &\qquad+[\lambda_{3,1}\lambda_{3,2}-\lambda_{1,1}\lambda_{1,2}+\lambda_{2,1}(\lambda_{2,2}+\lambda_{3,2}+\lambda'_{2,3})]\widetilde{\langle 2,1,2\rangle}\\ &\qquad+\lambda_{3,1}(\lambda_{1,2}+2\lambda_{2,2}+\lambda_{3,2}+\lambda'_{1,2}+\lambda'_{2,3})\widetilde{\langle 2,2,1\rangle}\\ &\qquad+(\lambda_{2,2}+\lambda_{3,2}+\lambda'_{2,3})(\lambda_{1,2}+\lambda_{2,2}+\lambda_{3,2}+\lambda'_{1,2}+\lambda'_{23})\widetilde{\langle 2,2,2\rangle},\end{aligned}$$

(6.22)

$$\begin{aligned}&-\lambda_{1,1}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2})(\lambda_{1,1}+\lambda_{2,1}+\lambda_{3,1}+\lambda'_{1,2}+\lambda'_{2,3})\widetilde{\langle 1,1,1\rangle}\\ &\quad=\big[\lambda_{1,1}\lambda_{1,2}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2})\\ &\qquad-\lambda_{2,2}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2})(\lambda_{2,1}+\lambda_{3,1}+\lambda'_{2,3})\\ &\qquad+\lambda_{3,1}\lambda_{3,2}(\lambda_{2,1}+\lambda_{3,1}+\lambda'_{2,3})]\widetilde{\langle 2,1,1\rangle}\\ &\qquad-\lambda_{3,2}[\lambda_{2,2}(\lambda_{1,1}+\lambda_{2,2}+\lambda'_{1,2})\\ &\qquad\qquad+\lambda_{1,1}\lambda_{1,2}-\lambda_{3,1}\lambda_{3,2}-\lambda_{2,1}(\lambda_{2,2}+\lambda_{3,2}+\lambda'_{2,3})]\widetilde{\langle 2,1,2\rangle}\\ &\qquad+[-\lambda_{2,2}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2})(\lambda_{1,2}+\lambda_{2,2}+\lambda'_{1,2})\\ &\qquad\quad+\lambda_{3,1}\lambda_{3,2}(\lambda_{1,2}+2\lambda_{2,2}+\lambda_{3,2}+\lambda'_{1,2}+\lambda'_{2,3})]\widetilde{\langle 2,2,1\rangle}\\ &\qquad+\lambda_{3,2}(\lambda_{2,2}+\lambda_{3,2}+\lambda'_{2,3})\\ &\qquad\quad\times(\lambda_{1,2}+\lambda_{2,2}+\lambda_{3,2}+\lambda'_{1,2}+\lambda'_{2,3})\widetilde{\langle 2,2,2\rangle}.\end{aligned}$$

As for the

$$B'_{1,2}\begin{pmatrix} j_1, & j_2, & j_3 \\ 1, & 1, & i_3 \end{pmatrix},$$

we have

$$(6.23) \quad -(\lambda'_{1,2}+\lambda_{1,1}+\lambda_{2,1})\langle\widetilde{1,1,2}\rangle = \lambda_{3,1}\langle\widetilde{1,2,1}\rangle + \lambda_{1,2}\langle\widetilde{2,1,2}\rangle + (\lambda_{2,2}+\lambda_{3,2}+\lambda'_{2,3})\langle\widetilde{1,2,2}\rangle,$$

$$(6.24) \quad -(\lambda_{1,1}+\lambda_{1,2}+\lambda'_{1,2})(\lambda_{1,1}+\lambda_{2,1}+\lambda_{3,1}+\lambda'_{1,2}+\lambda'_{2,3})\langle\widetilde{1,1,1}\rangle = [\lambda_{2,2}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2}) - \lambda_{3,1}\lambda_{3,2}]\langle\widetilde{1,2,1}\rangle - \lambda_{1,2}\lambda_{3,2}\langle\widetilde{2,1,2}\rangle - \lambda_{3,2}(\lambda_{2,2}+\lambda_{3,2}+\lambda'_{2,3})\langle\widetilde{1,2,2}\rangle + \lambda_{1,2}(\lambda_{1,1}+\lambda_{2,1}+\lambda'_{1,2})\langle\widetilde{2,1,1}\rangle.$$

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