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A NON-LINEAR METHOD FOR CONSTRUCTING CERTAIN BASIC SEQUENCES IN BANACH SPACES

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I. Introduction

A basic problem in Banach space theory is whether every infinite dimensional Banach space contains an isomorphic copy of c_0 or a subspace which is isomorphic to an infinite dimensional conjugate space. This is, of course, a first step toward the better known conjecture that every Banach space contains an isomorphic copy of c_0 or l_1 or an infinite dimensional reflexive subspace. In this paper, we exhibit a new technique for constructing infinite boundedly complete basic sequences and consequently separable conjugate subspaces. An interesting feature of this method is that it involves a non-linear approach, which is in sharp contrast with the linear nature of the problem we address. Our approach also combines techniques originating in the local theory of Banach spaces (Dvoretzky's theorem and concentration phenomenon) with infinite dimensional concepts like dentability and "transfinite slicing" of sets.

One application of this method is that Banach spaces with the *Analytic Radon-Nikodym Property* (ARNP) contain copies of infinite dimensional conjugate spaces. (Recall that a complex Banach space X is said to have the ARNP if every X-valued bounded analytic map on the open unit disc of the complex plane has radial limits almost surely). This class of spaces contains—besides those possessing the Radon-Nikodym property (RNP) (see [D-U])—all Banach lattices not containing c_0 [B-D] as well as all preduals of Von Neuman algebras [H-P]. What is needed for the proof is the following geometric characterization of such spaces established in [G-L-M]: *Every bounded subset of a Banach space with the ARNP has arbitrarily norm-small slices determined by Lipschitz and plurisubharmonic functions.* In the classical RNP setting (where the slices are determined by continuous linear functionals) the analogous statement (i.e. The existence of infinite dimensional conjugate spaces) was established in [G-M1]. Also shown there is the case where the "slices" are determined by a finite number of linear functionals i.e.

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when the space has the *Point of Continuity Property* (PCP). In Theorem (4) below, we formulate the general principle underlying the proof of the "analytic case" and we show how it can be used to recover these results and also to extend the class of Banach spaces containing infinite dimensional conjugate spaces.

For unexplained notations and terminology we refer to the books of Lindenstrauss-Tzafriri [L-T] and Milman-Schechtman [M-S]. We just recall that a continuous function $\varphi\colon X \to \mathbf{R}$ is said to be *plurisubharmonic* if

$$\varphi(x) \leq \int_0^{2\pi} \varphi(x + e^{i\theta}y)\frac{d\theta}{2\pi} \quad \text{for all } x, y \text{ in } X.$$

II. Main result

Recall that a *boundedly complete basic sequence* in a Banach space X is a sequence $(x_n)_n$ that verifies the following conditions:

(i) For some $K > 0$, we have $\|\Sigma_{i=1}^{n} a_i x_i\| \leq K\|\Sigma_{i=1}^{m} a_i x_i\|$ for all choices of scalars $(a_i)_i$ and integers $n < m$.

(ii) $\Sigma_i a_i x_i$ converges whenever $(a_i)_i$ is a sequence of scalars verifying $\sup_N \|\Sigma_{i=1}^{N} a_i x_i\| < \infty$.

It is well known that the closed linear span in X of such a sequence is isomorphic to a conjugate Banach space [L-T, Proposition 1.b.4].

Here is the main result of this paper.

THEOREM 1. *Every infinite dimensional Banach space with the Analytic Radon-Nikodym property contains infinite boundedly complete basic sequences.*

We shall need the following lemma which is actually a reformulation of some results of V. Milman concerning the *spectrum* of a uniformly continuous function (see for instance [M] and its list of references). We include a sketch of the proof for the sake of completeness.

LEMMA 2. *In every infinite dimensional complex Banach space X, there exists a net $(u_\beta)_\beta$ on the unit sphere such that for any Lipschitz function ϕ on X we have*

$$(*) \qquad \lim_{\beta} \sup_{0 \leq r \leq 1} \sup_{\theta, \theta' \in \mathbf{T}} \left|\phi\left(re^{i\theta}u_\beta\right) - \phi\left(re^{i\theta'}u_\beta\right)\right| = 0.$$

Proof. We need to combine the following three basic results that can be found in the book of Milman-Schechtman [M-S, Chapter (2)]. The first is the complex version of Dvoretzky's theorem:

(i) Every infinite dimensional complex Banach space contains a sequence of complex subspaces $(X_n)_n$ such that $\dim(X_n) = n$ and $\lim_n d(X_n, l_n^2) = 1$.

(ii) For any $\alpha > 0$ and integer k, there exists $\rho(k, \alpha) > 0$ such that for every $n \geq k$, if μ is Lebesgue measure on the n-dimensional euclidean sphere S, then for any subset $A \subset S$ verifying $\mu(S \setminus A) < \rho(k, \alpha)$, there exists a complex space Y of dimension k so that for all x in $Y \cap S$, $\operatorname{dist}(x, A) < \alpha$.

The last fact we need is the well known *concentration phenomenon*:

(iii) If ϕ is a 1-Lipschitz function on the n-dimensional complex euclidean sphere S and if m is its median on (S, μ) then for each $\alpha > 0$ we have

$$\mu\{x \in S; |\phi(x) - m| > \alpha\} \leq 2\exp(-n\alpha^2).$$

We can now prove the following result.

Claim. *Let* $(\varepsilon_N)_N$ *be a sequence of positive reals that decreases to* 0. *For every N-tuple*

$$\beta = (\phi_1, \dots, \phi_N)$$

of 1-*Lipschitz functions on* X, *there exists a complex subspace* Y_β *with* $d(Y_\beta, l_N^2) \leq 1 + \varepsilon_N$ *such that if* S_β *denotes the unit sphere of* Y_β, *then* $\max_{1 \leq j \leq N} \operatorname{osc}(\phi_j, S_\beta) \leq \varepsilon_N$.

For that, use (i) to choose n large enough so that $d(X_n, l_n^2) \leq 1 + \frac{1}{2}\varepsilon_N$ for some subspace X_n of X, and in such a way that

$$2N \exp\left(-n\frac{\varepsilon_N^2}{64}\right) < \rho\left(N, \frac{\varepsilon_N}{8}\right).$$

Here $\rho(k, \alpha)$ is the function given in (ii). Since $d(X_n, l_n^2) \leq 1 + \frac{1}{2}\varepsilon_N$, we can find an euclidean norm $|||\ \ |||$ on X_n such that $\|x\| \leq |||x||| \leq (1 + \frac{1}{2}\varepsilon_N)\|x\|$ for all x in X_n. The functions $(\phi_j)_{j=1}^N$ are then 1-Lipschitz on the euclidean sphere S_n. If m_j denotes the mean value of ϕ_j on (S_n, μ), from (iii) we obtain

$$\mu\left\{\bigcup_{j=1}^N \left(|\phi_j - m_j| > \frac{\varepsilon_N}{8}\right)\right\} \leq N \cdot 2\exp\left(-n\frac{\varepsilon_N^2}{64}\right) < \rho\left(N, \frac{\varepsilon_N}{8}\right).$$

Let $A = \bigcap_{j=1}^N\{|\phi_j - m_j| \leq \varepsilon_N/8\}$; by (ii) there exists a subspace Y_β of dimension N such that for each x in $Y_\beta \cap S_n$, $\operatorname{dist}_{l_2}(x, A) \leq \varepsilon_N/8$. It follows that

$$|\phi_j - m_j| \leq \frac{\varepsilon_N}{4} \quad \text{on } Y_\beta \cap S_n$$

since the ϕ_j's are 1-Lipschitz. Hence $\operatorname{osc}(\phi_j, Y_\beta \cap S_n) \le \frac{1}{2}\varepsilon_N$ and $\operatorname{osc}(\phi_j, S_\beta) \le \varepsilon_N$ for each $j = 1, 2, \ldots, N$. The claim is proved.

This now clearly implies the existence of a net $(u_\beta)_\beta$ on the unit sphere of X such that for every Lipschitz function ϕ we have

$$\lim_\beta \sup_{\theta, \theta' \in \mathbf{T}} \left|\phi\left(e^{i\theta}u_\beta\right) - \phi\left(e^{i\theta'}u_\beta\right)\right| = 0.$$

To get the full statement $(*)$, it is enough to apply the above to the functions $\phi_j(y) = \phi(r_j y)$ whenever ϕ is Lipschitz and for $r_j = 0, 1/J, 2/J, \ldots, 1$. Then one can use the Lipschitz property to get the statement for all r in $[0,1]$.

LEMMA 3. *Assume X is a separable Banach space with the Analytic Radon-Nikodym property. Then for every positive integer k, there exists a countable ordinal γ_k and a family $(F_{\alpha,k}, \phi_{\alpha,k})_{\alpha<\gamma_k}$ where $(F_{\alpha,k})_\alpha$ is a decreasing family of closed subsets of the unit ball of X and $(\phi_{\alpha,k})_\alpha$ is a family of plurisubharmonic and 1-Lipschitz functions on X such that*

(a) $F_{0,k} = B_X$,

(b) $F_{\alpha,k} \cap \{\phi_{\alpha,k} > 0\} \neq \varnothing$, $\operatorname{diam}(F_{\alpha,k} \cap \{\phi_{\alpha,k} > 0\}) < 2^{-k}$ *and* $F_{\alpha+1,k} = F_{\alpha,k} \setminus \{\phi_{\alpha,k} > 0\}$ *if* $\alpha < \gamma_k$,

(c) $F_{\alpha,k} = \bigcap_{\beta<\alpha} F_{\beta,k}$ *if α is a limit ordinal,*

(d) $F_{\gamma_{k},k} = \varnothing$ *for all* $k \in \mathbf{N}$.

Proof. By the results of [G-L-M], for every non empty closed bounded subset F of X and any $\varepsilon > 0$, there exists a Lipschitz and plurisubharmonic function ϕ such that

$$F \cap \{\phi > 0\} \neq \varnothing \quad \text{and} \quad \operatorname{diam}\left(F \cap \{\phi > 0\}\right) < \varepsilon.$$

For any $\varepsilon = 2^{-k}$, one can proceed with a straightforward transfinite induction—starting with $F_{0,k} = B_X$—to construct the family $(F_{\alpha,k}, \phi_{\alpha,k})_{\alpha<\gamma_k}$ claimed in the lemma.

Proof of Theorem 1. Since the ARNP is hereditarily stable, we can assume without loss of generality that X is separable. For each x in B_X and $k \ge 0$, we denote by $\alpha_k(x)$ the first ordinal α such that $x \in F_{\alpha,k} \cap \{\phi_{\alpha,k} > 0\}$ where $(F_{\alpha,k}, \phi_{\alpha,k})_{\alpha<\gamma_k}$ are given by Lemma (3). If $x \notin B_X$ we let $\alpha_k(x) = -1$.

Let R_j be a 2^{-j}-net in the unit disc Δ of the complex plane $\mathbf{C}$. Let $(u_\beta)_\beta$ be the net of points on the unit sphere of X constructed in Lemma (2).

Start with $x = 0$. Set $\varepsilon = \phi_{\alpha_0(0),0}(0) > 0$ and $\phi = \phi_{\alpha_0(0),0}$. Use $(*)$ to choose β large enough so that

$$\left|\phi\left(re^{i\theta}u_\beta\right) - \phi\left(re^{i\theta'}u_\beta\right)\right| < \varepsilon/4 \quad \text{for all } r \text{ in } [0,1] \text{ and } \theta, \theta' \text{ in } \mathbf{T}.$$

But ϕ is plurisubharmonic, hence

$$\varepsilon = \phi(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi\left(re^{i\theta}u_\beta\right) d\theta \quad \text{for all } r.$$

It follows that $\phi(zu_\beta) > 0$ for all $z \in \Delta$.

Set $e_1 = u_\beta$ and note that $\alpha_0(ze_1) \leq \alpha_0(0)$ for all z in Δ since $ze_1 \notin F_{\alpha_0(0)+1,0}$.

We now construct by induction a sequence of norm one elements $(e_n)_n$ in X and an increasing sequence of finite subsets $(A_n)_n$ in Ball(X^*) such that for each $n \geq 1$,

(i) $\sup_{x^* \in A_n} \langle x^*, x \rangle \geq \frac{1}{2} \|x\|$ for all x in the linear span of $\{e_1, e_2, \dots, e_n\}$,
(ii) $\sup_{x^* \in A_n} |\langle x^*, e_{n+1} \rangle| \leq 2^{-(n+1)}$,
(iii) $\alpha_j(\sum_{l=1}^n z_l e_l + ze_{n+1}) \leq \alpha_j(\sum_{l=1}^n z_l e_l)$ for all $j = 0, 1, \dots, n$, all $z_j \in R_j$ and all $z \in \Delta$.

To do that, suppose $\{e_1, \dots, e_n\}$ and A_{n-1} have been obtained and let E_n be the linear span of $\{e_1, \dots, e_n\}$. Find a finite subset A_n of Ball(X^*) verifying (i). Consider the finite family D_n of 1-Lipschitz and plurisubharmonic functions defined by

$$\psi(y) = \phi_{\alpha_j(z_1e_1+z_2e_2+\cdots+z_ne_n),\, j}(z_1e_1+z_2e_2+\cdots+z_ne_n+y)$$

for $j = 0, 1, 2, \dots, n$ and $z_j \in R_j$.

By the definition of α_j we have $\psi(0) > 0$ for all ψ in D_n. Let $\varepsilon = \min\{2^{-(n+1)}; \min\{\psi(0); \psi \in D_n\}\}$ and use $(*)$ to find β large enough so that for every ψ in $A_n \cup D_n$,

$$(**) \quad \left|\psi\left(re^{i\theta}u_\beta\right) - \psi\left(re^{i\theta'}u_\beta\right)\right| < \varepsilon/4 \quad \text{for all } r \text{ in } [0,1] \text{ and } \theta, \theta' \text{ in } \mathbf{T}.$$

Let $e_{n+1} = u_\beta$ and note that the above inequality applied to the elements of A_n implies (ii).

On the other hand, since each ψ in D_n is plurisubharmonic, $(**)$ also gives that $\psi(zu_\beta) > 0$ for all z in Δ, which in turn gives (iii). This finishes the construction.

The proof that (i) and (ii) imply that $(e_n)_n$ is a basic sequence is standard and is left to the interested reader.

To show that $(e_n)_n$ is boundedly complete, consider a sequence of complex numbers $(z_j)_j$ such that $\sup_N \|\sum_{j=1}^N z_j e_j\| \leq 1$. We can suppose without loss of generality that $z_j \in R_j$ for each j.

Now fix the integer j and note that the sequence $\{\alpha_j(\sum_{l=1}^n z_l e_l); n \geq j\}$ is a decreasing sequence of ordinals and hence there is n_0 so that it becomes stationary—say equal to α—for $n \geq n_0$. But this implies that for all $n \geq n_0$, the points $\sum_{l=1}^n z_l e_l$ belong to the slice $F_{\alpha,j} \cap \{\phi_{\alpha,j} > 0\}$ whose diameter is less than 2^{-j}. It follows that the sequence of such partial sums is Cauchy and consequently convergent. ■

We shall now formulate the general principle—for constructing boundedly complete basic sequence—which underlies the above proof. This will allow us to recover the known results and will also permit a further extension of the class of Banach spaces containing infinite dimensional conjugate spaces.

Let $\mathscr{C}$ be a class of functions on a Banach space X. Say that X is *$\mathscr{C}$-dentable* if for every bounded subset $F \subset X$ and any $\varepsilon > 0$, there exists $\varphi \in \mathscr{C}$ so that

$$F \cap \{\varphi > 0\} \neq \varnothing \quad \text{and} \quad \operatorname{diam}(F \cap \{\varphi > 0\}) < \varepsilon.$$

A careful consideration of the above proof gives the following result.

THEOREM 4. *Let X be a separable real (resp. complex) Banach space and let $\mathscr{C}$ be a class of continuous functions on X that contains all the affine functions. Assume that*

(a) *X is $\mathscr{C}$-dentable and*

(b) *$(u_\beta)_\beta$ is a net on the unit sphere of X so that $\varphi(x) \leq \liminf_\beta \varphi(x + \lambda u_\beta)$ for every $x \in X$ and any λ in $\mathbf{R}$ (resp. $\mathbf{C}$).*

Then there exists an infinite countable subset $(u_{\beta_n})_n$ that forms a boundedly complete basic sequence in X.

Here are some situations where Theorem (4) is applicable.

Examples. (1) Assume $\mathscr{C}_1$ is the space of affine and continuous functions on X. In this case, $\mathscr{C}_1$-dentability is equivalent to the Radon-Nikodym property [D-U]. On the other hand, any weakly null net $(u_\beta)_\beta$ which is not norm convergent automatically verifies condition (b) for the class $\mathscr{C}_1$. Actually, the class of functions verifying (b) for a fixed net $(u_\beta)_\beta$, is stable under suprema, finite infima, finite linear combinations with positive coefficients and of course uniform convergence on the unit ball of X. If we denote by $\mathscr{C}_2$ the "closure" of $\mathscr{C}_1$ under these operations, one can easily see that the $\mathscr{C}_2$-dentability of X is equivalent to say that X has PCP: *that is, bounded sets have relative weak neighborhoods of arbitrarily small diameter*. Theorem 4 then allows us to recover some of the results in [G-M1] and [G-M2], in particular, that Banach spaces with PCP are hereditarily separable duals.

(2) The conclusion of Theorem 1 can of course be obtained by applying Theorem 4 to the set $\mathscr{C}_3$ of plurisubharmonic and 1-Lipschitz functions. Any net $(u_\beta)_\beta$ verifying $(*)$ of Lemma (2) would satisfy condition (b) of Theorem (4), in view of the plurisubharmonicity of the functions in the class $\mathscr{C}_3$. Again, analogously to the linear case, if we denote by $\mathscr{C}_4$ the closure of $\mathscr{C}_3$ under the operations described in Example 1) we obtain that $\mathscr{C}_4$-dentable Banach

spaces contain infinite boundedly complete basic sequences. Such spaces could be called spaces with the *Analytic-PCP*. Larger classes of spaces can be shown to contain infinite dimensional conjugate spaces via Theorem (4). Whether one can reach the class of spaces not containing c_0 is still an open question.

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