ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 1990

THE EMBEDDING OF BANACH SPACES INTO SPACES WITH STRUCTURE

BY

M. ZIPPIN

1. Introduction

Let X be a separable Banach space. A sequence $\{Y_n\}_{n=1}^{\infty}$ of finite dimensional subspaces of X is called a *finite dimensional decomposition* (f.d.d., in short) of X if each $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} T_n x$ with $T_n x \in Y_n$. A basis of X is a f.d.d. where each Y_n is of dimension 1. It is well known and easy to prove that X has a f.d.d. if and only if there is a sequence $\{P_n\}_{n=1}^{\infty}$ of commuting projections on X such that each P_n is of finite rank, $\sup_n \|P_n\| < \infty$, $P_1 X \subset P_2 X \subset \cdots$ and $\bigcup_n P_n X$ is dense in X. The existence of Banach spaces without the approximation property makes it reasonable to investigate how "close" a given separable space is to spaces with a f.d.d. In this direction are the following three problems (the first two of which were previously solved (see [2] and [5])).

PROBLEM 1. *Given a separable Banach space X does there exist a subspace E of X such that both E and X/E have f.d.d.'s?*

PROBLEM 2. *Given a separable Banach space E does there exist a separable space X and a subspace Y of X, both with an f.d.d., such that E = X/Y?*

PROBLEM 3. *Given a separable space E does there exist a space X containing E such that both X and X/E have f.d.d.'s?*

The first problem is positively solved by W. B. Johnson and H. P. Rosenthal in [2]. The second one is answered by J. Lindenstrauss in [5] in the following strong sense: every separable space E is isomorphic to a quotient X^{**}/X where both X and X^{**} have bases. The purpose of this paper is to give a positive solution of Problem 3. Since every complemented subspace of a space with an f.d.d. has the bounded approximation property one does not expect a given separable space to be complemented in a space with an f.d.d.

Received May 1, 1988.

For a subspace E of X, being complemented in X is a very strong condition. It means that there is a number $\lambda \geq 1$ such that for every Banach space Z and every operator $T\colon E \to Z$ there is an extension $\tilde{T}\colon X \to Z$ with $\|\tilde{T}\| \leq \lambda\|T\|$. We find the following weaker property of a subspace E of X easier to handle yet interesting enough.

DEFINITION 1. A subspace E of X is said to be almost complemented in X if there is a number $\lambda > 0$ such that for every compact Hausdorff space K and every operator $T\colon E \to C(K)$ there is an extension $\tilde{T}\colon X \to C(K)$ with $\|\tilde{T}\| \leq \lambda\|T\|$.

We will prove the following result:

THEOREM. *Let E be a separable Banach space. Then there exists a Banach space X with an f.d.d. which contains E such that*

(1.1) *E is almost complemented in X*

and

(1.2) *X/E has an f.d.d.*

The proof of the theorem consists of three parts. The first part (Section 3) is mainly an algebraic construction of a normed space with an f.d.d. containing E. This construction is the foundation for some topological consequences given in Section 4. The last part is a variant of E. Michael's selection theorem [6] which leads to the operator extension property. Before starting the proof of the theorem we need some information about almost complemented subspaces.

Notation. Let X be a normed space and $A \subset X$. $[A]$ denotes the closed linear span of A; span A is the algebraic span. Conv A is the closed convex hull of A and A^+ is the annihilator of A in X^*.

2. Extension of operators into $C(K)$ spaces

J. Lindenstrauss investigated in [4] the extension of compact operators into $C(K)$ spaces. A special case of Theorem 6.1 in [4] is the following: for every Banach space X, every subspace $E \subset X$, any $\varepsilon > 0$ and every compact operator $T\colon E \to C(K)$ there is a compact extension

$$\tilde{T}\colon X \to C(K) \quad \text{with } \|\tilde{T}\| < (1+\varepsilon)\|T\|.$$

One should therefore expect the class of almost complemented subspaces of a Banach space to be rather large. Restricting the range space of an operator to be a $C(K)$ space is a considerable convenience. Indeed (see [1, p. 490]) if

Z is a Banach space then every operator

$$T\colon Z \to C(K)$$

determines the function

$$\varphi(T)\colon K \to \|T\| \cdot B(Z^*)$$

(where $B(Z^*)$ denotes the closed unit ball of Z^*) defined by $(\varphi(T)(k)z = (Tz)(k)$ which is ω^* continuous. Conversely, every ω^* continuous function $\varphi\colon K \to \lambda B(Z^*)$ determines an operator

$$T(\varphi)\colon Z \to C(K)$$

defined by

$$(T(\varphi)(z))(k) = \varphi(k)(z).$$

Clearly $\|T(\varphi))\| = \sup\{\|\varphi(k)\|: k \in K\} \le \lambda$. In the sequel a Banach space Z is regarded as a subspace of $C(B(Z^*))$ via the natural embedding $(Jz)(z^*) = z^*(z)$. The topology on $B(Z^*)$ is the ω^* topology which is metric when Z is separable. Let E be a subspace of X and $T\colon E \to X$ the corresponding isometric embedding and, for $\lambda \ge 1$ let

$$K(\lambda) = \{x^* \in \lambda B(X^*): \|x^*|_E\| \le 1\}.$$

We regard $C(B(E^*))$ as a subspace of $C(K(\lambda))$ via the natural embedding S defined by

$$(Sf)(x^*) = f(T^*x^*) \quad \text{for every } x^* \in B(X^*).$$

We say that a function $\varphi\colon B(E^*) \to X^*$ *extends functionals* if for every $e^* \in B(E^*)$ and $e \in E$, $\varphi(e^*)(e) = e^*(e)$.

Example 1. Every Banach space Z is almost complemented in $C(B(Z^*))$. Indeed, the function

$$\varphi_0\colon B(Z^*) \to B(C(B(Z^*))^*)$$

defined by

$$\varphi_0(z^*)(f) = f(z^*)$$

for every $f \in C(B(Z^*))$ and $z^* \in B(Z^*)$ is clearly ω^* continuous and extends functionals. If $T\colon Z \to C(K)$ is a given operator, then, using the notation above,

$$\varphi(T): K \to B(Z^*)$$

is ω^* continuous hence, the composition

$$\varphi_0 \circ \varphi(T): K \to B(C(B(Z^*))^*)$$

is ω^* continuous. Consider the operator

$$\tilde{T} = T(\varphi_0 \circ \varphi(T)): C(B(Z^*)) \to C(K).$$

It is easy to check that $\tilde{T}$ is a norm preserving extension of T.

Example 2. If H is a compact Hausdorff space then $C(H)$ is complemented in a space X if it is almost complemented in X because the identity I: $C(H) \to C(H)$ can be extended to a projection of X onto $C(H)$.

The following is a list of simple, well known facts brought here for the sake of completeness.

PROPOSITION 1. *Let E be a subspace of a Banach space X. Then the following properties are equivalent*:

(2.1) *E is almost complemented in X.*

(2.2) *The natural embedding J: $E \to C(B(E^*))$ has an extension $\tilde{J}$: $X \to C(B(E^*))$.*

(2.3) *There is an ω^* continuous function φ: $B(E^*) \to X^*$ which extends functionals.*

(2.4) *There is a $\lambda > 0$ such that if*

$$K(\lambda) = \{x^* \in \lambda B(X^*): \|x^*|_E\| \leq 1\}$$

and

S: $C(B(E^*)) \to C(K(\lambda))$ is the embedding defined by $Sg(x^*) = g(x^*|_E)$ then there is a projection P of $C(K(\lambda))$ onto $S(C(B(E^*)))$ with $\|P\| \leq \lambda$.

Proof. (2.1) ⇒ (2.2). Formal.

(2.2) ⇒ (2.3) Let φ_0: $B(E^*) \to B(C(B(E^*))^*)$ be defined by $\varphi_0(e^*)(f) = f(e^*)$ as above. Let φ_1 be the restriction of $\tilde{J}^*$ to $B(C(B(E^*))^*)$. Then φ_0 and φ_1 are ω^* continuous; hence, if $\lambda = \|\tilde{J}\|$, the function $\varphi = \varphi_1 \circ \varphi_0$: $B(E^*) \to \lambda B(X^*)$ is ω^* continuous and for every $e^* \in B(E^*)$ and $e \in E$ we have

$$\varphi(e^*)(e) = \tilde{J}^*(\varphi_0 e^*)(e) = \varphi_0(e^*)(\tilde{J}e) = \varphi_0(e^*)(e) = e^*(e).$$

It follows that φ extends functionals.

(2.3) $\Rightarrow$ (2.1) Let

$$\lambda = \sup\{\|\varphi(e^*)\|: e^* \in B(E^*)\}$$

and let T: $E \to C(K)$ be any operator. With the above notations,

$$\varphi \circ \varphi(T): K \to \lambda B(X^*)$$

is ω^* continuous. Let

$$\tilde{T} = T(\varphi \circ \varphi(T)): X \to C(K);$$

then $\|\tilde{T}\| \leq \lambda \|T\|$ and for every $e \in E$ and $k \in K$,

$$(\tilde{T}e)(k) = ((\varphi \circ \varphi(T))(k))(e) = (\varphi(T)(k))(e) = (Te)(k)$$

because φ extends functionals. It follows that $\tilde{T}$ extends T. This proves (2.1).

(2.4) $\Rightarrow$ (2.1) Suppose that $S(C(B(E^*)))$ is complemented in $C(K(\lambda))$ and let

$$P: C(K(\lambda) \to S(C(B(E^*)))$$

be a projection with $\|P\| \leq \lambda$. Let

$$\varphi_0: B(E^*) \to B(C(B(E^*))^*)$$

be the function defined by

$$\varphi_0(e^*)(f) = f(e^*)$$

and let

$$\varphi_1: B(C(B(E^*))^*) \to \|P\|B(C(K(\lambda))^*)$$

be the restriction of P^*. Then $\varphi = \varphi_1 \circ \varphi_0$ is ω^* continuous and extends functionals, and therefore, since (2.3) $\Rightarrow$ (2.1), very operator T: $E \to C(K)$ can be extended to an operator

$$T_1: C(K(\lambda)) \to C(K) \quad \text{with } \|T_1\| \leq \|P\|.$$

Regarding X as a subspace of $C(K(\lambda))$ (via the natural embedding U: $X \to C(K(\lambda))$ defined by $(Ux)(x^*) = x^*(x)$) we put $\tilde{T} = T_1|x$; then $\tilde{T}$ is the desired extension of T.

(2.3) ⇒ (2.4) Define V: $C(K(\lambda)) \to C(B(E^*))$ by $(Vf)(e^*) = f(\varphi(e^*))$ and put $P = SV$. Then P is a projection of $C(K(\lambda))$ onto $S(C(B(E^*)))$. This proves Proposition 1.

Proposition 1 suggests a general method of proving that a subspace E of X is almost complemented. All that has to be done is to construct a ω^* continuous function

$$\varphi\colon B(E^*) \to X^*$$

which extends functionals.

Example 3. Let $1 < p < \infty$ and let E be a subspace of l_p. Then $\varphi(e^*)$, the Hahn Banach extension of e^*, is a suitable function from $B(E^*)$ to $B(l_p^*)$ because, as is easily checked, φ is ω^* continuous. It follows that E is almost complemented in l_p.

3. The Algebraic construction

Let E be a separable Banach space and regard E as a subspace of a space Y' with a monotone basis (for example, we may let $Y' = C[0,1]$). Consider the space $Y = Y' + c_0$ where the norm is defined by $\|(x, z)\| = \max\{\|x\|, \|z\|\}$ for any $x \in Y'$ and $z \in c_0$. The space Y has a normalized monotone basis $\{y_n\}_{n=1}^{\infty}$ with biorthogonal functionals $\{y_n^*\}_{n=1}^{\infty}$ such that $\{y_{2n-1}\}_{n=1}^{\infty}$ and $\{y_{2n}\}_{n=1}^{\infty}$ are monotone bases of Y' and c_0 respectively. We may assume that $E_0 = E \cap \text{Span}\{y_{2n-1}\}_{n=1}^{\infty}$ is norm dense in E. Let $\{P_n\}_{n=1}^{\infty}$ denote the natural basis projections so that $\|P_n\| = 1$ for all n. Put $E_n = E_0 \cap P_n(Y)$.

Now select a subsequence of even integers $\{\alpha(n)\}_{n=1}^{\infty}$ satisfying the following conditions.

(3.1) $\alpha(1)$ is so large that $E_{\alpha(1)} \neq \phi$.

(3.2) $\alpha(n+1)$ is an even integer so large that $E_{\alpha(n+1)}$ is strictly larger than $E_{\alpha(n)}$ and if $e \in E_0$ and $P_{\alpha(n)}e \neq 0$ then there is an $e_0 \in E_{\alpha(n+1)}$ such that $P_{\alpha(n)}e = P_{\alpha(n)}e_0$. For every n let $\tilde{G}_n = \{e \in E_0\colon P_{\alpha(n)}e = 0\}$ and $G_n = \tilde{G}_n \cap E_{\alpha(n+1)}$.

Now we divide the construction to five steps.

Step 1. We find a subspace W_n of $E_{\alpha(n+1)}$ such that the following two conditions are satisfied.

(3.3) $E_{\alpha(n)} + W_n + G_n = E_{\alpha(n+1)}$ is a direct sum (and hence $P_{\alpha(n)}|_{W_n}$ and $I - P_{\alpha(n)}|_{W_n}$ are isomorphic mappings);

(3.4) $P_{\alpha(n-1)}\omega = 0$ for every $\omega \in W_n$.

Indeed, start with any subspace U_n of $E_{\alpha(n+1)}$ such that $E_{\alpha(n)} + U_n + G_n = E_{\alpha(n+1)}$ is a direct sum. Let $\{u_i\}_{i=1}^N$ be a basis of U_n. If $P_{\alpha(n-1)}u_i \neq 0$ then, by condition (3.2), there is a $v_i \in E_{\alpha(n)}$ with $P_{\alpha(n-1)}v_i = P_{\alpha(n-1)}u_i$. Put $\omega_i = u_i$ if $P_{\alpha(n-1)}u_i = 0$ and $\omega_i = u_i - v_i$ if $P_{\alpha(n-1)}u_i \neq 0$. Let $W_n = \operatorname{span}\{\omega_i\}_{i=1}^N$ then W_n is the desired subspace.

So far we have an algebraic separation between $E_{\alpha(n)}$ and W_n in the sense that

$$E_{\alpha(n)} \cap W_n = \{0\}.$$

For future arguments we need the stronger separation property:

$$(I - P_{\alpha(n-1)})E_{\alpha(n)} \cap P_{\alpha(n)}W_n = \{0\},$$

which need not hold in general. In order to achieve this kind of separation we will have to perturb E_0 slightly. The only reason for starting with $Y = Y' + c_0$ (instead of Y') at the beginning of this section is to ensure that this perturbation process is possible. To achieve this perturbation, we will construct a certain linear mapping $S\colon E_0 \to Y$. Before doing this, let us consider the motivation for this construction. Since E_0 is supported on $\{y_{2n-1}\}_{n=1}^\infty$ and $\alpha(n) = 2N$ for some N we have by (3.4) that $P_{\alpha(n)}W_n$ is a subspace of

$$(P_{\alpha(n)} - P_{\alpha(n-1)})Y$$

supported only on the odd basis elements

$$\{y_{2i-1}\}_{i=1}^\infty \quad \text{with } \alpha(n-1) < 2i-1 < \alpha(n).$$

$(I - P_{\alpha(n-1)})E_{\alpha(n)}$ is also a subspace of

$$(P_{\alpha(n)} - P_{\alpha(n-1)})Y$$

supported on the same

$$\{y_{2i-1}\}, \quad \alpha(n-1) < 2i-1 < \alpha(n),$$

and

$$\dim(I - P_{\alpha(n-1)})E_{\alpha(n)} \le \tfrac{1}{2}(\alpha(n) - \alpha(n-1)).$$

If we can achieve a perturbation so that W_n is still supported on the odd basis vectors while for each $x \in (I - P_{\alpha(n-1)})E_{\alpha(n)}$, $x \neq 0$, we could have

$$y_{2i}^*(x) \neq 0 \quad \text{for some } \tfrac{1}{2}\alpha(n-1) < i < \tfrac{1}{2}\alpha(n)$$

then it would follow that

$$(I - P_{\alpha(n-1)})E_{\alpha(n)} \cap P_{\alpha(n)}W_n = \{0\}.$$

Step 2. Given $\varepsilon > 0$, we construct a linear mapping $S\colon E_0 \to Y$ satisfying the following conditions:

$$\|Se - e\| \leq \varepsilon\|e\| \quad \text{for all } e \in E_0, \tag{3.5}$$

$$S(E_{\alpha(n)}) = S(E_0) \cap P_{\alpha(n)}Y, \quad n \geq 1, \tag{3.6}$$

$$S(G_n) = (I - P_{\alpha(n)})Y, \quad n \geq 1, \tag{3.7}$$

$$S(E_{\alpha(n)}) + S(W_n) + S(G_n) = S(E_{\alpha(n+1)}) \quad \text{is a direct sum} \tag{3.8}$$

and

$$P_{\alpha(n-1)}Sw = 0 \quad \text{for all } w \in W_n,$$

$$(I - P_{\alpha(n-1)})S(E_{\alpha(n)}) \cap P_{\alpha(n)}S(W_n) = \{0\}. \tag{3.9}$$

The construction of S is a simple but tedious process. The reader is referred to the appendix for details.

Step 3. Clearly the space $E_0' = S(E_0)$ is isomorphic to E_0 with

$$\|S\|\,\|S^{-1}\| \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}$$

and if we put $E_{\alpha(n)}' = S(E_{\alpha(n)})$, $W_n' = S(W_n)$ and $G_n' = S(G_n)$ then these new subspaces satisfy conditions (3.2), (3.3) and (3.4) in addition to the following condition:

$$(I - P_{\alpha(n-1)})E_{\alpha(n)}' \cap P_{\alpha(n)}W_n' = \{0\}. \tag{3.10}$$

In order to avoid complicated notation we will assume that $E_0 = E_0'$, $E_{\alpha(n)} = E_{\alpha(n)}'$ $W_n = W_n'$ and $G_n = G_n'$ for every $n \geq 1$. Now let $Y_0 = \operatorname{span}\{y_i\}_{i=1}^{\infty}$,

$\eta > 0$ and $\delta(i) > 0$ so small that

$$\prod_{k=1}^{\infty}\prod_{i=k}^{\infty}(1+\delta(i)) < 1+\eta.$$

Let $\{n(k)\}_{k=1}^{\infty}$ be an increasing sequence of integers such that $n(1) = 1$ and $n(k+1)$ is so large that the following condition is satisfied:

(3.11) Let $e \in E_0$ and $P_{\alpha(n(k))}e \neq 0$ and put

$$\nu_k(e) = \inf\{\|e_1\| : e_1 \in E_0,\ P_{\alpha(n(k))}e_1 = P_{\alpha(n(k))}e\}.$$

Then there is an $e_0 \in E_{\alpha(n(k+1))}$ such that

$$P_{\alpha(n(k))}e_0 = P_{\alpha((k))}e \quad \text{and} \quad \|e_0\| \leq (1+\delta(k+1))\nu_k(e).$$

Step 4. Let $F_0 = E_0$, $F_k = E_{\alpha(n(k))}$, $Q_k = P_{\alpha(n(k))}$, $\tilde{H}_k = F_0 \cap (I - Q_k)Y_0$, $H_k = F_{k+1} \cap \tilde{H}_k$ and $U_k = W_{n(k)}$. We claim that the following conditions hold:

$$F_{k+1} = F_k + U_k + H_k \quad \text{is a direct sum,} \tag{3.12}$$

$$Q_{k-1}u = 0 \quad \text{for all } u \in U_k, \tag{3.13}$$

$$(I - Q_{k-1})F_k \cap Q_k U_k = \{0\}, \tag{3.14}$$

Indeed, (3.12) and (3.13) are evident; to prove (3.14) note that

$$\begin{aligned} F_k &+ U_k + H_k \\ &= E_{\alpha(n(k))} + W_{n(k)} + G_{n(k)} + W_{n(k)+1} + G_{n(k)+1} + \cdots \\ &\quad + W_{n(k+1)-1} + G_{n(k+1)-1} \\ &= F_{k+1}. \end{aligned}$$

By (3.4), if $u \in U_k = W_{n(k)}$ then $P_{\alpha(n(k)-1)}u = 0$; hence, clearly,

$$Q_{k-1}u = P_{\alpha(n(k-1))}u = 0.$$

Finally, if $x \in (I - Q_{k-1})F_k \cap Q_k U_k$ then $x \in P_{\alpha(n(k))}W_{n(k)}$ hence, by (3.4),

$$P_{\alpha(n(k)-1)}x = 0.$$

It follows that $x \in (I - P_{\alpha(n(k)-1)})E_{\alpha(n(k))}$ and so, by (3.10) $x = 0$. This proves (3.14).

Step 5. To complete the construction, let

$$X_0 = \text{span } F_0 \cup \left(\bigcup_{k=1}^{\infty} Q_k F_0 \right)$$

let

$$C = \text{convex hull of } B(F_0) \cup \bigcup_{k=1}^{\infty} Q_k(B(F_0)),$$

let μ be the gauge functional of C and define $\|x\| = \mu(x)$ for every $x \in X_0$. The space X_0 thus becomes a normed space and F_0 is a subspace of X_0. If $x \in X_0$, $\|x\| = 1$ and $\varepsilon > 0$ is given then there exist elements $\{e_i\}_{i=1}^N \subset B(F_0)$, positive numbers $\{\lambda_i\}_{i=1}^N$ and integers $\{j(i)\}_{i=1}^N$ such that

$$x = \sum_{i=1}^{N} \lambda_i Q_{j(i)} e_i \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i \leq 1 + \varepsilon. \tag{3.15}$$

It follows from (3.15) that, as a projection on X_0, $\|Q_n\| = 1$. Note that, by (3.11), if $x \in Q_n(X_0)$, at the small cost of allowing

$$\sum_{i=1}^{N} \lambda_i < 1 + \delta(n+1))$$

(instead of $\Sigma_{i=1}^N \lambda_i < 1 + \varepsilon$) we may assume that

$$N = n, \quad e_i \in B(F_{n+1}) \quad \text{and} \quad j(i) = i \quad \text{for all } 1 \leq i \leq N \tag{3.16}$$

We have thus constructed a Banach space X = the completion of X_0 with an f.d.d. which contains E_0 and hence contains E. In the next section we will show that X/E has an f.d.d.

4. Topological consequences

Our algebraic construction yields the following two results.

LEMMA 1. *Let $X_n = Q_n X_0$ and assume that $x^* \in X_0^*$ is a functional which satisfies the inequality $|x^*(x)| \leq \|x\|$ for all $x \in X_{n-1} \cup F_n$. Then*

$$|x^*(x)| \leq (1 + \delta(n+1))\|x\| \quad \textit{for all } x \in [X_{n-1} + F_n]$$

Proof. Assume that $x = y + e$ with $y \in X_{n-1}$ and $e \in F_n$ and that $\|x\| = 1$. Then, by (3.15) and (3.16),

$$y + e = \sum_{i=1}^{n} \lambda_i Q_i e_i \quad \text{where } e_i \in B(F_{n+1}),\ \lambda_i \geq 0$$

and

$$\sum_{i=1}^{n} \lambda_i \leq 1 + \delta(n+1).$$

Applying $Q_n - Q_{n-1}$ to both sides of the above equation we get

$$(I - Q_{n-1})e = (Q_n - Q_{n-1})(y + e) = \lambda_n (Q_n - Q_{n-1}) e_n.$$

Suppose that $e_n = z + u + h$ where $z \in F_n$, $u \in U_n$ and $h \in H_n$. Then

$$(I - Q_{n-1})e = \lambda_n Q_n u + \lambda_n (I - Q_{n-1}) z$$

and so

$$(I - Q_{n-1})(e - \lambda_n z) = \lambda_n Q_n u$$

where $u \in U_n$ and $e - \lambda_n z \in F_n$. It follows from (3.12) and (3.14) that $\lambda_n u = 0$, $e - \lambda_n z \in F_{n-1}$. Therefore $Q_n e_n = Q_n z = z$ and $\|z\| \leq 1$. Hence

$$|x^*(x)| \leq \sum_{i=1}^{n} \lambda_i x^*(Q_i e_i) \leq (1 + \delta(n+1))$$

as claimed.

The statement of Lemma 1 means, in fact, that

$$(4.1) \qquad B[X_{n-1} + F_n] \subset (1 + \delta(n+1)) \operatorname{conv}(B(X_{n-1}) \cup B(F_n)).$$

LEMMA 2. *Let $q: X \to X/E$ be the quotient map; let $x \in X_{n-1}$ and $\omega_1 = Q_n \omega$ where $\omega \in U_n$. If $\|q(x + \omega_1)\| < 1$ then $\|q(x)\| < \prod_{i=n+1}^{\infty}(1 + \delta(i))$.*

Proof. Pick $e \in F_0$ such that $\|x + \omega_1 + e\| \leq 1$.

Suppose that $e \in F_m$ with $m > n$. Then, by (3.15) and (3.16) there exist $\{e_i\}_{i=1}^m \subset B(F_{m+1})$ and positive numbers $\{\lambda_i\}_{i=1}^m$ such that

$$\sum_{i=1}^{m} \lambda_i < 1 + \delta(m+1) \quad \text{and} \quad x + \omega_1 + e = \sum_{i=1}^{m} \lambda_i Q_i e_i.$$

Let $e_m = f + u + g$ where $f \in F_m$, $u \in U_m$ and $g \in H_m$; then

$$(Q_m - Q_{m-1})(x + \omega_1 + e) = (I - Q_{m-1})e.$$

On the other hand,

$$\begin{aligned}(Q_m - Q_{m-1})\left(\sum_{i=1}^{m} \lambda_i Q_i e_i\right) &= (Q_m - Q_{m-1})\lambda_m e_m \\ &= (Q_m - Q_{m-1})(\lambda_m(f + u + g)) \\ &= \lambda_m(I - Q_{m-1})f + \lambda_m Q_m u.\end{aligned}$$

It follows that

$$(I - Q_{m-1})(e - \lambda_m f) = Q_m(\lambda_m u)$$

and, therefore, by (3.14), $e - \lambda_m f \in F_{m-1}$ and $\lambda_m u = 0$. This means that

$$f = Q_m f = Q_m(f + g) = Q_m e_m \quad \text{and} \quad \|f\| \le \|f + g\| = \|e_m\| \le 1.$$

Thus we have the equality $x + \omega_1 + e = \sum_{i=1}^{m-1}\lambda_i Q_i e_i + \lambda_m f$ and so

$$x + \omega_1 + (e - \lambda_m f) = \sum_{i=1}^{m-1} \lambda_i Q_i e_i, \quad e - \lambda_m f \in F_{m-1}$$

and

$$\|x + \omega_1 + (e - \lambda_m f)\| \le 1 + \delta(m + 1).$$

We now repeat the procedure $m - n$ times to get an $e_0 \in F_n$ such that

$$\|x + \omega_1 + e_0\| < (1 + \delta(m + 1))(1 + \delta(m)) \cdots (1 + \delta(n + 2)) = \mu$$

It follows that there exist $\{e_i\}_{i=1}^{n} \subset B(F_{n+1})$ and non-negative $\{\mu_i\}_{i=1}^{n}$ such that

$$\sum_{i=1}^{n} \mu_i < \mu(1 + \delta(n + 1)) \quad \text{and} \quad x + e_0 + \omega_1 = \sum_{i=1}^{n} \mu_i Q_i e_i.$$

Let $e_n = f + u + g$ where $f \in F_n$, $u \in U_n$ and $g \in H_n$. Then

$$\begin{aligned}(Q_n - Q_{n-1})(x + e_0 + \omega_1) &= (I - Q_{n-1})e_0 + Q_n\omega_1 \\ &= (I - Q_{n-1})e_0 + Q_n\omega.\end{aligned}$$

On the other hand,

$$(Q_n - Q_{n-1})\left(\sum_{i=1}^{n} \mu_i Q_i e_i\right) = \mu_n(I - Q_{n-1}f + \mu_n Q_n u.$$

It follows from (3.14) that $e_0 - \mu_n f \in F_{n-1}$ and $\omega = \mu_n u$. Consider the element

$$y_1 = \sum_{i=1}^{n-1} \mu_i Q_i e_i + \mu_n(f + u + q).$$

Then, clearly, $\|y_1\| \leq \mu(1 + \delta(n+1))$ and

$$\begin{aligned} y_1 &= \sum_{i=1}^{n-1} \mu_i Q_i e_i + \mu_n Q_{n-1} e_n + (I - Q_{n-1})(e_0 + \omega) \\ &= x + e_0 + \omega + \mu_n g \in x + E_0 \end{aligned}$$

This proves Lemma 2.

COROLLARY. *X/E has an f.d.d. determined by $\{X_n + E\}_{n=1}^{\infty}$*

Proof. Let $y \in X_n$ then $y = x + e + \omega_1$ where $x \in X_{n-1}$, $e \in F_n$ and $\omega_1 = Q_n \omega$ with $\omega \in W_n$. Moreover, if y has another representation $y = x' + e' + \omega_1'$ of the same type then $x + e = x' + e'$, $\omega = \omega'$ and $\omega_1 = \omega_1'$. It follows from Lemma 2 that the map

$$s_{n-1}\colon X_n + E \to X_{n-1} + E$$

defined by $s_{n-1}q(y) = q(x)$ is a projection of norm

$$\|s_{n-1}\| \leq \prod_{i=n+1}^{\infty} (1 + \delta(i))$$

(as above, $q\colon X \to X/E$ is the quotient map). It follows that there is a projection S_{n-1} of X/E onto $X_{n-1} + E$ (defined for $x \in X_k + E$ by $S_{n-1}x = s_{n-1} \cdots s_{k-2} s_{k-1} x$ and extended by continuity to all of X/E) with

$$\|S_{n-1}\| \leq \sum_{k=n+1}^{\infty} \prod_{i=k}^{\infty} (1 + \delta(i)) < 1 + \eta.$$

It is easy to see that $S_k S_n = S_m$ with $m = \min\{k, n\}$ and hence these projections determine an f.d.d. for X/E. This proves Corollary 1. This result plus Step V of the construction of Section 3 finishes the proof of (1.2) of the main theorem.

5. The operator extension property

We will proceed to prove (1.1).

Recall that, by Proposition 1, the existence of a number $\lambda > 0$ such that every operator

$$T: E \to C(K)$$

has an extension $\tilde{T}: X \to C(K)$ with $\|\tilde{T}\| \leq \lambda \|T\|$ is equivalent to the existence of a ω^* continuous function $\varphi: B(E^*) \to \lambda B(X^*)$ which extends functionals. If E is a subspace of a separable Banach space X and $S: E \to X$ is the isometric embedding, let

$$\psi: B(E^*) \to 2^{X^*}$$

be defined by $\psi(e^*) = S^{*-1}(e^*)$. We search for a ω^* continuous selection φ: $B(E^*) \to X^*$ of ψ. Since Michael's selection theorem [6] does not hold in the ω^* topology we need certain modifications. Let us first make three easy observations.

Observation 1. Let X_1 be a finite dimensional subspace of X. Then $S^*(X_1^+) = (X_1 \cap E)^+$ (Z^+ denotes the annihilator of Z in X^*).

Observation 2. S^* is an ω^* open mapping. Indeed, if $\mathring{B}(X^*)$ denotes the open unit ball of X^* then the collection $\{\varepsilon\mathring{B}(X^*) + X_1^+: \varepsilon > 0$ and $X_1 \subset X$ a finite dimensional subspace$\}$ is a base for the ω^* neighborhood system of 0 in X^*. By Observation 1,

$$S^*\left(\varepsilon\mathring{B}(X^*) + X_1^+\right) = \varepsilon\mathring{B}(E^*) + (X_1 \cap E)^+$$

is a ω^* neighborhood of 0 in E^*.

Observation 3. The carrier $\psi: B(E^*) \to 2^{X^*}$ defined by $\psi(e^*) = S^{*-1}(e^*)$ is an ω^* lower semicontinuous carrier into the collection of the closed convex subsets of X^* (i.e., for every $e^* \in B(E^*)$ and ω^* open set $V \subset X^*$ for which $\varphi(e^*) \cap V \neq \emptyset$ there is an ω^* open $U \subset E^*$ such that $e^* \in U$ and for each $e_0^* \in U$, $\varphi(e_0^*) \cap V = \emptyset$). Indeed $U = S^*(V)$ is the desired ω^* open neighborhood of e^* in E^*, by Observation 2.

We will first prove:

PROPOSITION 2. *Let E be a subspace of a separable Banach space X. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of finite dimensional subspaces of X with $X_1 \subset X_2 \subset \cdots$, $\bigcup_{n=1}^{\infty} X_n$ dense in X and $\bigcup_{n=1}^{\infty} X_n \cap E$ dense in E. Let $\{\beta(n)\}_{n=1}^{\infty}$ and*

$\{\varepsilon(n)\}_{n=1}^{\infty}$ *be decreasing sequences of positive numbers with*

$$\prod_{n=1}^{\infty}(1+\varepsilon(n)) < \lambda \quad \text{and} \quad \sum_{n=1}^{\infty} 2^n\beta(n) \le 1$$

and let $V_n = \beta(n+1)B(X^*) + X_n^+$. *Finally, put* $\lambda(n) = \prod_{i=1}^{n}(1+\varepsilon(i))$, $\lambda(0) = 1$ *and make the following assumption*:

(5.1) *For every* $n \ge 1$, $e_0^* \in B(E^*)$, $x_0^* \in X^*$ *and* $v^* \in V_n$ *for which* $x_0^* + v^* \in \psi(e_0^*)$ *and* $\|x_0^*|x_n\| \le \lambda(n-1)$ *there is an* $\omega^* \in V_n$ *such that*

$$x_0^* + \omega^* \in \lambda(n)B(X^*) \cap \psi(e_0^*).$$

Then there is an ω^* *continuous function* φ: $B(E^*) \to \lambda B(X^*)$ *which extends functionals.*

Proof. Our argument is a modification of the proof of Theorem 2.3 of [6]. We will construct a sequence of ω^* continuous functions φ_n: $B(E^*) \to \lambda(n-1)B(X^*)$ such that the following two conditions are satisfied for every $e^* \in B(E^*)$:

$$\varphi_i(e^*) \in \psi(e^*) + V_i \quad i = 1,2,\dots, \tag{5.2}$$

$$\varphi_i(e^*) \in \varphi_{i-1}(e^*) + 2V_{i-1} \quad i = 2,3,\dots. \tag{5.3}$$

Once this is proved, the ω^* compactness of $\lambda \cdot B(X^*)$ will yield a uniform limit function

$$\varphi: B(E^*) \to \lambda B(X^*)$$

which is an ω^* continuous selection of ψ. We will construct the ϕ_n by induction. Our first step is to construct ϕ_1. To do this, for each $e_0^* \in B(E^*)$ pick an

$$x_0^* = x_0^*(e_0^*) \in B(X^*) \cap \psi(e_0^*)$$

and consider the set

$$\begin{aligned} U^1(e_0^*) &= \{e^* \in B(E^*) : x_0^*(e_0^*) \in \psi(e^*) + V_1\} \\ &= \{e^* \in B(E^*) : \psi(e^*) \cap (x_0^*(e_0^*) + V_1) \ne \emptyset\}. \end{aligned}$$

Since ψ is ω^* l.s.c., $U^1(e_0^*)$ is an ω^* open set and the collection $\{U^1(e_0^*): e_0^* \in B(E^*)\}$ covers the ω^* compact $B(E^*)$. Hence there is a finite subcover $U^1(e_{1,1}^*), \dots, U^1(e_{1,N(1)}^*)$ and a partition of the unit consisting of the ω^*

continuous non-negative functions $p_1^1, \dots, p_{N(1)}^1$ such that

$$\sum_{i=1}^{N(1)} p_i^1(e^*) = 1 \quad \text{for all } e^* \in B(E^*)$$

and for every $1 \le i \le N(1)$, p_i^1 vanishes outside $U^1(e_{1,i}^*)$. Let $x_{1,i}^* = x^*(e_{1,i}^*)$ then the function

$$\varphi_1(e^*) = \sum_{i=1}^{N(1)} p_i^1(e^*) x_{1,i}^*$$

is ω^* continuous and if $p_i^1(e^*) \neq 0$ then $e^* \in U^1(e_{1,i}^*)$ and hence $x_{1,i}^* \in \psi(e^*) + V_1$. Since V_1 is convex we get $\varphi_1(e^*) \in \psi(e^*) + V_1$ and, clearly $\|\varphi_1(e^*)\| \le \max\|x_{1,i}^*\| \le 1$. This completes the construction of ϕ_1. Suppose now that the ω^* continuous functions $\varphi_1, \varphi_2, \dots, \varphi_n$ have been constructed with

$$\varphi_i \colon B(E^*) \to \lambda(i-1) B(X^*)$$

so that (5.2) and (5.3) are satisfied for $1 \le i \le n$ and proceed by induction. Let $e_0^* \in B(E^*)$ and put $x_0^* = \varphi_n(e_0^*)$. Then $\|x_0^*\| \le \lambda(n-1)$ and, by (5.2), there is a $v^* = v^*(e_0^*) \in V_n$ such that $x_0^* + v^* \in \psi(e^*)$. By (5.1) there is an $\omega^* = \omega^*(e_0^*) \in V_n$ such that

$$x_0^* + \omega^* \in \lambda(n) B(X^*) \cap \psi(e_0^*).$$

Since φ_n is ω^* continuous, it follows from Proposition 2.5 of [6] that the carrier

$$\big(\varphi_n(e^*) + V_n\big) \cap \psi(e^*)$$

is ω^* l.s.c. and therefore the set

$$\begin{aligned} U^{n+1}(e_0^*) &= \{e^* \in B(E^*) \colon \varphi_n(e_0^*) + \omega^*(e_0^*) \\ &\qquad \in \big(\varphi_n(e^*) + V_n\big) \cap \psi(e^*) + V_{n+1}\} \\ &= \{e^* \in B(E^*) \colon \big[\big(\varphi_n(e^*) + V_n\big) \cap \psi(e^*)\big] \\ &\qquad \cap \big(\varphi_n(e_0^*) + \omega^*(e_0^*) + V_{n+1}\big) \neq \varnothing\} \end{aligned}$$

is ω^* open. Since the collection $\{U^{n+1}(e_0^*) \colon e_0^* \in B(E^*)\}$ covers $B(E^*)$, there is a finite subcover

$$U^{n+1}(e^*_{n+1,1}),\dots,U^{n+1}(e^*_{n+1,N(n+1)})$$

and a partition of the unit which consists of the ω^* continuous non-negative functions $p_1^{n+1},\dots,p_{N(n+1)}^{n+1}$ with $\sum_{i=1}^{N(n+1)}p_i^{n+1}(e^*)=1$ for every $e^*\in B(E^*)$ such that each p_i^{n+1} vanishes outside $V^{n+1}(e^*_{n+1,i})$. Define

$$\varphi_{n+1}(e^*)=\sum_{i=1}^{N(n+1)}p_i^{n+1}(e^*)x^*_{n+1,i}\quad\text{where } x^*_{n+1,i}=\varphi_n(e^*_{n+1,i})+\omega^*(e^*_{n+1,i}).$$

It follows that

$$\|\varphi_{n+1}(e^*)\|\le\max\|x^*_{n+1,i}\|\le\lambda(n)\quad\text{for all } e^*\in B(E^*).$$

If $p_i^{n+1}(e^*)\neq 0$ then $e^*\in U^{n+1}(e^*_{n+1,i})$ and so

$$x^*_{n+1,i}\in(\varphi_n(e^*)+V_n)\cap\psi(e^*)+V_{n+1}.$$

Since V_n and V_{n+1} are convex sets we get

$$\varphi_{n+1}(e^*)\in\varphi_n(e^*)+V_n+V_{n+1}\subset\varphi_n(e^*)+2V_n\quad\text{and}$$
$$\varphi_{n+1}(e^*)\in\psi(e^*)+V_{n+1}.$$

This completes the induction step and the proof of Proposition 2.

In order to complete the proof of Theorem 1 we only have to show that condition (5.1) holds for the spaces X and E constructed in Sections 3 and 4. Let

$$1+\varepsilon(n)=\prod_{i=n+1}^{\infty}(1+\delta(i))\quad\text{and}\quad\lambda(n)=\prod_{i=1}^{n}(1+\varepsilon(i))$$

as above; then

$$\prod_{n=1}^{\infty}(1+\varepsilon(n))\le 1+\eta$$

by the definition of $\delta(i)$ in Section 3. Suppose that

$$e_0^*\in B(E^*),\quad x_0^*\in X^*,\quad \|x_0^*|x_n\|\le\lambda(n-1)$$

and

$$x^*=x_0^*+v^*\in\psi(e_0^*).$$

This means that $|x^*(x)| \leq \lambda(n-1)$ for all $x \in B(X_n)$ and $|x^*(x) \leq 1$ for all $x \in B(E)$. It follows from Lemma 3.1 that

$$|x^*(x)| \leq (1+\delta(n+2))\lambda(n-1)\|x\| \quad \text{for all } x \in [X_n + F_{n+1}].$$

Let y_0^* denote the restriction of x^* to $[X_n + F_{n+1}]$; then by the Hahn-Banach Theorem, there is a $y^* \in X_{n+1}^*$ which extends y_0^* and

$$\|y^*\| \leq (1+\delta(n+2))\lambda(n-1).$$

Let $z^* = Q_{n+1}x^* - y^*$; then $z^* \in X_{n+1}^* \cap [X_n + F_{n+1}]^+$. Extend z^* to $X_{n+1} + E_0$ by putting

$$z^*(h) = 0 \quad \text{for all } h \text{ with } Q_{n+1}h = 0$$

and

$$z^*(\omega) = 0 \quad \text{for } \omega \in W_{n+1}$$

(this can be done by defining $z^*(I - Q_{n+1})\omega = -z^*Q_{n+1}\omega$ for $\omega \in U_{n+1}$). Now use Hahn Banach's Theorem again to get an extension u_{n+1}^* of z^* to all of X. Clearly

$$u_{n+1}^* \in [X_n + E]^+,$$
$$\|Q_{n+1}^*(x^* + u_{n+1}^*)\| \leq (1+\delta(n+2))\lambda(n-1)$$

and

$$x^* + u_{n+1}^* \in \psi(e_0^*).$$

We now have

$$\|(x^* + u_{n+1}^*)|_{X_{n+1}}\| \leq (1+\delta(n+2))\lambda(n-1) \quad \text{and} \quad x^* - u_{n+1}^* \in \psi(e_0^*)$$

so, repeating the above procedure we can find $u_{n+2}^* \in [X_{n+1} + E]^+$ such that

$$\|(x^* + u_{n+1}^* + u_{n+2}^*)|_{X_{n+2}}\| \leq (1+\delta(n+3))(1+\delta(n+2))\lambda(n-1).$$

Proceeding by induction we can find a sequence $\{u_{n+i}^*\}_{i=1}^\infty \subset X^*$ such that

$$u_{n+i}^* \in [X_{n+i-1} + E]^+$$

and

$$\left\| x^* + \sum_{j=1}^{i} u^*_{n+j} \Big|_{X_{n+i}} \right\| \le \lambda(n-1) \prod_{j=n+2}^{n+2+i} (1+\delta(j))$$

$$\le \lambda(n-1) \cdot \prod_{j=n+2}^{\infty} (1+\delta(j)) \le \lambda(n).$$

Let $u^* = \omega^* \lim_i \Sigma_{j=1}^{i} u^*_{n+j}$; then

$$u^* \in X_n^+ \subset V_n, \quad x^* + u^* \in \psi(e_0^*) \quad \text{and} \quad \|x^* + u^*\| \le \lambda(n).$$

It follows that $\omega^* = v^* + u^*$ is the desired functional. Condition (5.1) is thus satisfied and so, by Proposition 2 and Proposition 1, (1.1) of the main theorem is established.

Remark. We can easily strengthen this result to show that every separable space E is contained in a space X so that both X and X/E have bases. To see this, let $\{E_n\}_{n=1}^{\infty}$ be a sequence of finite dimensional spaces which is dense in the family of all finite dimensional spaces in the Banach-Mazur distance and let $C_p = (\Sigma \oplus E_n)_{l_p}$, for $1 < p < \infty$. It is known (see e.g. [3]) that $Y \oplus C_p$ has a basis for any Banach space Y with an f.d.d. so if we replace the X above with $X' = X \oplus C_p$, both X' and $X'/E = X/E \oplus C_p$ have bases.

Appendix. We construct the operator S: $E_0 \to Y$ by using a suitable biorthogonal system. Let $m(k) = \dim E_{\alpha(k)}$, $p(k) = \dim W_k$ and $q(k) = \dim G_k$ and suppose that the sequences

$$\{e_i\}_{i=1}^{m(k)} \subset E_{\alpha(k)} \quad \text{and} \quad \{f_i\}_{i=1}^{m(k)} \subset Y^*$$

have been constructed such that:

(a) $f_i(e_j) = \delta_{ij}$ and $\|e_i\| = 1$ for all $1 \le i, j \le m(k)$.
(b) For each $1 \le j \le k-1$,

$$\{e_i\}_{i=1}^{m(j)}, \quad \{e_i\}_{i=m(j+1)-q(j)+1}^{m(j+1)} \quad \text{and} \quad \{e_i\}_{i=m(j+1)-p(j)-q(j)+1}^{m(j+1)-q(j)}$$

are bases of $E_{\alpha(j)}$, G_j and W_j respectively.
(c) For all $1 \le j \le k-1$, if $1 \le i \le m(j) + p(j)$ then $P^*_{\alpha(j)} f_i = f_i$ and if $m(j) + p(j) < i \le m(j+1)$ then $P^*_{\alpha(j)} f_i = 0$.
(d) For all $1 \le j \le k-1$ and $1 \le i \le m(j)$ $f_i(\omega) = 0$ for all $\omega \in W_{j+1}$.

We proceed by induction to construct $\{e_i\}_{i=m(k)+1}^{m(k+1)}$ and $\{f_i\}_{i=m(k)+1}^{m(k+1)}$. Pick a basis $\{u_i\}_{i=1}^{p(k)}$ of W_k with $\|u_i\| = 1$ and put $v_i = P_{\alpha(k)}u_i$. Then, by (3.3), $\{v_i\}_{i=1}^{p(k)}$ is a basis of $P_{\alpha(k)}W_k$. It follows that there exist functionals $\{u_i^*\}_{i=1}^{p(k)}$ in $[y_n^*]_{n=\alpha(k)+1}^{\alpha(k+1)}$ such that $u_i^*(v_j) = u_i^*(u_j) = \delta_{i,j}$ for all $1 \le i, j \le p(k)$. For each $1 \le i \le p(k)$ let

$$e_{m(k)+i} = u_i \quad \text{and} \quad f_{m(k)+i} = u_i^* - \sum_{j=1}^{m(k)} u_i^*(e_j)f_j.$$

Then, by (d),

$$f_i(e_j) = \delta_{i,j} \quad \text{for all } 1 \le i, j \le m(k) + p(k)$$

and

$$P_{\alpha(k)}^* f_i = f_i \quad \text{if } 1 \le i \le m(k) + p(k) = m(k+1) - q(k).$$

Since $E_{\alpha(k+2)} = E_{\alpha(k+1)} + W_{k+1} + G_{k+1}$ is a direct sum we have

$$P_{\alpha(k+1)}W_{k+1} \cap G_k = \{0\}.$$

Therefore there exists a basis $\{g_i\}_{i=1}^{q(k)}$ of G_k with $\|g_i\| = 1$ and functionals $\{g_i^*\}_{i=1}^{q(k)}$ in $[y_i^*]_{i=\alpha(k)+1}^{\alpha(k+1)}$ such that $g_i^*(g_j) = \delta_{i,j}$ for $1 \le i, j \le q(k)$ and $g_i^*(\omega) = 0$ for every $\omega \in W_{k+1}$. Put

$$e_{m(k)+p(k)+i} = g_i \quad \text{and} \quad f_{m(k)+p(k)+i} = g_i^* - \sum_{j=1}^{m(k)+p(k)} g_i^*(e_j)f_j$$
$$\text{for } 1 \le i \le q(k).$$

Then $f_i(e_j) = \delta_{i,j}$ for all $1 \le i, j \le m(k+1)$ and $P_{\alpha(k+1)}^* f_i = f_i$ if $1 \le i \le m(k+1)$. Moreover, if $1 \le i \le m(k+1)$ then $f_i(\omega) = 0$ for all $\omega \in W_{k+1}$. This completes the induction step in the construction of the biorthogonal system. Let $\{\varepsilon(n)\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that

$$\varepsilon(n) \cdot \sum_{j=1}^{m(n+1)} \|f_j\| \le 2^{-n}\varepsilon.$$

Recall that each $e \in E_0$ is supported on $\{y_{2i-1}\}_{i=1}^{\infty}$ and

$$\dim\big(I - P_{\alpha(k)}E_{\alpha(k+1)} \le \tfrac{1}{2}(\alpha(k+1)) - \alpha(k)\big).$$

For each $k \geq 1$ and $m(k) < i \leq m(k+1)$ put

$$e_i' = e_i + \varepsilon(k) y_{j(i)} \quad \text{where } j(i) = \alpha(k) + 2(i - m(k)).$$

Then $\|e_i' - e_i\| \leq \varepsilon(k)$. Let $Se_i = e_i'$ and extend S to a linear operator from E_0 into Y. Condition (3.5) is clearly satisfied by the definition of $\varepsilon(k)$. Since

$$P_{\alpha(k+1)} e_i' = e_i' \quad \text{for all } m(k) < i \leq m(k+1)$$

we get (3.6). If $m(k) + p(k) < i \leq m(k+1)$ (i.e., $e_i \in G_k$) then $P_{\alpha(k)} e_i' = 0$; hence

$$S(G_k) = (I - P_{\alpha(k)})Y \quad (\text{cf. } (7)).$$

If $m(k) < i \leq m(k) + p(k)$ (i.e., $e_i \in W_k$) then $P_{\alpha(k)} e_i' = P_{\alpha(k)} e_i$ is an element supported on $\{y_{2i-1}\}_{i=1}^{\infty}$; hence

$$(I - P_{\alpha(k-1)}) S E_{\alpha(k)} \cap P_{\alpha(k)} W_k = \{0\} \quad (\text{cf. } (3.9)),$$

and $P_{\alpha(k-1)} e_i' = 0$ and so (3.8) holds. This concludes the construction of S.

Acknowledgement. We thank the referee for many valuable remarks which made this paper readable and for making the remark preceding the appendix.

References

1. N. Dunford and J. Schwartz, *Linear Operators, Vol. I*, Interscience, New York, 1958.
2. W.B. Johnson and H.P. Rosenthal, *On ω*-basic sequences and their applications to the study of Banach spaces*, Studia Math., vol. 43 (1972), pp. 77–92.
3. W.B. Johnson, H.P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math., vol. 9 (1971), pp. 488–506.
4. J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc., vol. 48, 1964.
5. ______, *On James' paper "Separable conjugate spaces"*, Israel J. Math., vol. 9 (1971), pp. 279–284.
6. E. Michael, *Continuous selections I*, Ann. of Math., vol. 63 (1956), pp. 361–382.

The Hebrew University of Jerusalem
Jerusalem, Israel