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FUNCTIONS WITH A UNIQUE MEAN VALUE

BY

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Section 1

Let G be a Hausdorff locally compact group. An *admissible* subspace $S \subset L_\infty(G)$ is a subspace containing the constants such that if $f \in S$, then ${}_g f(x) = f(g^{-1}x)$ defines ${}_g f \in S$. A function $f \in L_\infty(G)$ *potentially has a unique left invariant mean* if there is a constant c such that whenever $f \in S \subset L_\infty(G)$, S an admissible subspace, then any left invariant mean M on S has $M(f) = c$. A function $f \in L_\infty(G)$ *has a unique left invariant mean value* if it potentially has a unique left invariant mean value, and also there is an admissible subspace $S \subset L_\infty(G)$ with $f \in S$ and there is a left invariant mean on S. If G is amenable, the above two notions are the same, but in general a function may potentially have a unique mean value without actually having one. The analogous notions for right translations or translations on left and right are easy to formulate.

A function $f \in L_\infty(G)$ *left averages* (*to* c) if there is a constant c in the $\|\cdot\|_\infty$- closed convex hull of $\{{}_g f: g \in G\}$. Any function which left averages to a constant must potentially have a unique left invariant mean value. The following is well known.

1.1. THEOREM. *If G is amenable as a discrete group, then the following are equivalent for $f \in L_\infty(G)$:*

(1) *f has a unique left invariant mean value;*

(2) *f left averages;*

(3) *$f \in \|\cdot\|_\infty$-closed span $C \cup \{{}_g f - f: g \in G\}$;*

(4) *$f \in \|\cdot\|_\infty$-closed span $C \cup \{{}_g \zeta - \zeta: \zeta \in L_\infty(G),\ g \in G\}$.*

Remark. The implications (2) implies (3) and (3) implies (4) are always true. The implications (3) implies (1), (2) implies (1) and (1) implies (4) only need the assumption that G is amenable as a locally compact group. However, all the other implications need the hypothesis that G is amenable as a discrete group. For example, if G is a compact group with a unique

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invariant mean on $L_\infty(G)$, then an open dense set $V \subset G$ with $\lambda_G(V) < 1$ will give $f = 1_V$ satisfying (4) but not (1), (2), or (3) because both 1 and $\lambda(V)$ can be left-invariant mean values of f on $C + \text{span}\{_g f \colon g \in G\}$.

This result does provide examples that distinguish left and right translations. In Rosenblatt and Talagrand [6], it was shown that a discrete amenable group admits a left invariant mean that is not right invariant if and only if there is an infinite conjugacy orbit $\{xyx^{-1} \colon x \in G\}$ for some $y \in G$. This provides a class of examples of groups for which the following hold.

1.2. THEOREM. *Let G be an amenable discrete group. The following are equivalent:*

(1) *every f which left averages must also right average and/or vice versa;*
(2) *every left invariant mean is right invariant and/or vice versa;*
(3) $\|\cdot\|_\infty$*-closed span of* $\{_g\zeta - \zeta \colon g \in G,\ \zeta \in l_\infty(G)\}$ *is* $\|\cdot\|_\infty$*-closed span of* $\{\zeta_g - \zeta \colon g \in G,\ \zeta \in l_\infty(G)\}$.

Proof. By Theorem 1 and its obvious analogue for right translations, (3) and (1) are equivalent. Clearly (3) also implies (2). Conversely, assume (2). If

$$F \in \text{span}\{_g\zeta - \zeta \colon g \in G,\ \zeta \in l_\infty(G)\}$$

is not in

$$\|\cdot\|_\infty\text{-closed span}\{\zeta_g - \zeta \colon g \in G,\ \zeta \in l_\infty(G)\},$$

then there exists $M \in l_\infty^*(G)$, $M(F) \neq 0$, with $M(\zeta_g - \zeta) = 0$ for all $g \in G$, $\zeta \in l_\infty(G)$. It follows that M^+ and M^- are both right invariant and $M^+(F) \neq M^-(F)$. But by the assumption (2), M^+ and M^- are left invariant and so $M^+(F) = M^-(F) = 0$ because

$$F \in \text{span}\{_g\zeta - \zeta \colon g \in G,\ \zeta \in l_\infty(G)\}.$$

This shows (2) implies (3). □

Remark. Theorem 1.2 shows that left averaging and right averaging are not the same in general. However, this theorem does not address non-amenable groups. We conjecture that if G is non-amenable, then there is $f \in L_\infty(G)$ such that f left averages, but f does not right average. See Section 2 for a proof of this fact in the case that G is discrete.

If G is not amenable, then the set of functions $\mathscr{U}$ with a unique left mean value is not well understood. It is not known if these functions form a subspace of $L_\infty(G)$. Indeed, a related problem is not resolved: does there exist a largest admissible subspace of $L_\infty(G)$ on which there exists a unique G-invariant mean? See Section 2.

1.3. PROPOSITION. *A function $f \in L_\infty(G)$ admits a unique left invariant mean value if and only if there is a unique constant c such that whenever $\alpha \in C$, $\alpha_i \in C$, $g_i \in G$, $i = 1, \ldots, n$, and $\alpha + \sum_{i=1}^n \alpha_{i\,g_i}f \geq 0$, then*

$$\alpha + \sum_{i=1}^{n} \alpha_i c \geq 0 \qquad (*)$$

Proof. If f admits a unique left invariant mean value c, then there exists an admissible subspace $S \subset L_\infty(G)$, with $f \in S$, and a left invariant mean M on S with $M(f) = c$. Now if

$$\alpha + \sum_{i=1}^{n} \alpha_{i\,g_i}f \geq 0,$$

then

$$\alpha + \sum_{i=1}^{n} \alpha_i c = M\left(\alpha + \sum_{i=1}^{n} \alpha_{i\,g_i}f\right) \geq 0.$$

But also, this constant c is unique with this property. Indeed, if c_0 has this property, then we can define M_0 on $S_0 = C + \operatorname{span}\{_g f\colon g \in G\}$ by

$$M_0\left(\alpha + \sum_{i=1}^{n} \alpha_{i\,g_i}f\right) = \alpha + \sum_{i=1}^{n} \alpha_i c_0$$

and obtain a left invariant mean M_0 on an admissible subspace S_0 with $M_0(f) \neq c$, contrary to assumption.

Conversely, this same construction shows that if c has property $(*)$, then there exists an invariant mean M on $C + \operatorname{span}\{_g f\colon g \in G\}$ with $M(f) = c$. Also, if S_0 is any other admissible subspace of $L_\infty(G)$ with $f \in S_0$ and S_0 admits a left invariant mean M_0, then $M_0(f)$ satisfies property $(*)$ and by uniqueness, $M_0(f) = c$. Hence, c is potentially and actually the unique left invariant mean value of f. □

1.4. COROLLARY. *If f left averages and right averages, then f left and right averages to a unique constant c and f has a unique left and/or right invariant mean value c.*

Proof. If f right averages to c and $F = \alpha + \sum_{i=1}^n \alpha_{i\,g_i}f \geq 0$, then by averaging F on the right, we see $\alpha + \sum_{i=1}^n \alpha_i c \geq 0$. So c is a left mean value of f. By the usual argument, if f both left and right averages, then there is a unique constant c to which it averages. But then if

$$\alpha + \sum_{i=1}^{n} \alpha_{i\,g_0}f \geq 0$$

always implies

$$\alpha + \sum_{i=1}^{n} \alpha_i \gamma \geq 0,$$

we can argue $\gamma = c$. Indeed, choosing an average

$$Af = \sum_{i=1}^{n} \lambda_{i g_i} f$$

with $\|Af - c\|_\infty < \varepsilon$, $c + \varepsilon - \sum_{i=1}^{n} \lambda_{i g_i} f \geq 0$ and $-c + \varepsilon + \sum_{i=1}^{n} \lambda_{i g_i} f \geq 0$. So

$$c + \varepsilon - \gamma \geq 0 \quad \text{and} \quad -c + \varepsilon + \gamma \geq 0.$$

Since $\varepsilon > 0$ is arbitrary, $c = \gamma$. □

Remark. This is the abstract principle that enables one to construct a unique G-invariant mean on WAP(G) given Ryll-Nardzewski's theorem.

Here are some of the unresolved questions:

(a) Does there exist f which left averages to a unique constant, but f does not have a unique left invariant mean value, or vice versa?
(b) Is $\mathscr{U}$ a subspace if G is not amenable?
(c) If G is not amenable does there exist $f_1, f_2 \in L_\infty(G)$ such that f_1 and f_2 have unique left invariant mean values, but $f_1 + f_2$ does not in the sense that $(*)$ is satisfied for more than one constant? Such an example would resolve b) for the group in question.
(d) Is there a largest admissible subspace with a unique left invariant mean value? By Zorn's Lemma, there are always maximal spaces of this type. Is there a maximum such space?
(e) How different, if at all, are $\mathscr{U}$, $\{f \in L_\infty(G)\colon f$ left averages$\}$, $\{f \in L_\infty(G)\colon f$ right averages$\}$, and $\{f \in L_\infty(G)\colon f$ left and right averages$\}$? These questions are related to (a).

Note that there is a possible phenomenon related to (d) here. One can possibly have admissible subspaces $S_1 \subset S_2$ such that S_1 admits more than one left invariant mean, but S_2 admits a unique left invariant mean. For this reason, if there is a largest subspace W with a unique left invariant mean, then $\mathscr{U} \subset W$ but possibly $\mathscr{U} \neq W$. Hence, if a) is shown, it is not clear that then W does not exist. See Section 2 for answers to some of the above, in case G contains non-abelian free groups.

A property related to the above is easy to show in general: the functions that left average do not in general form a subspace. Indeed, we have this theorem; it should be compared with Emerson [2].

1.5. THEOREM. *For a discrete group G, G is amenable if and only if whenever f_1 and f_2 left average, then $f_1 + f_2$ left averages. Indeed, if G is not amenable, then there are f_1 and f_2 which left average to 0 such that $f_1 + f_2$ does not left average.*

Proof. One direction above is proved by Theorem 1.1. Conversely, if G is not amenable, then $l_\infty(G) = \|\cdot\|_\infty$-closed span $\{{}_gf - f\colon g \in G,\ f \in l_\infty(G)\}$. Since G is infinite, there exists $A \subset G$ such that both A and $A^c = G \setminus A$ are permanently positive. See Pier [4] or Rosenblatt [6] for references. Hence, any average $\sum_{i=1}^n \lambda_i 1_{g_iA}$ is 1 somewhere and 0 somewhere. Thus, 1_A does not left average. It is easy to see if

$$f \in l_\infty(G), \quad (f_n) \subset l_\infty(G), \quad \lim_{n\to\infty} \|f - f_n\|_\infty = 0,$$

and each f_n left averages, then f left averages. So some

$$F \in \operatorname{span}\{{}_gf - f\colon g \in G,\ f \in l_\infty(G)\}$$

does not left average. But each ${}_gf - f$ left averages to 0 because

$$\lim_{N\to\infty} \left\| (1/N) \sum_{n=1}^{N} {}_{g^n}({}_gf - f) \right\|_\infty = 0.$$

Therefore, the set $\{f \in l_\infty(G)\colon f \text{ left averages}\}$ is not a subspace of $l_\infty(G)$, i.e., there exists f_1 and f_2 which left average such that $f_1 + f_2$ does not. By subtracting suitable constants c_1 and c_2, $f_1 - c_1$ and $f_2 - c_2$ left average to 0, but $f_1 - c_1 + f_2 - c_2 = f_1 + f_2 + c$, where $c = -c_1 - c_2$, does not left average. □

Remark. (1) It is probably the case in general for non-amenable groups that there exist functions which left average to more than one constant. See Section 2 for a proof of this in the case that G is a discrete group.

(2) One question here is whether for non-amenable groups

$$\operatorname{span}\{{}_gf - f\colon g \in G,\ f \in l_\infty(G)\}$$

is closed; i.e. does every TILF have to be 0? Woodward [10] resolved this for amenable groups in the negative, Saeki [7] resolved it affirmatively for the free group F_2 and Willis [9] showed this for all non-amenable groups. See Section 2, Proposition 2.12 ff.

(3) The previous theorem is almost true for all groups. If G is a non-amenable locally compact group, then the conclusion above is true. However, if G is amenable, but not amenable as a discrete group, it is not

clear whether $\{f \in L_\infty(G): f \text{ left averages}\}$ forms a subspace. Would this imply G_d amenable?

A stronger averaging property gives an interesting variant of Theorem 1.5. A function $f \in L_\infty(G)$ *strongly left averages* if every linear combination $\sum_{i=1}^m \alpha_{i\,g_i} f$ left averages. If one only knows the same for convex combinations $\sum_{i=1}^m \lambda_{i\,x_i} f$, then write

$$\sum_{i=1}^{m} \alpha_{i\,x_i} f = c_1 A_1 - c_2 A_2 \quad \text{where } c_1 = \sum \{\alpha_i \colon \alpha_i > 0\},$$

unless all $\alpha_i \leq 0$ and then $c_1 = 0$, and $c_2 = -\Sigma\{\alpha_i\colon \alpha_i < 0\}$, unless all $\alpha_i \geq 0$ and then $c_2 = 0$, and A_1, A_2 are the appropriate convex combinations. If we can choose a constant a_1 and an average

$$\sum_{j=1}^{n} \lambda_{j\,g_j} A_1 \quad \text{with} \left\| a_1 - \sum_{j=1}^{m} \lambda_{j\,g_j} A_1 \right\|_\infty < \varepsilon,$$

then $A = \sum_{j=1}^m \lambda_{j\,g_j} A_2$ is an average of translates of f too. So if we can choose a constant a_2 and an average

$$\sum_{k=1}^{p} \gamma_{k\,h_k} A \text{ with} \left\| a_2 - \sum_{j=1}^{p} \gamma_{k\,h_k} A \right\|_\infty < \epsilon,$$

we then would have

$$\begin{aligned}
&\left\| c_1 a_1 - c_2 a_2 - \sum_{k=1}^{p} \gamma_{k\,h_k} \sum_{j=1}^{n} \lambda_{j\,g_j} \left(\sum_{i=1}^{n} \alpha_{i\,x_i} f \right) \right\|_\infty \\
&\quad \leq \left\| c_1 a_1 - \sum_{k=1}^{p} \gamma_{k\,h_k} \left(\sum_{j=1}^{n} \lambda_{j\,g_j} c_1 A_1 \right) \right\|_\infty \\
&\qquad + \left\| -c_2 a_2 + \sum_{k=1}^{p} \gamma_{k\,h_k} \left(\sum_{j=1}^{n} \lambda_{j\,g_j} c_2 A_2 \right) \right\|_\infty \\
&\quad = \left\| \sum_{j=1}^{m} \gamma_{j\,h_j} \left(c_1 \left(a_1 - \sum \lambda_{i\,g_i} A_1 \right) \right) \right\|_\infty \\
&\qquad + \left\| c_2 \left(\sum_{k=1}^{p} \gamma_{k\,h_k} A - a_2 \right) \right\|_\infty \\
&\quad \leq \epsilon |c_1| + \varepsilon |c_2|.
\end{aligned}$$

Hence, we see that f strongly left averages if and only if every convex combination $\Sigma_{i=1}^{n}\lambda_{i\,g_i}f$ left averages.

The same type of argument as the one above shows that if f_1 left averages and f_2 strongly left averages, then $f_1 + f_2$ left averages. This gives:

1.6. THEOREM. *A discrete group G is amenable if and only if whenever f left averages, then any average $\Sigma_{i=1}^{n}\lambda_{i\,g_i}f$ also left averages.*

Proof. By Theorem 1, if G is amenable and f left averages, then so does any average $\Sigma_{i=1}^{n}\lambda_{i\,g_i}f$. If G is not amenable, then there are f_1 and f_2 which left average to 0, but $f_1 + f_2$ does not left average. By the remark above, f_2 cannot strongly left average. □

Remark. By approximating λ_i by rationals, it is easy to see that if G is not amenable, then there is f which left averages to 0 such that some average $(1/N)\Sigma_{i=1}^{N}{}_{g_i}f$ does not left average.

Again, it is easy to see that if f_1 and f_2 strongly left average, then $f_1 + f_2$ strongly left averages. However, it is not clear whether $\{f \in L_\infty(G)$: f strongly left averages$\}$ admits a (unique) left invariant mean. It is clear, just as for WAP(G), that $\{f \in L_\infty(G)$: f strongly left and right averages$\}$ admits a unique left invariant mean. More generally,

$$\mathscr{A}_u = \{f \in L_\infty(G) : f \text{ strongly left averages to a unique constant } c\}$$

is a subspace admitting a unique left invariant mean M_u. The problem is whether every function which strongly left averages, must average to a unique constant. We will see in Section 2 that this is not the case. It is worthwhile to observe here that $\mathscr{A}_u$ is in some sense relatively small.

1.7. THEOREM. *If $\mathscr{S}$ is an admissible subspace admitting a (unique) left invariant mean M, then the subspace $\mathscr{S} + \mathscr{A}_u$ admits a (unique) left invariant mean M.*

Proof. If M is unique, there is only one possible value for $M(f_1 + f_2)$ if $f_1 \in \mathscr{S}$ and $f_2 \in \mathscr{A}_u$, namely $M(f_1) + M_u(f_2)$. We show that if $f_1 + f_2 \geq 0$, then $M(f_1) + M_u(f_2) \geq 0$. Indeed, for all $\varepsilon > 0$, there is an average

$$A(f_2) = \sum_{i=1}^{n} \alpha_{i\,g_i}f_2 \quad \text{with } \|A(f_2) - M_u(f_2)\|_\infty \leq \epsilon.$$

So $A(f_1) + M_u(f_2) \geq A(f_1) + A(f_2) - \varepsilon \geq -\varepsilon$. But then

$$M(f_1) + M_u(f_2) = M(A(f_1)) + M_u(f_2) = M(A(f_1) + M_u(f_2)) \geq -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $M(f_1) + M_u(f_2) \geq 0$. □

Remark. Is $\mathscr{A}_u$ the largest left-invariant subspace with the property of Theorem 1.7?

Section 2

It is possible to answer many of the previous questions about functions with unique left invariant mean values in the class of discrete groups which are non-amenable because they contain non-abelian free groups.

First, consider a free group F_2 on free generators x and y. Let Y be the words in reduced form which begin on the left with y or y^{-1} and let X be the same set with x playing the role of y. Then $\{x^nY: n \in Z\}$ and $\{y^nX: n \in Z\}$ are partitions of $F_2 \setminus \{e\}$. Let $f = 1_Y$. Then

$$\frac{1}{N} \sum_{n=1}^{N} {}_{x^n}f = \frac{1}{N} 1_{\bigcup_{n=1}^{N} x^n Y}$$

and hence f left averages to 0. But similarly 1_X left averages to 0. Now

$$1 = 1_{\{e\}} + 1_Y + 1_X.$$

Hence,

$$\frac{1}{N} \sum_{n=1}^{N} {}_{y^n}f = 1 - \frac{1}{N} 1_{\{y_0, \dots, y^N\}} - \frac{1}{N} 1_{\bigcup_{n=1}^{N} y^n X}.$$

Therefore, f left averages to 1.

2.1. THEOREM. *If G is a discrete group containing F_2, then there is a set $A \subset G$ such that 1_A left averages to any constant c, $0 \leq c \leq 1$.*

Proof. Let $\{x_\alpha: \alpha \in \mathscr{A}\}$ be a set of right cosets representatives of F_2 in G. Let Y be as above and let $A = \bigcup\{Yx_\alpha: \alpha \in \mathscr{A}\}$. From the argument above, it is clear that

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} {}_{x^n}1_A \right\|_\infty = 0$$

and

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}{}_{y^n}1_A - 1\right\|_\infty = 0.$$

So both 0 and 1 are in $O_l(1_A)$, the $\|\cdot\|_\infty$-closed convex hull of the left translates of 1_A. Hence, $[0,1]\subset O_l(1_A)$. □

2.2. Corollary. *If G is a discrete group containing F_2, then there is a set $A\subset G$ such that 1_A left averages, but does not right average.*

Proof. Use Corollary 1.4 and the example of $A\subset G$ from Theorem 2.1. □

Remark. See 2.12 following.

An example like the previous one will have an even stronger property. Let $X_0 = F_2\setminus X$ and $Y_0 = F_2\setminus Y$.

2.3. Lemma. *If $f\in l_\infty(F_2)$ is constant on some gX_0, then f left averages to that constant.*

Proof. If $f=c$, a constant, on gX_0, then $|f-c|\le K1_{F_2\setminus gX_0}$ where $K=\|f\|_\infty + c$. So

$$|_{g^{-1}}f - c| \le K1_{F_2\setminus X_0} = K1_X.$$

As above, there is a sequence A_n of left averages such that

$$\lim_{n\to\infty}\|A_n 1_X\|_\infty = 0.$$

So

$$\begin{aligned}\limsup_{n\to\infty}\left\|A_n({}_{g^{-1}}f) - c\right\|_\infty &\le \limsup_{n\to\infty}\left\|A_n(|{}_{g^{-1}}f - c|)\right\|_\infty \\ &\le K\limsup_{n\to\infty}\|A_n(1_X)\|_\infty = 0.\end{aligned}$$ □

Remark. If f is constant, except for finitely many values on some gX_0, then the same conclusion holds. Indeed, then $f+h$ is constant c on some gX_0 for some h with finite support. Hence, for all $\varepsilon>0$, there is an average $A(f+h)$ such that $\|A(f+h)-c\|_\infty\le\varepsilon$. But then

$$|A(f) - c| \le A(|h|) + \varepsilon.$$

Since $A(h)$ has finite support, there is another average B with $B(A(|h|)) \leq \varepsilon$. Hence

$$|B(A(f)) - c| \leq B(|Af - c|) \leq B(A(|h|)) + \varepsilon \leq 2\varepsilon.$$

2.4. LEMMA. *In the free group F_2, if finitely many left translates of X_0 have non-empty intersection, then there is a left translate of X_0 contained in that intersection.*

Proof. Let $g \in \bigcap_{i=1}^{n} g_i X_0$, i.e., $g_i^{-1}g \in X_0$ for $i = 1, \ldots, n$. Write

$$X_0 = \{e\} \cup \bigcup_{k \neq 0} Y_k \quad \text{where } Y_k = y^k(\{e\} \cup X).$$

We claim that for each $i = 1, \ldots, n$, $g_i^{-1}gY_k \subset X_0$ for all but finitely many values of k. Indeed, if there is an x or x^{-1} in the reduced form of $g_i^{-1}g$, then $g_i^{-1}gY_k \subset X_0$ if $|k|$ is sufficiently large. Otherwise, $g_i^{-1}g = y^{k_0}$ and then $g_i^{-1}gY_k \subset X_0$ for all $k \neq -k_0$.

Thus, there is some $k \neq 0$ such that $g_i^{-1}gY_k \subset X_0$ for $i = 1, \ldots, n$. That is,

$$gY_k \subset \bigcap_{i=1}^{n} g_i X_0.$$

But gY_k contains $gy^k x X_0$. □

2.5. THEOREM. *Suppose $f \in l_\infty(F_2)$ is such that for all $g \in F_2$, there exists $h \in F_2$ such that $g \in hX_0$ and f is constant on hX_0. Let $\bar{f}$ be any left average of f. Then $\bar{f}$ left averages to any value between* $\sup \bar{f}$ *and* $\inf \bar{f}$.

Proof. Let $g \in F_2$. It is enough to show that $\bar{f}$ left averages to $\bar{f}(g)$. Suppose

$$\bar{f} = \sum_{i=1}^{n} \alpha_{i\, g_i} f$$

is a left average of f. Then

$$\bar{f}(g) = \sum_{i=1}^{n} \alpha_i f(g_i^{-1}g).$$

By the conditions on f, for each i there is some $h_i X_0$ containing $g_i^{-1}g$ on which f is constant. Since

$$g \in \bigcup_{i=1}^{n} g_i h_i X_0,$$

Lemma 2.4 shows there is some

$$hX_0 \subset \bigcap_{i=1}^{n} g_i h_i X_0.$$

Hence, $\bar{f}$ is constantly $\bar{f}(g)$ on hX_0. By Lemma 2.3, $\bar{f}$ left averages to $\bar{f}(g)$. □

Remark. If f satisfies the hypotheses of Theorem 2.5 except f is only constant on xX_0 excluding a finite number values, then f will strongly left average again, although perhaps not to any c in $[\inf \bar{f}, \sup \bar{f}]$.

The example 1_X above has the property needed in 2.5. Indeed, $\{X_0, X\}$ is a partition of F_2 and $\{x^k X_0 \colon k \neq 0\}$ is a partition of X. Hence, 1_X strongly left averages too. By using the right coset construction of Theorem 2.1, this shows:

2.6. THEOREM. *If G is a discrete group containing F_2, then there is a set $A \subset G$ such that 1_A strongly left averages, but does not right average.*

A refinement of the previous arguments gives even more. Again, let x and y be free generators of F_2. Let

$$X' = \bigcup \{x^n X_0 \colon n \in Z, n \text{ odd}\}.$$

If $f \in l_\infty(F_2)$, $0 \leq f \leq 1_{F_2 \setminus X'}$, then f averages to 0 by Lemma 2.3 because $xX_0 \subset X'$. Choose any $B \subset F_2$ such that both B and $B^c = F_2 \setminus B$ are permanently positive. There is no harm in assuming $e \notin B$. Let

$$B' = B \cap X' \quad \text{and} \quad B'' = B \cap (F_2 \setminus X')$$

and let

$$A = x^{-1}B' \cup B''.$$

Then xA and A are disjoint and $xA \cup A \supset B$; hence, $xA \cup A$ is permanently positive. Also, $F_2 \setminus A$ is permanently positive. Indeed, if $g_1, \ldots, g_n \in G$, there exists

$$g \in \bigcap_{i=1}^{n} g_i B^c \cap \bigcap_{i=1}^{n} g_i x^{-1} B^c.$$

So for all i, $g_i^{-1}g \notin B$ and so $g_i^{-1}g \notin B''$, while $xg_i^{-1}g \notin B$ and so $g_i^{-1}g \notin x^{-1}B'$. Hence,

$$g_i^{-1}g \notin x^{-1}B' \cup B'' = A \quad \text{for all } i,$$

and therefore, $g \notin \bigcup_{i=1}^{n} g_i A$.

This shows that $f = 1_A$ has the following properties:

(1) f left averages to 0;
(2) any left average of f has minimum equal to 0;
(3) f has a unique left invariant mean value of 0;
(4) any left average of ${}_xf + f$ has maximum value of 1.

We have observed (1) and constructed A so that (2) and (4) holds. But (2) shows

$$\|\cdot\|_\infty\text{-closed span } C \cup \{{}_gf \colon g \in G\}$$

admits a left invariant mean m with $m(f) = 0$, while (1) shows f potentially has a unique left invariant mean value of 0. Hence, (3) holds too. Note that this f then is an explicit example of Theorem 1.6 in that ${}_xf + f$ cannot left-average since, by (2) it can only left average to 0, and by (4) it can only left average to 1.

This function f also has the property that by (3), ${}_xf + f$ can have a left invariant mean value of 0, and by (4), ${}_xf + f$ can have a left invariant mean value of 1. So for any c, $0 \le c \le 1$, ${}_xf + f$ can have a left invariant mean value of c. This answers (c), and hence (b) of Section 1 in this case.

2.7. THEOREM. *If G is a discrete group containing F_2, then there is a set $A \subset G$ and $x \in G$ such that 1_A and ${}_x1_A$ have 0 as a unique left invariant mean value, but $1_A + {}_x1_A$ does not have a unique left invariant mean value; hence, $\mathscr{U}$ is not a subspace.*

Proof. Let $\{x_\alpha\}$ be right coset representatives of F_2 in G. Let

$$X' = \bigcup \{x^n X_0 x_\alpha \colon n \text{ odd}, \alpha \in \mathscr{A}\}.$$

Let $B \subset G$ be such that B and B^c are permanently positive in G. The rest of the construction proceeds similarly to give $f \in l_\infty(G)$ with properties (1)–(4). □

Furthermore, let f' be defined by $f' = (1 - f)1_{F_2\setminus X'}$. Then f' also satisfies (1)–(4). Property (1) is clear. To see (2), choose any average $Lf = \sum_{i=1}^{n}\alpha_{i\,g_i}f$ where each $\alpha_i > 0$ and $g_i, \ldots, g_n \in F_2$. Then $L({}_xf + f)(g) = 1$ for some $g \in F_2$ by (4). If $f(g_i^{-1}g) = 1$, then $f'(g_i^{-1}g) = 0$. If $f(g_i^{-1}g) = 0$, then $f(x^{-1}g_i^{-1}g) = 1$ and so

$$x^{-1}g_i^{-1}g \in F_2 \setminus X'.$$

Hence, $g_i^{-1}g \in X'$ and $f'(g_i^{-1}g) = 0$ again. Thus, $Lf'(g) = 0$. This proves (2). Since f' left averages to 0, this also proves (3). Moreover, to see (4), observe that $\frac{1}{2}L({}_xf + f)$ is an average of f and so

$$\tfrac{1}{2}L({}_xf + f)(g) = 0 \quad \text{for some } g \in G.$$

But $f' = 1_{F_2\setminus X'} - f$ and so

$${}_xf' + f' = 1_{F_2\setminus xX'} + 1_{F_2\setminus X'} - ({}_xf + f) = 1 - ({}_xf + f).$$

Thus,

$$L({}_xf' + f')(g) = 1 - L({}_xf + f)(g) = 1.$$

But now f and f' have 0 as a unique left mean value, while $f + f' = 1_{F_2\setminus X'}$. Clearly $1_{F_2\setminus X'}$ averages to 0. But also, ${}_x1_{X'} = 1_{F_2\setminus X'}$ so, $1_{X'}$ averages to 0. Since

$$1 = 1_{F_2\setminus X'} + 1_{X'},$$

$1_{F_2\setminus X'}$ must average to 1 too. Hence, $f + f'$ averages to 0 and to 1; therefore $f + f'$ cannot be in any admissible subspace admitting a left invariant mean. Actually $f + f'$ also strongly averages by Theorem 2.5 because $f + f' = 1_{F_2\setminus X'}$ and $\{x^nY_0\colon n \text{ even}\}$ is a partition of $F_2 \setminus X'$.

Now the same right coset construction of Theorem 2.7 shows this partial answer to (d) in Section 1. It also resolves (b) in a different manner than the above.

2.8. THEOREM. *If G is a discrete group containing F_2 as a subgroup, then $\mathcal{U}$ is not a subspace and there is no maximum admissible subspace of $l_\infty(G)$ admitting a (unique) left invariant mean.*

It is worth observing that the above construction gives four sets A, B, C, D which form a partition of F_2 where $1_A = f$, $1_B = f'$, $1_C = {}_xf$, and $1_D = {}_xf'$, so that 1_A, 1_B, 1_C, and 1_D each left averages to 0 and each has any left average with a minimum equal to 0. But then for any $h \in l_\infty(F_2)$, $h = h1_A + h1_B +$

$h1_C + h1_D$, a sum of four functions each of which averages to 0 and has a unique mean value of 0.

2.9. COROLLARY. *If G is a discrete group containing F_2 and $h \in l_\infty(G)$, then*

$$h = f_1 + f_2 + f_3 + f_4$$

where each f_i left averages to 0 and has a unique mean value of 0.

Some constructions related to the above can give us other important examples. Let f be as before in the proof of Theorem 2.8. Define $h \in l_\infty(F_2)$ by

$$h(g) = (1 + f(g) + {}_xf(g))1_{F_2 \setminus X_0}(g).$$

Then h left averages to 0. But $h + {}_xh \geq 1_{F_2 \setminus X_0} + 1_{x(F_2 \setminus X_0)}$. Since

$$x^{-1}X_0 \subset F_2 \setminus X_0 \quad \text{and} \quad x(F_2 \setminus X_0) \supset X_0.$$

Hence, $h + {}_xh \geq 1$ on F_2. Therefore, h cannot be in any admissible subspace which admits a left invariant mean.

However, h left averages to a unique constant. Indeed, suppose h left averages to $c > 0$. Then for $\varepsilon > 0$ there is a convex combination $A(h) = \sum_{i=1}^{n} \alpha_{i\,g_i}h$ with $\|A(h) - c\|_\infty < \varepsilon$. For any $g \in F_2$, gX_0 either contains all but one x^mX_0 (if g ends on the right in reduced form with $y^{\pm 1}$) or gX_0 is contained in some one x^mX_0. Therefore, there exists some x^mX_0 which is contained in g_iX_0 or misses g_iX_0 for all $i = 1, \ldots, n$. Hence, ${}_{g_i}h$ is either constantly 0 or equal to $1 + f + {}_xf$ on x^mX_0. We can assume that g_i, $i = 1, \ldots, n_0$, are such that ${}_{g_i}h = 0$ on x^mX_0 exactly for $i = n_0 + 1, \ldots, n$. Thus, $\sum_{i=1}^{n_0} \alpha_{i\,g_i}(f + {}_xf)$ is a constant c_ε within ε on x^mX_0. But

$$\left\| \sum_{i=1}^{n} \alpha_{i\,g_i}h - c \right\|_\infty \leq \varepsilon \quad \text{on } x^mX_0$$

shows

$$\left| \sum_{i=1}^{n_0} \alpha_{i\,g_i}h - c \right| < \varepsilon \quad \text{on } x^mX_0.$$

Hence,

$$c - \varepsilon \leq \sum_{i=1}^{n_0} \alpha_{i\,g_i}h = \sum_{i=1}^{n_0} \alpha_{i\,g_i}(1 + f + {}_xf) \quad \text{on } x^mX_0.$$

Thus, $c - \varepsilon \le 2\sum_{i=1}^{n_0}\alpha_i$ and so $\sum_{i=1}^{n_0}\alpha_i \ge \frac{1}{2}(c - \varepsilon)$. Hence, $f +_x f$ can be averaged to c_ε within $2\varepsilon/(c - \varepsilon)$. But any left average of $f +_x f$ has minimum value 0 and maximum value 1. This is a contradiction as soon as $2\varepsilon/(c - \varepsilon) < 1/2$.

This construction gives the following result.

2.10. THEOREM. *If G contains F_2, then there is a function $f \in l_\infty(G)$ which left averages to* 0, *and only to* 0, *but f does not have a* (*unique*) *left invariant mean value.*

Conversely, we can construct $f \in \mathscr{U}$ which does not left average. Construct a characteristic function $f \in l_\infty(F_2)$ such that for any $g_1, \dots, g_k$, $\eta_1, \dots, \eta_l$ distinct, there is $g \in F_2$ with $f(g_i^{-1}g) = 0$ for $i = 1, \dots, k$ and $f(\eta_j^{-1}g) = 1$ for $j = 1, \dots, l$. Let $h = f - 1_{X_0}$. To see $h \in \mathscr{U}$, just note that since $h \le 0$ on X_0, for all $\varepsilon > 0$, there is a left average $A(h)$ with $A(h) \le \varepsilon$. Since $h > 0$ on $F_2 \setminus X_0$, for all $\varepsilon > 0$, there is a left average $A(h) \ge -\varepsilon$. Hence, h potentially has 0 as a left invariant mean value. But also, for any linear combination

$$\lambda = \sum_{i=1}^{k} \alpha_{i g_i} h - \sum_{j=1}^{l} \beta_{j \eta_j} h,$$

with $g_1, \dots, g_k, \eta_1, \dots, \eta_l$ distinct, and $\alpha_i, \beta_i \ge 0$, there is a $g \in F_2$ with $f(g_i^{-1}g) = 0$ and $f(\eta_j^{-1}g^{-1}) = 1$ for all i, j. Hence, $h(g_i^{-1}g) \le 0$ and $h(\eta_j^{-1}g) \ge 0$. That is, $\lambda(g) \le 0$. But then if $c + \lambda \ge 0$, $c + \lambda(g) \ge 0$ and so $c \ge 0$. That is, 0 is the unique left-invariant mean value of h.

Now suppose h can be left averaged to c. Then $c = 0$ is the only possibility by the above. Let $A(h) = \sum_{i=1}^{n}\alpha_{i\,g_i}h$ be an average with $\|A(h)\|_\infty \le \varepsilon$. Since there is $g \in F_2$ with $f(g_i^{-1}g) = 1$ for all $i = 1, \dots, n$,

$$\sum \{\alpha_i \colon g_i^{-1}g \notin X_0\} < \varepsilon.$$

Assume $\alpha_1, \dots, \alpha_m$ represent those α_i with $g_i^{-1}g \in X_0$. Then $\sum_{i=1}^{m}\alpha_i > 1 - \varepsilon$. By Lemma 2.4, since $g \in \bigcap_{i=1}^{m} g_i X_0$, there is $\bar{g} \in F_2$ with

$$\bar{g}X_0 \subset \bigcap_{i=1}^{m} g_i X_0.$$

Thus, on $\bar{g}X_0$,

$$\sum_{i=1}^{m} \alpha_{i\,g_i} h = \sum_{i=1}^{m} \alpha_{i\,g_i} f - \sum_{i=1}^{m} \alpha_i.$$

Since

$$\left|\left(\sum_{i=1}^{m} \alpha_{i\,g_i}h\right) \Big/ \left(\sum_{i=1}^{m} \alpha_i\right)\right| \le \varepsilon/(1-\varepsilon) \quad \text{on } \bar{g}X_0,$$

this shows

$$\left|\left(\sum_{i=1}^{m} \alpha_{i\,g_i}f\right) \Big/ \left(\sum_{i=1}^{m} \alpha_i\right) - 1\right| \le \varepsilon/(1-\varepsilon) \quad \text{on } \bar{g}X_0.$$

Since $\varepsilon > 0$ is arbitrary, and 1_{X_0} left averages to 0, this shows f left averages to 1. But this is impossible by the choice of f.

2.11. THEOREM. *If G contains F_2, then there exists $f \in l_\infty(G)$ such that f has a unique left invariant mean value, but f does not left average.*

The examples provided by 2.2, 2.10, and 2.11 show that generally

$$\{f\colon f \text{ left averages}\} \ne \{f\colon f \text{ right averages}\},$$
$$\mathscr{U} \setminus \{f\colon f \text{ left averages}\} \ne \phi$$

and

$$\{f\colon f \text{ left averages to a unique constant}\} \setminus \mathscr{U} \ne \phi.$$

This answers most of (a) and (e) in Section 1, except it does not relate $\mathscr{U}$ and $\{f\colon f \text{ right averages}\}$. It was essentially already observed that if f right averages to c, then $M(f) = c$ defines a left invariant mean on $\|\cdot\|_\infty$-closed span $C + \{_gf\colon g \in G\}$. So if X^0 is the words in reduced form that do not end with $x^{\pm 1}$, then $f = 1_{X^0}$ right averages to any c, $0 \le c \le 1$, and so has different left-invariant mean values. The question should rightly be to relate $\mathscr{U}$ and $\{f\colon f \text{ right averages to a unique constant}\}$. But if G is amenable, these are not the same by Theorem 1.2.

A related question is whether functions which left and right average, admit a two-sided invariant mean on $G \cup \text{span}\,\{_gf\colon g \in G\}$. This is not generally the case. Let

$$f = 1_{F_2 \setminus (X_0 \cup X_0^{-1})}.$$

Then $f = 0$ on X_0 and X_0^{-1}, so f left and right averages to 0, the only possible two-sided invariant mean value. But

$$f(g) + f(xg) + f(gx) + f(xgx) \ge 1 \quad \text{for all } g \in F_2,$$

so f does not admit a two-sided invariant mean value. Notice $f' = 1_{F_2 \setminus (X_0 \cup (F_2 \setminus X_0)^{-1})}$ has the same properties as f but $f + f' = 1_{F_2 \setminus X_0}$ does not right average. So

$$\{f \in l_\infty(F_2): f \text{ left and right averages}\}$$

is not a subspace. This, and the previous examples of this section, show that $\mathscr{A}_u$ seems to be the only reasonable subspace on which there is a unique left invariant mean value. But Theorem 1.7 shows how it is essentially the heart of the class of admissible subspaces admitting a unique left-invariant mean, and hence to be considered a small subspace.

The class of groups that has been considered here has another property relevant to the remarks in the first section. Saeki [7] showed that if $f \in l_\infty(F_2)$ then there is

$$f_1, f_2 \in l_\infty(F_2) \quad \text{with } f = {}_x f_1 - f_1 + {}_y f_2 - f_2.$$

By the coset construction of Proposition 2.1, this proves:

2.12. Proposition. *If G is a discrete group containing F_2, then for every $f \in l_\infty(G)$, there exist $f_1, f_2 \in l_\infty(G)$ such that $f = {}_x f_1 - f_1 + {}_y f_2 - f_2$.*

Remark. When is this two term representation possible? Does it imply that G contains F_2? Note that by Tarski's characterization of non-amenable groups G, there exists sets

$$\{A_1, \ldots, A_n\} \quad \text{and} \quad \{B_1, \ldots, B_n\}$$

which partition G, and some $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ such that

$$\{g_1 A_1, \ldots, g_n A_n, h_1, B_1, \ldots, h_n B_n\}$$

is also a partition of G. Hence,

$$1 = \sum_{i=1}^{n} 1_{A_i} - 1_{g_i A_i} + \sum_{i=1}^{n} 1_{B_i} - 1_{h_i B_i}.$$

That is, if G is non-amenable, there are $f_1, \ldots, f_n \in l_\infty(G)$ and $x_1, \ldots, x_n \in G$ such that

$$1 = \sum_{i=1}^{n} {}_{x_i} f_i - f_i.$$

The question above is in part when is $n = 2$ possible? See the article by Krom and Krom [3] for an analogous question.

Proposition 2.12 clearly shows that if G contains F_2, then the only TILF on $l_\infty(G)$ is 0. But Willis [9] shows that this is true for any amenable group (discrete or not).

2.13. THEOREM (Willis). *If G is a non-amenable locally compact group then*

$$L_\infty(G) = \text{span}\{f - {}_g f\colon g \in G, f \in L_\infty(G)\},$$

and the only TILF on $L_\infty(G)$ is 0.

Another consequence of the argument in [9] is this proposition for discrete non-amenable groups.

2.14. PROPOSITION (Willis). *If G is a discrete non-amenable group, then there exist $g_1, \dots, g_n \in G$ such that $\{g_1, \dots, g_{n-1}\}$ generates an amenable group, and there exist $f_1, \dots, f_n \in l_\infty(G)$ such that $1 = \Sigma_{i=1}^n f_i - {}_{g_i} f_i$.*

2.15. COROLLARY. *If G is a discrete non-amenable group, then there exists $f \in l_\infty(G)$ which left averages to any c, $0 \le c \le 1$, and so f left averages but does not right average.*

Proof. Use 2.14 to write $1 = \Sigma_{i=1}^n f_i - {}_{g_i} f_i$ with $\{g_1, \dots, g_{n-1}\}$ generating an amenable group. Then $\Sigma_{i=1}^{n-1} f_i - {}_{g_i} f_i$ left averages to 0. Hence $f = f_n - {}_{g_n} f_n$ left averages to 1, while it obviously left averages to 0. So f left averages to any c, $0 \le c \le 1$. This f proves the corollary. □

Remark. The extension of this corollary to non-discrete non-amenable groups is open.

Section 3

Another interesting aspect of functions with unique left invariant mean values for amenable groups is that they do not usually form an algebra of functions. Let $\mathcal{U}$ be as before and let $\mathcal{U}^* = \{f \in L_\infty(G)\colon M(f)$ is uniquely determined if M is a left invariant mean on $L_\infty(G)\}$. Hence $\mathcal{U} \subset \mathcal{U}^*$ and $\mathcal{U} = \mathcal{U}^*$ if G is amenable as a discrete group. This was observed in Theorem 1.1 since (4) describes $\mathcal{U}^*$ and (1) describes $\mathcal{U}$.

The basic question is whether $\mathcal{U}^*$ or $\mathcal{U}$ can be an algebra. Forms of this question were considered by Chou [1] and Talagrand [8]. Let $\mathcal{N} = \{f \in L_\infty(G)\colon M(|f|) = 0$ for all left invariant means M on $L_\infty(G)\}$. Clearly if $f \in C + \mathcal{N}$, then $fh \in \mathscr{U}^*$ for all $h \in \mathscr{U}^*$. If $C + \mathscr{N} \neq \mathscr{U}^*$, then the converse below would prove $\mathscr{U}^*$ is not an algebra.

3.1. THEOREM. *If G is amenable as a discrete group and $f \in L_\infty(G)$ with $fh \in \mathscr{U}^*$ for all $h \in \mathscr{U}^*$, then $f \in C + \mathscr{N}$.*

3.2. COROLLARY. *If G is a compact group which is amenable as a discrete group, then only the constant functions $f \in L_\infty(G)$ have the property that $fh \in \mathscr{U}^*$ for all $h \in \mathscr{U}^*$.*

Proof of Theorem 3.1. First, $f({}_g\zeta - \zeta) \in \mathscr{U}^*$ for all $g \in G$ and $\zeta \in L_\infty(G)$. But if m is a left invariant mean,

$$m\big(f({}_{g^{-1}}\zeta - \zeta)\big) = m(f_{g^{-1}}\zeta) - m(f\zeta)$$
$$= m\big(({}_gf)\zeta\big) - m(f\zeta) = m\big(({}_gf - f)\zeta\big).$$

Hence,

$$|{}_gf - f|^2\zeta = ({}_gf - f)\big[\big(\overline{{}_gf - f}\big)\zeta\big\} \in \mathscr{U}^*$$

also.

Let

$$\zeta = \big(1/|{}_gf - f|^2\big)1_{\{|gf-f|^2 \geq \varepsilon\}}.$$

Let

$$E \subset \big\{|gf - f|^2 \geq \varepsilon\big\}$$

be a measurable set. Then

$$|{}_gf - f|^2\zeta 1_E = 1_E$$

is in $\mathscr{U}^*$ for all measurable $E \subset \{|{}_gf - f|^2 \geq \varepsilon\}$. It follows that $1_{\{|{}_gf-f|^2 \geq \varepsilon\}} \in \mathscr{N}$. To see this let

$$E_0 = \big\{|{}_gf - f|^2 \geq \varepsilon\big\}.$$

Since G is amenable as a discrete group, there are left invariant means θ_1, θ_2 and $A \subset G$ with $\theta_1(1_A) = 0$ and $\theta_2(1_{A^c}) = 0$. See Rosenblatt [5]. Let

$$E_1 = E_0 \cap A, \quad E_2 = E_0 \cap A^c, \quad \text{and } \theta = \tfrac{1}{2}(\theta_1 + \theta_2).$$

Then

$$\theta(1_{E_1}) = \theta_2(1_{E_1}) = \theta_2(1_{A^c}1_{E_1}) = 0 \quad \text{and} \quad \theta(1_{E_2}) = \theta_1(1_{E_2}) = \theta_1(1_A 1_{E_2}) = 0.$$

So $\theta(1_{E_0}) = \theta(1_{E_1}) + \theta(1_{E_2}) = 0$ and hence $M(1_{E_0}) = 0$ for all left invariant means M.

But now we have $1_{\{|gf-f| \geq \varepsilon\}} \in \mathcal{N}$ for all $\varepsilon > 0$ and hence $gf - f \in \mathcal{N}$ too. Let c be the unique constant with $M(f) = c$ for all left invariant means M. Let $f_0 = f - c$. Then f_0 has a unique left invariant mean value of 0 and ${}_gf_0 - f_0 \in \mathcal{N}$ for all $g \in G$. But then f_0 averages to 0 by Theorem 1.1. Hence, for all $\varepsilon > 0$, there are $g_1, \dots, g_N \in G$ with

$$\left\| \frac{1}{N} \sum_{i=1}^{N} {}_{g_i}f_0 \right\|_\infty \leq \varepsilon.$$

But

$$\frac{1}{N} \sum_{i=1}^{N} {}_{g_i}f_0 = \frac{1}{N} \sum_{i=1}^{N} \left({}_{g_i}f_0 - f_0 \right) + f_0.$$

This shows $f_0 \in \| \cdot \|_\infty$-closed span $\{ {}_gf_0 - f_0 : g \in G \}$ and so $f_0 \in \mathcal{N}$ by the above. That is $f = c + f_0 \in C + \mathcal{N}$. □

Remark. (1) To show $C + \mathcal{N} \neq \mathcal{U}^*$, and hence show $\mathcal{U}^*$ is not an algebra using Theorem 3.1, requires showing that if G is an amenable discrete group, then there exists $M \subset G$ and $g \in G$ such that $1_{gM \Delta M} \notin \mathcal{N}$. Although this is easy for certain groups, no general argument for it is known. However, an unpublished theorem of Granirer (cf. Chou [1], p. 182) shows in another fashion that $\mathcal{U}^*$ is not an algebra. So $C + \mathcal{N} \neq \mathcal{U}^*$ and the set M and $g \in G$ above exists in general. Granirer's argument uses his theorem that amenable groups do not admit multiplicative invariant means.

(2) Some assumption besides amenability of G as a locally compact group is needed here since if G is a compact group with a unique left invariant mean, then $\mathcal{U}^* = L_\infty(G)$ is an algebra. However, the above does not resolve if $\mathcal{U}$ can be an algebra. Moreover, it is possible that if $f \in \mathcal{U}^*$ and $fh \in \mathcal{U}^*$ for all $h \in \mathcal{U}$, then $f \in C + \mathcal{U}_0$ where $\mathcal{U}_0 = \{f \in \mathcal{U}: f$ has a unique left invariant mean value of $0\}$.

Added in Proof. Tianxuan Miao, *Amenability of locally compact groups and subspaces of $L^\infty(G)$*, Proc. Amer. Math. Soc. (to appear), contains a solution for general non-amenable groups of a number of the questions from Sections 1 and 2.

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