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HOMOGENIZABLE RELATIONAL STRUCTURES

BY

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In this paper we consider certain classes of relational structures which do not have the Amalgamation Property, and hence do not have a universal homogeneous structure, but which can be made to satisfy the Amalgamation Property by the imposition of some additional structure. We thus obtain a class which is the age of a homogeneous structure, and which has the original class as 'underlying structures.' We show that the underlying structure of the countable universal homogeneous structure has a model complete theory, so is a model comparison for its universal theory.

In Section 1 we first define homogenizations and homogenizable classes then introduce the notion of Local Failure of Amalgamation. Section 2 is devoted to stating and proving Theorem 2.1, which gives a sufficient condition for a class of structures to be homogenizable. This condition involves Local Failure of Amalgamation. In Section 3 we show that the theory of the 'homogenized' structure is model complete.

1. Homogenizable structures

DEFINITION 1.1. Let $\mathscr{L}$ be a purely relational first-order language, and let $\mathscr{C}$ be a class of $\mathscr{L}$-structures. The age of an $\mathscr{L}$-structure Γ is the class of structures isomorphic to finite substructures of Γ. Γ is homogeneous if every isomorphism between finite substructures of Γ extends to an automorphism of Γ. $\mathscr{C}$ has the *Hereditary Property* (HP) if all substructures of members of $\mathscr{C}$ belong to $\mathscr{C}$. $\mathscr{C}$ has the *Joint Embedding Property* (JEP) if whenever $A, B \in \mathscr{C}$, there is a structure $D \in \mathscr{C}$ embedding both A and B. $\mathscr{C}$ has the *Amalgamation Property* (AP) if for all embeddings α: $C \to A$ and β: $C \to B$ between C-structures A, B and C, there is a structure $D \in \mathscr{C}$ and embeddings γ: $A \to D$ and δ: $B \to D$ such that $\alpha\gamma = \beta\delta$.

In 1953, Fraïssé showed that the Amalgamation Property is a crucial condition for the existence of a homogeneous structure with a given age.

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THEOREM 1.2 (Fraïssé [4]). *Let $\mathscr{L}$ be a purely relational first-order language. Let $\mathscr{C}$ be an isomorphism-closed class of finite $\mathscr{L}$-structures. Then the following four conditions are necessary and sufficient for there to be a countable homogeneous structure with age $\mathscr{C}$.*

(i) *$\mathscr{C}$ has only countably many isomorphism classes;*
(ii) *$\mathscr{C}$ has the Hereditary Property;*
(iii) *$\mathscr{C}$ has the Joint Embedding Property;*
(iv) *$\mathscr{C}$ has the Amalgamation Property.*

Moreover, this structure is unique up to isomorphism.

We now give the notation and terminology we use to discuss amalgamation of diagrams. We then state a Compactness Lemma for amalgamation.

Notation 1.3. Let $\mathscr{C}$ be a class of $\mathscr{L}$-structures, and let $\alpha: C \to A$ and $\beta: C \to B$ be $\mathscr{L}$-embeddings between $\mathscr{C}$-structures. We denote the diagram shown in Figure 1 by $\mathfrak{D}[\alpha: C \to A;\ \beta: C \to B]$ or $\mathfrak{D}[\alpha; \beta]$ or $\mathfrak{D}[C; A, B]$ depending as the structures and embeddings involved are clear from the context.

Let $\mathscr{C}_f$ denote the class of finite $\mathscr{C}$-structures.

DEFINITION 1.4. Let $\alpha: C \to A$, $\beta: C \to B$, $\alpha_0: C_0 \to A_0$ and $\beta_0: C_0 \to B_0$ be embeddings between $\mathscr{C}$-structures and let $\mathfrak{D} = \mathfrak{D}[\alpha; \beta]$ and $\mathfrak{D}_0 = \mathfrak{D}[\alpha_0; \beta_0]$. Then (ϵ, η) is an embedding of α_0 in α if $\epsilon: C_0 \to C$ and $\eta: A_0 \to A$ are $\mathscr{L}$-embeddings such that $\epsilon\alpha = \alpha_0\eta$ as in Figure 2. We say that $\mathfrak{D}_0$ is a subdiagram of $\mathfrak{D}$ if there are embeddings $\epsilon: C_0 \to C$, $\eta: A_0 \to A$ and $\zeta: B_0 \to B$ such that (ϵ, η) embeds α_0 in α and (ϵ, ζ) embeds β_0 in β. That is, the diagram in Figure 3 commutes.

LEMMA 1.5 (Compactness Lemma). *Let $\mathscr{C}$ be a class of relational structures axiomatized by a set of universal sentences. Then a diagram*

$$\mathfrak{D}[\alpha: A \to B;\ \beta: A \to C]$$

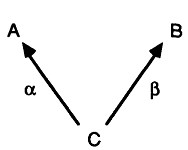

FIG. 1

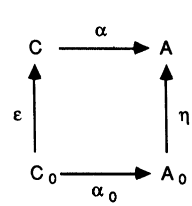

FIG. 2

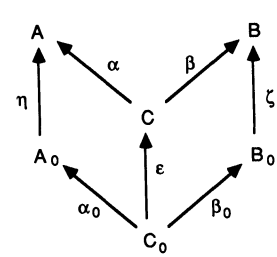

FIG. 3

can be amalgamated in $\mathscr{C}$ if and only if every finite subdiagram can be amalgamated in $\mathscr{C}_f$. Similarly, JEP holds for C if and only if it holds for $\mathscr{C}_f$.

We consider structures which are not necessarily homogeneous, but can be made homogeneous by expanding the language by finitely many relation symbols.

DEFINITION 1.6. Let $\mathscr{L}$ be a finite purely relational language. Let Σ be a set of universal $\mathscr{L}$-sentences and let $\mathscr{C}$ be the class of models of Σ. (Then there are only countably many isomorphism classes of finite $\mathscr{L}$-structures and $\mathscr{C}$ satisfies HP.) Assume $\mathscr{C}$ satisfies JEP. Then we say that $\mathscr{C}$ is *homogenizable* if there is a finite relational language $\mathscr{L}' \supset \mathscr{L}$ and a countable set $\Sigma' \supset \Sigma$ of universal axioms in $\mathscr{L}'$ such that:

(i) The class $\mathscr{C}'$ of models of Σ' satisfies AP and JEP;
(ii) Every $\mathscr{C}$-structure admits a $\mathscr{C}'$-structure;
(iii) If Γ' is the unique countable homogeneous $\mathscr{L}'$-structure with age $\mathscr{C}_f'$ and Γ is its $\mathscr{L}$-reduct then $\mathrm{Aut}_{\mathscr{L}}(\Gamma) = \mathrm{Aut}_{\mathscr{L}'}(\Gamma')$.

We call $\mathscr{C}'$ (resp. Γ') a *homogenization* of $\mathscr{C}$ (resp. Γ).

Example 1.7. The homogeneous undirected graphs were classified by Lachlan and Woodrow [6], [8]. None of the homogeneous graphs has age equal to the class of finite bipartite graphs. But the class of bipartite graphs is homogenizable. It is homogenized by adding to the language of graph theory a binary relation B such that $B(x, y)$ holds if and only if x and y lie in the same bipartite block.

We would like an analogy of Fraïssé's theorem for homogenizable classes. We will replace the Amalgamation Property in his theorem by the following property. This gives us sufficient conditions for a class to be homogenizable.

DEFINITION 1.8. Let $\mathscr{C}$ be a class of structures in which there are finitely many diagrams

$$\mathfrak{D}_j = \mathfrak{D}[\xi_{j,0}: X_j \to Y_{j,0}; \xi_{j,1}: X_j \to Y_{j,1}] \quad (j = 0, \dots, p),$$

such that any diagram $\mathfrak{D}$ in $\mathscr{C}$ fails to be amalgamated in $\mathscr{C}$ if and only if one of $\mathfrak{D}_j$ $(j = 0, \dots, p)$ is embeddable in $\mathfrak{D}$. (Note that each $\mathfrak{D}_j$ must fail AP.) Then we say that $\mathscr{C}$ has *Local Failure of Amalgamation* (LFA).

Example 1.9. The class of N-free graphs has LFA. A graph is *N-free* if it has no subgraph isomorphic to the graph shown in Figure 4. There are two complementary diagrams which fail AP, and any diagram which fails AP in the class of N-free graphs contains one of these as a subdiagram. They are the diagrams $\mathfrak{D}[C; A, B]$ shown in Figures 5 and 6, where $C = \{c_0, c_1, c_2\}$,

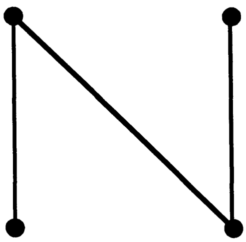

FIG. 4

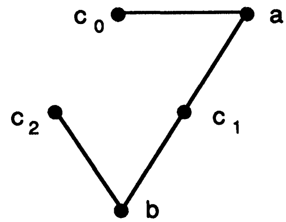

FIG. 5

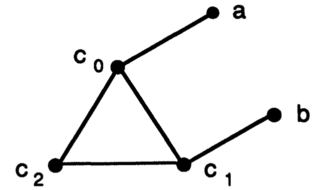

FIG. 6

$A = C \cup \{a\}$ and $B = C \cup \{b\}$. Thus, for any triple $\{x, y, z\}$ carrying complete (resp. null) induced subgraph, there is a distinguished vertex in $\{x, y, z\}$ such that any vertex joined to exactly one (resp. two) of x, y and z is joined (resp. not joined) to the distinguished vertex. This class can be homogenized by adding a ternary relation to distinguish one vertex from each triple carrying null or complete induced subgraph [3].

Example 1.10. The class of bipartite graphs does not have LFA. For each pair of positive integers (m, n) consider the bipartite graphs A and B consisting of paths of length $2n + 1$ and $2m + 2$ respectively. Then the induced subgraphs on the two endpoints of each of A and B are null, but cannot be amalgamated, while any proper subdiagram can be amalgamated.

2. Homogenization theorem

This section is devoted to proving the following theorem:

THEOREM 2.1. *Let $\mathscr{L}$ be a finite purely relational language. Let Σ be a set of universal $\mathscr{L}$-sentences. Let $\mathscr{C}$ be the class of models of Σ. Assume that $\mathscr{C}_f$ satisfies the Joint Embedding Property and that it has Local Failure of Amalgamation. Then $\mathscr{C}$ is homogenizable.*

The idea behind the proof is as follows. We homogenize $\mathscr{C}$ by adding to $\mathscr{L}$ a relation for each of the embeddings involved in the diagrams $\mathfrak{D}_j$ $(j = 0, \ldots, p)$. The relation R associated with the embedding $\xi_{j,k}$: $X_j \to Y_{j,k}$ will hold on the tuple $\bar{x}$ in a $\mathscr{C}'$-structure Γ' if and only if the $\mathscr{C}$-structure induced on $\bar{x}$ is isomorphic to X_j and $\bar{x}$ acts as if it were already embedded in $Y_{j,k}$. That is, for any $\mathscr{C}'$-structure Δ' containing Γ', the diagram $\mathfrak{D}[\bar{x}; \Delta', Y_{j,k}]$ in Figure 7 can be amalgamated.

We now define the terminology and notation used in the proof.

By the Compactness Lemma, we may assume that all the structures X_j, $Y_{j,k}$ $(j = 0, \ldots, p,\ k = 0, 1)$ are finite. We use $\bar{x}$ to denote a tuple of arbitrary

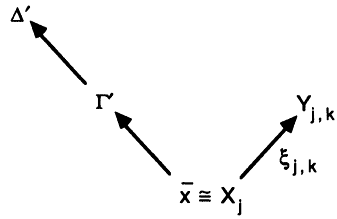

FIG. 7

length, the length being clear from context. We also abuse notation by identifying the ordered tuple $\bar{x} = (x_0, \ldots, x_s)$ and the unordered set $\{x_0, \ldots, x_s\}$ whenever convenient.

Let $\{\delta_i: C_i \to A_i\}_{i=0}^n$ be an enumeration of the distinct (up to isomorphism of embeddings) embeddings involved in $\mathfrak{D}_0, \ldots, \mathfrak{D}_p$. Let $q_i = |C_i|$ and $r_i = |A_i|$. Let

$$\mathscr{L}' = \mathscr{L} \cup \{R_i: 0 \leq i \leq n\},$$

where each R_i is a new relation symbol of parity q_i. The relation R_i is to be defined so that $R_i(\bar{x})$ holds if the induced $\mathscr{L}$-structure on $\bar{x}$ is isomorphic to C_i and can be extended to be embedded in A_i.

Let $\{c_{ij}: j = 0, \ldots, q_i - 1\}$ be a fixed labelling of C_i and $\{a_{ij}: j = 0, \ldots, r_i - 1\}$ a fixed labelling of A_i, with $a_{ij} = c_{ij}\delta_i$ $(j = 0, \ldots, q_i - 1)$. Let $C_i(\bar{x})$ denote the formula which says that ψ_i: $x_j \mapsto c_{ij}$ $(j = 0, \ldots, q_i - 1)$ is an $\mathscr{L}$-isomorphism, i.e., says that $\bar{x}$ carries the induced $\mathscr{C}$-structure C_i with the chosen labelling. Similarly, let $A_i(\bar{x})$ denote the formula which says that φ_i: $x_j \mapsto a_{ij}$ $(j = 0, \ldots, r_i - 1)$ is an $\mathscr{L}$-isomorphism. Note that $C_i(\bar{x})$ and $A_i(\bar{x})$ are quantifier-free $\mathscr{L}$-formulae.

Let $\mathrm{Wit}(\bar{x}; \delta_i) = C_i(\bar{x}) \wedge (\exists \bar{w}) A_i(\bar{x}^\frown \bar{w})$. This formula says that $C_i(\bar{x})$ holds and there is a tuple $\bar{w}$ which is a witness for the embedding δ_i extending the structure C_i on $\bar{x}$ to A_i. Note that $\mathrm{Wit}(\bar{x}; \delta_i)$ is an existential $\mathscr{L}$-formula.

Now, any diagram $\mathfrak{D}[\alpha: C \to A; \beta: C \to B]$ can be amalgamated unless it embeds one of the diagrams $\mathfrak{D}_j = \mathfrak{D}[\xi_{j,0}; \xi_{j,1}]$ $(j = 0, \ldots, p)$ from Definition 1.8. In particular, if $\alpha = \delta_i$ then there is an embedding (η, χ) of $\xi_{j,k}$ in δ_i, as in Figure 8. Then

$$\mathfrak{D}\left[\eta\delta_i: X_j \to A_i; \xi_{j,1-k}: X_j \to Y_{j,1-k}\right]$$

cannot be amalgamated in $\mathscr{C}$. We call such a diagram an *incompatibility diagram* and we say that $\xi_{j,1-k}$ is *δ_i-incompatible*.

There may be more than one embedding of $\xi_{j,k}$ in δ_i, but since C_i and A_i are finite there are, up to isomorphism, only finitely many subembeddings of

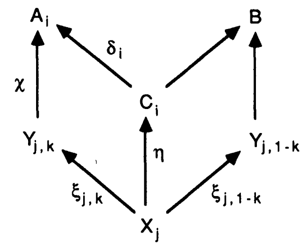

FIG. 8

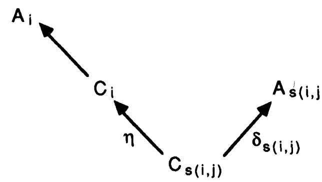

FIG. 9

δ_i, hence only finitely many incompatibility diagrams. List these as $(\mathfrak{D}_{ij}$: $j = 0, \dots, m_i)$. Since each δ_i-incompatible embedding is one of $\delta_0, \delta_1, \dots, \delta_n$ the δ_i-incompatible embedding in $\mathfrak{D}_{ij}$ is $\delta_{s(i,j)}$ for some $0 \leq s(i,j) \leq n$. (See Figure 9). This defines a function $s(i,j)$ $(0 \leq i \leq n, 0 \leq j \leq m_i)$. If $\Gamma \vDash C_i(\bar{x})$ then the structure induced on $\bar{x}$ is isomorphic to C_i. Then η embeds $C_{s(i,j)}$ in C_i, hence in $\bar{x}$. Let $\bar{x}(i,j)$ denote the image of $C_{s(i,j)}$ in $\bar{x}$.

Let

$$\text{Against}_j(\bar{x}; \delta_i) = C_i(\bar{x}) \wedge \text{Wit}\big(\bar{x}(i,j); \delta_{s(i,j)}\big).$$

If $\Gamma \vDash \text{Against}_j(\bar{x}; \delta_i)$ then $\mathfrak{D}[\bar{x}; A_i, \Gamma]$ cannot be amalgamated. That is, the witness for $\delta_{s(i,j)}$ on $\bar{x}(i,j)$ is a witness against δ_i on $\bar{x}$. Let

$$\text{Against}(\bar{x}; \delta_i) = \bigvee_{j=0}^{m_i} \text{Against}_j(\bar{x}; \delta_i).$$

This says that there is some witness against δ_i on $\bar{x}$. Note that Against$(\bar{x}; \delta_i)$ is equivalent to an existential $\mathscr{L}$-formula.

A structure Γ is existentially closed in $\mathscr{C}$ (see [5]) if any existential formula consistent with $\Sigma \cup \Delta(\Gamma)$ is realized in Γ. The next lemma says that unless there is a witness against δ_i on $\bar{x}$ in Γ then a witness for δ_i can be added to Γ and that an existentially closed structure will already contain such a witness.

LEMMA 2.2. *Let Γ be a $\mathscr{C}$-structure and let $\Delta(\Gamma)$ be the elementary diagram of Γ, i.e., the set of all atomic $\mathscr{L}_\Gamma$-sentences and negations of atomic $\mathscr{L}_\Gamma$-sentences true in Γ. Then $\Gamma \vDash C_i(\bar{x}) \wedge \neg\text{Against}(\bar{x}; \delta_i)$ if and only if $\Delta(\Gamma) \cup \text{Wit}(\bar{x}; \delta_i)$ is consistent.*

In particular, if Γ is existentially closed then

$$\Gamma \vDash \text{Wit}(\bar{x}; \delta_i) \leftrightarrow C_i(\bar{x}) \wedge \neg\text{Against}(\bar{x}; \delta_i).$$

Proof. Since $\mathscr{C}$ has LFA, $\mathfrak{D} = \mathfrak{D}[\bar{x}; \Gamma, A_i]$ fails to be amalgamated if and only if

$$\Gamma \models \mathrm{Against}(\bar{x}; \delta_i).$$

But $\mathfrak{D}$ fails to be amalgamated if and only if $\Delta(\Gamma) \cup \mathrm{Wit}(\bar{x}; \delta_i)$ is inconsistent.

Now assume Γ is existentially closed. Then, since Wit $(\bar{x}; \delta_i)$ is an existential $\mathscr{L}$-formula, $\Delta(\Gamma) \cup \mathrm{Wit}(\bar{x}; \delta_i)$ is consistent if and only if $\Gamma \models \mathrm{Wit}(\bar{x}; \delta_i)$. ■

Now, if $\Gamma \models C_i(\bar{x})$ then there are three possibilities.

(a) There is a witness for δ_i, i.e., $\bar{x}$ is already embedded in a copy of A_i in Γ;

(b) There is a witness against δ_i, i.e., some sub-tuple of $\bar{x}$ is already embedded via one of the δ_i-incompatible embeddings in $A_{s(i,j)}$;

(c) There is neither a witness for δ_i nor one against, i.e., $\bar{x}$ can be embedded in A_i, but is not.

We are now ready to state the axioms which homogenize $\mathscr{C}$. The relation R_i is to be defined in such a way that $\Gamma \models R_i(\bar{x})$ if a witness for δ_i on $\bar{x}$ can be added to Γ. So we need axioms to say that when $R_i(\bar{x})$ holds the structure behaves as if it already had a witness for δ_i on $\bar{x}$. We eliminate problems caused by possibility (c) with axioms saying that either $R_i(\bar{x})$ holds or one of the relations $R_{s(i,j)}$ corresponding to a δ_i-incompatible embedding holds. Note that (c) does not arise in an existentially closed structure. We will use this fact to impose a $\mathscr{C}'$-structure on an existentially closed $\mathscr{C}$-structure.

AXIOMS 2.3. *Let Σ' consist of the following set of axioms in the language $\mathscr{L}'$ and let $\mathscr{C}'$ be the class of models of Σ'.*

(I) Σ;

(II) $R_i(\bar{x}) \to C_i(\bar{x}) \quad (i = 0, \ldots, n)$;

(III) $\mathrm{Wit}(\bar{x}; \delta_i) \to R_i(\bar{x}) \quad (i = 0, \ldots, n)$;

(IV) $C_i(\bar{x}) \to R_i(\bar{x}) \vee \bigvee_{j=0}^{m_i} R_{s(i,j)}(\bar{x}(i,j)) \quad (i = 0, \ldots, n)$;

(V) *Whenever $k > 0$ and $\varphi\, (\bigcup_{j=0}^{k} \bar{x}_j)$ is a quantifier-free $\mathscr{L}$-formula and*

$$\Sigma \models \bigwedge_{j=0}^{k} \mathrm{Wit}(\bar{x}_j; \delta_{i_j}) \to \varphi\left(\bigcup_{j=0}^{k} \bar{x}_j\right)$$

then

$$\bigwedge_{j=0}^{k} R_{i_j}(\bar{x}_j) \to \varphi\left(\bigcup_{j=0}^{k} \bar{x}_j\right)$$

is an axiom.

By (I), the underlying $\mathscr{L}$-structure of a $\mathscr{C}'$-structure is a $\mathscr{C}$-structure. Axioms (II) say that $R_i(\bar{x})$ holds only when the $\mathscr{C}$-structure induced on $\bar{x}$ is C_i. Axioms (III) say that if $\bar{x}$ is already embedded in an isomorphic copy of A_i then $R_i(\bar{x})$ must hold. Axioms (IV) ensure that whenever $\bar{x}$ has induced substructure C_i then either R_i or one of the relations corresponding to an δ_i-incompatible embedding holds for $\bar{x}$. Scheme (V) means that $R_i(\bar{x})$ is consistent with the existence of a witness for δ_i on $\bar{x}$.

In particular, since $\delta_{s(i,j)}$ is δ_i-incompatible,

$$\Sigma \models \mathrm{Wit}(\bar{x};\delta_i) \to \neg \mathrm{Against}_j(\bar{x};\delta_i).$$

Thus

$$\Sigma \models \mathrm{Wit}(\bar{x};\delta_i) \wedge \mathrm{Against}_j(\bar{x};\delta_i) \to x_0 \neq x_0$$

Then by Axiom Scheme (V),

$$\Sigma' \models R_i(\bar{x}) \to \neg R_{s(i,j)}(\bar{x}(i,j)). \tag{1}$$

for $0 \leq i \leq n$ and $0 \leq j \leq m_i$.

Note that all the axioms are equivalent to universal $\mathscr{L}'$-formulae.

We now show how to put a $\mathscr{C}'$-structure on an existentially closed $\mathscr{C}$-structure.

LEMMA 2.4. *Any existentially closed C-structure* Γ *admits a unique* $\mathscr{C}'$*-structure.*

Proof. We first show for any $\mathscr{C}'$-structure Γ' on Γ,

$$\Gamma' \models R_i(\bar{x}) \leftrightarrow \mathrm{Wit}(\bar{x};\delta_i).$$

By (III), $\Gamma' \models \mathrm{Wit}(\bar{x};\delta_i) \to R_i(\bar{x})$. So assume now that $\Gamma' \models R_i(\bar{x})$. Then by (II), $\Gamma \models C_i(\bar{x})$ and by Equation (1), $\Gamma' \models \neg R_{s(i,j)}(\bar{x}(i,j))$. But then again by (III),

$$\Gamma \models \neg \mathrm{Wit}\big(\bar{x}(i,j);\delta_{s(i,j)}\big) \quad \text{for all } j = 0,\dots,m_i,$$

i.e., $\Gamma \models \neg\mathrm{Against}(\bar{x};\delta_i)$. Then by Lemma 2.2, $\Gamma \models \mathrm{Wit}(\bar{x};\delta_i)$.

So define an $\mathscr{L}'$-structure Γ' on Γ by

$$\Gamma' \models R_i(\bar{x}) \text{ if and ony if } \Gamma \models \mathrm{Wit}(\bar{x};\delta_i) \tag{2}$$

We need to verify that this structure satisfies the axioms Σ'.

(I) $\Gamma' \models \Sigma$ since $\Gamma \models \Sigma$.

(II) $\Gamma' \models R_i(\bar{x}) \to C_i(\bar{x})$ because $\mathrm{Wit}(\bar{x};\delta_i)$ holds only if $C_i(\bar{x})$ holds.

(III) $\Gamma' \models \mathrm{Wit}(\bar{x};\delta_i) \to R_i(\bar{x})$ by the definition.

(IV) By Lemma 2.2,

$$\Gamma \models C_i(\bar{x}) \to \mathrm{Wit}(\bar{x};\delta_i) \vee \mathrm{Against}(\bar{x};\delta_i).$$

That is,

$$\Gamma \models C_i(\bar{x}) \to \mathrm{Wit}(\bar{x};\delta_i) \wedge \bigvee_{j=0}^{m_i} \mathrm{Wit}(\bar{x}(i,j);\delta_{S(i,j)}).$$

Then by Equation (2),

$$\Gamma' \models C_i(\bar{x}) \to R_i(\bar{x}) \vee \bigvee_{j=0}^{m_i} R_{s(i,j)}(\bar{x}(i,j)).$$

(V) This follows directly from Equation (2).
Thus (2) defines a $\mathscr{C}'$-structure. ■

The following three lemmas show that $\mathscr{C}'$ is a homogenization of $\mathscr{C}$. Note that Lemma 2.5 implies that Σ' is consistent.

LEMMA 2.5. *Every model of* Σ *can be expanded to a model of* Σ'.

LEMMA 2.6. $\mathscr{C}_f'$ *has AP and JEP.*

LEMMA 2.7. *If* Γ' *is the unique countable homogeneous* $\mathscr{L}'$*-structure with age* $\mathscr{C}_f'$ *then*

$$\mathrm{Aut}_{\mathscr{L}}(\Gamma) = \mathrm{Aut}_{\mathscr{L}'}(\Gamma').$$

Proof of 2.5. Let Γ be a model of Σ. By [5, Corollary 3.2.2.], Γ can be embedded in an existentially closed $\mathscr{C}$-structure Δ. By Lemma 2.4, Δ admits a unique $\mathscr{C}'$-structure Δ'. Since Σ' consists of universal formulae, the restriction Γ' of Δ' to Γ is a $\mathscr{C}'$-structure. ■

Proof of 2.6. Let $\mathfrak{D}[C'; A', B']$ be a finite diagram in $\mathscr{C}'$. Let A, B and C be the $\mathscr{L}$-reducts of A', B' and C' respectively. We first embed A and B in $\mathscr{C}$-structures $\tilde{A}$ and $\tilde{B}$ respectively, such that any $\mathscr{C}'$-structures on $\tilde{A}$ and $\tilde{B}$ coincide with the original $\mathscr{C}'$-structures on A and B, i.e., if $\bar{x} \in A$ then $\tilde{A} \models \mathrm{Wit}(\bar{x};\delta_i)$ if and only if $A' \models R_i(\bar{x})$ and similarly for $\tilde{B}$ and B'. We then amalgamate the diagram $\mathfrak{D}[\mathscr{C}; \tilde{A}, \tilde{B}]$ in $\mathscr{C}$, and show that any $\mathscr{C}'$-structure on the amalgam yields an amalgam of $\mathfrak{D}[\mathscr{C}'; A', B']$.

We show that A can be embedded in such a structure $\tilde{A}$. Let $\{a_i: i = 0, \dots, p\}$ be a fixed labelling of A. Let $A(\bar{x})$ denote the formula which says that $\varphi\colon x_i \mapsto a_i$ $(i = 0, \ldots, p)$ is an $\mathscr{L}$-isomorphism, and let $A'(\bar{x})$ be the formula which says that φ is an $\mathscr{L}'$-isomorphism. Consider pairs $(i, \bar{w})$ where $0 \leq i \leq n$ and $\bar{w}$ is a subtuple of $\bar{x}$ such that $A(\bar{x}) \models C_i(\bar{w})$. List such pairs as $\{(i_k, \bar{x}_k)\colon k = 0, \ldots, t\}$ so that

$$A'(\bar{x}) \models R_{i_k}(\bar{x}_k) \text{ for } 0 \leq k \leq s \quad \text{and} \quad A'(\bar{x}) \models \neg R_{i_k}(\bar{x}_k) \text{ for } s+1 \leq k \leq t.$$

We need to show that

$$\Sigma \cup A(\bar{x}) \cup \{\mathrm{Wit}(\bar{x}_k; \delta_{i_k})\colon k = 0, \ldots, s\} \cup \{\neg \mathrm{Wit}(\bar{x}_k; \delta_{i_k})\colon k = s+1, \ldots, t\} \tag{3}$$

is consistent.

By Axioms Scheme (IV), for each $(k = s+1, \ldots, t)$ there is some $0 \leq j_k \leq m_{i_k}$ such that

$$A'(\bar{x}) \models R_{s(i_k, j_k)}(\bar{x}_k(i_k, j_k)).$$

We show that

$$\Sigma \cup A(\bar{x}) \cup \{\mathrm{Wit}(\bar{x}_k; \delta_{i_k})\colon k = 0, \ldots, s\} \cup \{\mathrm{Against}_{j_k}(\bar{x}_k; \delta_{i_k})\colon k = s+1, \ldots, t\} \tag{4}$$

is consistent. Clearly, any model of (4) is a model of (3).

Assume (4) is inconsistent. Then

$$\Sigma \models \bigwedge_{k=0}^{s} \mathrm{Wit}(\bar{x}_k; \delta_{i_k}) \wedge \bigwedge_{k=s+1}^{t} \mathrm{Wit}(\bar{x}_k(i_k, j_k); \delta_{s(i_k, j_k)}) \rightarrow \neg A(\bar{x})$$

$A(\bar{x})$ is a quantifier-free $\mathscr{L}$-formula, so by Axiom Scheme (V),

$$\Sigma' \models \bigwedge_{k=0}^{s} R_{i_k}(\bar{x}_k) \wedge \bigwedge_{k=s+1}^{t} R_{s(i_k, j_k)}(\bar{x}_k(i_k, j_k)) \rightarrow \neg A(\bar{x}).$$

That is, we have replaced each $\mathrm{Wit}(\bar{x}; \delta_i)$ by $R_i(\bar{x})$. But this contradicts the fact that $A' \models \Sigma'$. Thus (4) is consistent, as required.

We now embed A in a model $\tilde{A}$ of (3) and similarly embed B in a structure $\tilde{B}$. Assume the diagram $\mathfrak{D}[C; \tilde{A}, \tilde{B}]$ cannot be amalgamated in $\mathscr{C}$. Then by LFA, there is a tuple $\bar{x}$ in C such that for some $0 \leq i \leq n$, $C \models C_i(\bar{x})$ and $A \models \mathrm{Wit}(\bar{x}; \delta_i)$ and $\tilde{B} \models \mathrm{Against}_j(\bar{x}; \delta_i)$ for some $0 \leq j \leq m_i$. Then $A' \models R_i(\bar{x})$ and $B' \models R_{s(i,j)}(\bar{x}(i, j))$. Since C' is a substructure of both

A' and B', we now have

$$C' \models R_i(\bar{x}) \wedge R_{s(i,j)}(\bar{x}(i,j)).$$

By (1) this contradicts the fact that C' is a model of Σ'.

Thus, there is an amalgam D of $\mathfrak{D}[C; \tilde{A}, \tilde{B}]$ in $\mathscr{C}$. By Lemma 2.5, D admits a $\mathscr{C}'$-structure D'. We must show that the restrictions of D' to A and B coincide with A' and B'. So take $\bar{x} \subseteq A'$ such that $A' \models R_i(\bar{x})$. Then $\tilde{A} \models \mathrm{Wit}(\bar{x}; \delta_i)$ and so since $\mathrm{Wit}(\bar{x}; \delta_i)$ is an existential formula, $D \models \mathrm{Wit}(\bar{x}; \delta_i)$. Then by Axiom (III), $D' \models R_i(\bar{x})$. If $A' \models \neg R_i(\bar{x})$ then for some $0 \leq j \leq m_i$, $A' \models R_{s(i,j)}(\bar{x}(i,j))$. Then by the above,

$$D' \models R_{s(i,j)}(\bar{x}(i,j)),$$

and so by (1),

$$D' \models \neg R_i(\bar{x})$$

Similarly for B'.

The proof that $\mathscr{C}$ has the Joint Embedding Property is entirely analogous. ■

Proof of 2.7. Let Γ' be the countable homogeneous structure with age $\mathscr{C}'$ and let Γ be its $\mathscr{L}$-reduct. We need to show that $\mathrm{Aut}_{\mathscr{L}'}(\Gamma') = \mathrm{Aut}_{\mathscr{L}}(\Gamma)$.

Γ is existentially closed so by (2),

$$\Gamma' \models R_i(x) \leftrightarrow Wit(\bar{x}; \delta_i).$$

Take $\varphi \in \mathrm{Aut}_{\mathscr{L}}(\Gamma)$. Then, since $\mathrm{Wit}(\bar{x}; \delta_i)$ is an $\mathscr{L}$-formula,

$$\Gamma \models \mathrm{Wit}(\bar{x}; \delta_i) \text{ if and only if } \Gamma \models \mathrm{Wit}(\bar{x}\varphi; \delta_i).$$

Thus it follows that $\Gamma' \models R_i(\bar{x})$ if and only if $\Gamma' \models R_i(\bar{x}\varphi)$, and hence φ is an $\mathscr{L}'$-automorphism. ■

3. Model completeness

In this section we show that the $\mathscr{L}$-theory of the 'homogenized' structure is model complete. Thus it is the model companion (see [5]) for the theory Σ of the class $\mathscr{C}$.

COROLLARY 3.1. *Let $\mathscr{L}$, $\mathscr{L}'$, $\mathscr{C}$ and $\mathscr{C}'$ be as in Theorem* 2.1. *Let Γ' be the countable homogeneous structure with age $\mathscr{C}'$ and let Γ be its $\mathscr{L}$-reduct. Then* $\mathrm{Th}(\Gamma)$ *is model complete.*

Proof. A theory T is model complete if every formula is T-equivalent to a universal (or to an existential) formula. (See [2, p. 111].) Thus, every $\mathscr{L}'$-formula φ is Th(Γ')-equivalent to an existential $\mathscr{L}'$-formula, i.e.,

$$\Gamma' \vDash_{\mathscr{L}'} \varphi(\bar{x}) \leftrightarrow (\exists \bar{y}) \psi(\bar{x}, \bar{y})$$

for some quantifier-free formula ψ. We can assume that ψ is in disjunctive normal form.

Now, for each relation R_i of $\mathscr{L}' \setminus \mathscr{L}$, we have

$$\Gamma' \vDash_{\mathscr{L}'} R_i(\bar{x}) \leftrightarrow \mathrm{Wit}(\bar{x}; \delta_i)$$

and

$$\Gamma' \vDash_{\mathscr{L}'} \neg R_i(\bar{x}) \leftrightarrow \mathrm{Against}(\bar{x}; \delta_i)$$

Thus we can replace each occurrence of R_i or $\neg R_i$ ($i = 0, \ldots, n$), in ψ by an existential $\mathscr{L}$-formula and then bring all the existential quantifiers forward to obtain a quantifier-free $\mathscr{L}$-formula χ such that

$$\Gamma' \vDash_{\mathscr{L}'} \varphi(\bar{x}) \leftrightarrow (\exists \bar{y}) \chi(\bar{x}, \bar{y}).$$

In particular, this is true if φ is an $\mathscr{L}$-formula. Thus

$$\Gamma \vDash_{\mathscr{L}} \varphi(\bar{x}) \leftrightarrow (\exists \bar{y}) \chi(\bar{x}, \bar{y}).$$

This proves that Γ is model complete. ■

Given a class $\mathscr{C}$ of structures, we would like to choose a 'favourite' countable structure with age $\mathscr{C}$. When $\mathscr{C}$ satisfies the hypotheses of Fraïssé's Theorem, we choose the unique countable homogeneous structure. Homogeneous structures are model complete, so are their own model companions. When there is no homogeneous structure, the model companion, when it exists, is a candidate for favourite. But a homogenizable structure is also a possibility. However, there is no unique homogenizable structure associated with a homogenizable class. Thus we need to compare homogenizations and to choose a favourite homogenization of each class. In particular, we would want this to be the homogeneous structure when it exists.

Albert and Burris [1] have studied model companions of universal classes. They too looked at the ways in which AP fails. They defined the *Strong Bounded Obstruction Property* (SBOP) for classes of finite structures to hold whenever any diagram $\mathfrak{D}[C; A, B]$ which fails to be amalgamated contains a subdiagram which fails AP whose size depends only on the size of C. Thus LFA implies SBOP. Albert and Burris proved that a class satisfying SBOP and JEP has a model companion. The results of this paper show that if LFA holds then the model companion is homogenizable.

Now, every homogenizable structure is $\aleph_0$-categorical, and *D*. Saracino proved in [7] that every $\aleph_0$-categorical theory has a model companion. This raises the following question. Is there a homogenizable class whose model companion is not homogenizable? If so, then we have two different countable structures having a good claim to be our favourite countable structure for the class. If not, then we have arrived at the same structure in two seemingly different ways.

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