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RIEMANNIAN FOLIATIONS ON SIMPLY CONNECTED MANIFOLDS AND ACTIONS OF TORI ON ORBIFOLDS

BY

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1. Introduction

Basic properties of Riemannian foliations on simply connected manifolds have been established by P. Molino [Mol-1] and E. Ghys [Ghy]. In this paper we complete their results by showing a close relationship between such foliations and actions of tori on orbifolds.

As a general reference on Riemannian foliations, we refer to the book of P. Molino [Mol].

1.1. We first give a typical example where tori actions on orbifolds arise naturally.

Let H be a connected subgroup of the Lie group of isometries of an orientable Riemannian manifold Y. Let us assume that H acts locally freely on Y. Then the orbits under H of the points of Y are the leaves of a Riemannian foliation $\mathscr{F}$ on Y.

Assume that the closure $\overline{H}$ of H is compact. Let K be a maximal compact subgroup of H. As the Lie algebra of H is a compact Lie algebra ([Bki], Chap. IX), this maximal compact subgroup is unique, hence invariant in H. The quotient group $L = H/K$ is a dense abelian contractible subgroup of the compact group $\overline{L} = \overline{H}/K$ which must be isomorphic to a torus T^N of dimension N. The action of K on Y is also locally free; hence the orbits under K are the fibers of a generalized Seifert fibration on Y (i.e. a foliation whose leaves are compact with finite holonomy); its base space is naturally an oriented orbifold X whose underlying topological space is Y/K. The torus $T^N = \overline{H}/K$ acts effectively on X and the restriction of this action to the dense subgroup L is locally free. The orbits under L are the leaves of a foliation $\mathscr{F}_X$ on X, and the foliation $\mathscr{F}$ is the pull back of $\mathscr{F}_X$ by the projection p of Y on X.

Conversely, given an action of a torus T^N on an orientable orbifold X of dimension n (see 3.1 and 3.2) and a dense contractible subgroup L of T^N

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acting locally freely on X, one can replace X by the manifold $\hat{X}$ of direct orthonormal frames on X (with respect to a T^N-invariant metric on X) and get a locally free action of $H = L \times SO(n)$ on $\hat{X}$ by isometries giving a foliation $\hat{\mathscr{F}}$ on $\hat{X}$. The above construction leads back to the action of L on X (in that case, the maximal compact subgroup is $SO(n)$).

Note that the holonomy pseudogroups of $\mathscr{F}$, $\mathscr{F}_X$ and $\hat{\mathscr{F}}$ are differentiably equivalent (in the sense of 2.2).

The foliation $\mathscr{F}$ on Y is a particular case of a Killing foliation, i.e., a Riemannian foliation on a complete manifold for which the Molino central sheaf is trivial (cf. [Mol-2]).

Other examples of Killing foliations are given by Riemannian foliations on simply connected manifolds (cf. [Mol-1]). We prove in this paper the following theorem:

1.2. THEOREM. *Let $\mathscr{F}$ be a Riemannian foliation on a compact simply connected manifold Y. Then one can construct, in a unique way up to isomorphism, an effective smooth action of a torus T^N on a simply connected compact orbifold X and a dense contractible subgroup L of T^N whose action on X is locally free. Let us denote by $\mathscr{F}_X$ the foliation of X whose leaves are the orbits of L; then the holonomy pseudogroups of $\mathscr{F}$ and $\mathscr{F}_X$ are equivalent.*

In fact we prove a more general version of this theorem for Killing foliations (see Cor. 3.5) which can be interpreted as a sort of converse to 1.1.

The torus T^N and its subgroup L are related to $\mathscr{F}$ as follows: there is an open dense subset of Y which is the union of the leaves with trivial holonomy. These leaves are all isomorphic to each other and are called generic leaves. Let F be a generic leaf of $\mathscr{F}$. The holonomy pseudogroup of the foliation restricted to the closure of F is equivalent to the pseudogroup generated by a dense subgroup Γ of $\mathbf{R}^k$ acting by translation on $\mathbf{R}^k$, where k is the codimension of F in its closure. The torus T^N is isomorphic to $\Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$; its dimension N is the rank of Γ. The subgroup L is the image in the quotient $\Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$ of the kernel of the homomorphism from $\Gamma \otimes \mathbf{R}$ on $\mathbf{R}^k$ sending $\gamma \otimes r$ on $r\gamma$. The dimension of X is $N + \dim W$, where W is the space of leaf closures on $\mathscr{F}$ (or the space of orbit closures of T^N).

As a consequence, the problem of classifying the possible holonomy pseudogroups of Riemannian foliations on compact simply connected manifolds is essentially reduced to the problem of classifying tori actions on compact 1-connected orbifolds. This will be the subject of a forthcoming paper.

1.3. *Realization of the classifying space*. For any foliation $\mathscr{F}$ on Y, one can construct the classifying space $B\mathscr{H}$ of its holonomy pseudogroup $\mathscr{H}$. It can be thought of as a space with a foliation $B\mathcal{F}$ whose holonomy pseudogroup is equivalent to $\mathcal{H}$, and such that the holonomy covering of each leaf is contractible (cf. [Hae-1]). Moreover one has a classifying map $p: Y \to B\mathcal{H}$ transverse to $B\mathcal{F}$ such that $\mathcal{F} = p^*(B\mathcal{F})$. The homotopy class of p is unique up to an homotopy along the leaves.

For a Riemannian foliation on a complete Riemannian manifold Y, the classifying space $B\mathcal{H}$ plays the role of a "base space". Indeed the homotopic theoretical fiber of the classifying map p of Y in $B\mathcal{H}$ has the homotopy type of the common holonomy covering F of the leaves (cf. [Hae-1], 3.1.5). When Y is compact and 1-connected, F is the generic leaf. Hence it is important to know the structure of the classifying space.

Now, let us suppose that Y is compact and 1-connected, and let us look at the foliation $\mathcal{F}_X$ constructed in theorem 1.2. The holonomy covering of each leaf is isomorphic to L, hence contractible. So the orbifold X acts as the classifying space (see 3.3) for the holonomy pseudogroup of $\mathcal{F}_X$ (and therefore of $\mathcal{F}$). More precisely one has the following result (see §6):

1.4. THEOREM. *With the notations of Theorem* 1.2, *the pair* $(X, \mathcal{F}_X)$ *is a model for the classifying space of the holonomy pseudogroup of* $\mathcal{F}$. *In particular, there is a smooth map* p *of* Y *on* X (*in the orbifold sense*) *such that* $\mathcal{F} = p^*(\mathcal{F}_X)$. *The cohomological dimension of the generic leaf* F *of* $\mathcal{F}$ *is equal to* $\dim Y - \dim X = \dim \overline{F} - N$. *This positive integer vanishes if and only if* F *is contractible*; *in that case* p *is a homotopy equivalence and* X *is a smooth manifold.*

One can prove in a number of cases that the generic leaf F has the homotopy type of a finite Poincaré complex F_0. This occurs for instance when F is simply connected (see § 6), thus in particular if $\pi_2(X) = 0$.

The existence of the map $p: Y \to X$ implies a theorem of E. Ghys [Ghy] asserting that any Riemannian foliation on a compact simply connected manifold can be approximated by a Riemannian foliation with compact leaves (i.e., a generalized Seifert fibration). To get such an approximation, one replaces the dense subgroup L of T^N by a nearby closed subgroup L'; if L' is close enough to L, then it will act also locally freely on X defining a foliation $\mathcal{F}'_X$ on X, close to $\mathcal{F}$ and whose leaves are compact; the map p will still be transverse to $\mathcal{F}'_X$ so that $p^*(\mathcal{F}'_X)$ is a Riemannian foliation with compact leaves which is close to $\mathcal{F}$.

The following result shows that a map of a compact manifold Y' in a compact 1-connected manifold Y which is transverse to a Riemannian foliation on Y can exist only if the dimension of Y' is big enough.

1.5. THEOREM. *Let* $\mathcal{F}$ *be a Riemannian foliation on a compact simply connected manifold* Y. *Let* f *be a smooth map of a compact connected manifold* Y' *in* Y *which is transverse to* $\mathcal{F}$.

Then the dimension of Y' is bigger than or equal to the dimension of the orbifold X associated to $\mathcal{F}$.

In particular, if the generic leaf of $\mathcal{F}$ is contractible, then $\dim Y' \geq \dim Y$, *and* $\dim Y' = \dim Y$ *if and only if f is a homotopy equivalence.*

If $\dim \mathcal{F} = 1$ *and if* $\dim Y' \leq \dim Y$, *then f is homotopic to a diffeomorphism by an homotopy transverse to $\mathcal{F}$.*

1.6. *A conjecture.* Examples of Riemannian foliations on simply connected manifolds whose associated orbifold is X are obtained as follows: consider a simply connected smooth manifold Y which is a generalized Seifert fiber space with base space an orbifold X like in Theorem 1.2, and generic fiber a compact manifold F_0, and let us denote by p the projection of Y on the base space X. Then the foliation $p^*(\mathcal{F}_X)$ on Y is such a foliation.

From the remark following Theorem 1.4, one is lead naturally to the following:

CONJECTURE. *For a Riemannian foliation $\mathcal{F}$ on a compact simply connected manifold Y, the classifying map $p\colon Y \to X$ of Theorem* 1.4 *can be chosen to be the projection of a Seifert fibration with generic fiber a compact manifold F_0 having the homotopy type of the generic leaf F of $\mathcal{F}$.*

This is obviously the case if the leaves of $\mathcal{F}$ are compact (in which case $F_0 = F$). The conjecture is also true if the dimension of the leaves is one, or if $\dim Y \leq 4$ by an explicit classification (see 6.3).

A particular case of the conjecture would be that if $\mathcal{F}$ is a Riemannian foliation on a compact simply connected manifold whose generic leaf is contractible, the leaves would be the orbits of a locally free action of a group L isomorphic to $\mathbf{R}^m$.

This paper is organized as follows.

In §2 we recall the notion of a Killing foliation, and define the associated notion of a Killing pseudogroup. We begin §3 by defining the notion of a smooth action of a Lie group on an orbifold. We state in Theorem 3.4 a more general version of Theorem 1.2 which relates Killing foliations, Killing pseudogroups and actions of tori on manifolds. Corollary 3.7 of Theorem 3.4 gives Theorem 1.2 and the first part of Theorem 1.4.

In § 4 we describe a local model for a Killing pseudogroup and its local realization as the holonomy pseudogroup of a foliation on an orbifold. This foliation is given by a locally free action of a dense contractible subgroup of a torus acting on the orbifold. The glueing up of these local realizations is done in § 5, and ends up the proof of Theorem 3.4.

In § 6 we prove the last part of Theorem 1.4 and make some remarks on Conjecture 1.6. In the last paragraph, we prove Theorem 1.5.

2. Review of some basic notions on Riemannian foliations and complete pseudogroups of isometries

We begin by recalling some definitions and notations.

2.1. DEFINITION. A foliation $\mathscr{F}$ of codimension m on a smooth manifold Y can be given by:

(i) an open covering $\{V_i\}_{i\in I}$ of Y,
(ii) for each $i \in I$, a surjective submersion with connected fibers f_i of V_i on a manifold T_i (of dimension m),
(iii) local diffeomorphisms h_{ji}: $f_i(V_i \cap V_j) \to f_j(V_i \cap V_j)$ with $f_j = h_{ji} \circ f_i$.

Such data describing the foliation will be called a 1-cocycle.

The connected components of the intersections of the leaves of $\mathscr{F}$ with V_i are the fibers of the submersion f_i. The transverse changes of coordinates h_{ji} generate a pseudogroup $\mathscr{H}$ of transformations, called the *holonomy pseudogroup of* $\mathscr{F}$ (associated to the given cocycle). It acts on the transverse manifold T which is the disjoint union of the T_i's.

It is clear that two 1-cocycles defining the same foliation $\mathscr{F}$ give rise to two holonomy pseudogroups $\mathscr{H}$ and $\mathscr{H}'$ which are equivalent in the following sense:

2.2. *A differentiable equivalence.* Φ between two pseudogroups $\mathscr{H}$ and $\mathscr{H}'$ acting respectively on differentiable manifolds T and T' is a maximal collection Φ of diffeomorphisms φ from open sets of T to open sets of T' such that:

(i) the sources (resp. the targets) of the elements of Φ cover T (resp. T');
(ii) if $\varphi, \psi \in \Phi$, $h \in \mathscr{H}$ and $h' \in \mathscr{H}'$, then $\psi \circ h \circ \varphi^{-1} \in \mathscr{H}'$, $\psi^{-1} \circ h' \circ \varphi \in \mathscr{H}$ and $h' \circ \varphi \circ h \in \Phi$.

2.3. The weak homotopy type of the *classifying space* $B\mathscr{H}$ of the holonomy pseudogroup of $\mathscr{F}$ depends only on the equivalence class of $\mathscr{H}$ (cf. [Hae-1]). One can define the homotopy groups $\pi_i(\mathscr{H})$ of $\mathscr{H}$ as the homotopy groups $\pi_i(B\mathscr{H})$ of $B\mathscr{H}$ (see [Sal-2] for a direct definition of $\pi_1(\mathscr{H})$). We shall say that the pseudogroup $\mathscr{H}$ is connected if $\pi_0(\mathscr{H}) = 1$ (or equivalently if the space of orbits of $\mathscr{H}$ is connected) and 1-connected if $\pi_0(\mathscr{H}) = \pi_1(\mathscr{H}) = 1$.

If $\mathscr{H}$ is the holonomy pseudogroup of a foliation $\mathscr{F}$ on a manifold Y, the classifying map p: $Y \to B\mathscr{H}$ induces a surjection of $\pi_1(Y)$ on $\pi_1(B\mathscr{H}) = \pi_1(\mathscr{H})$. In particular if Y is simply connected, then its holonomy pseudogroup is 1-connected.

2.4. The holonomy pseudogroup of a Riemannian foliation is a pseudogroup of local isometries of the transverse Riemannian manifold T. Moreover if the Riemannian foliation is defined on a complete Riemannian manifold, then the holonomy pseudogroup $\mathscr{H}$ is *complete* in the following sense: for any two points x and y of T, there exists open neighbourhoods U and V of x and y respectively such that each germ of an element of $\mathscr{H}$ with source in U and target in V is the germ of an element of $\mathscr{H}$ defined on the whole of U.

We remark that if $\mathscr{H}$ is a complete pseudogroup of local isometries, then the closure $\overline{\mathscr{H}}$ of $\mathscr{H}$ in the space of local isometries with the C^1-topology is also a complete pseudogroup of local isometries of T. The orbit $\overline{\mathscr{H}}x$ of a point x of T under $\overline{\mathscr{H}}$ is the closure of the orbit of x under $\mathscr{H}$, and the space W of orbit closures under $\mathscr{H}$ is a Hausdorff space.

2.5. For a complete pseudogroup $\mathscr{H}$ of local isometries of T, one can associate the structural sheaf $\mathfrak{h}$ of $\mathscr{H}$: for an open set U of T, one defines $\mathfrak{h}(U)$ to be the set of vector fields ξ on U verifying the following condition: for any point x in U, there exists an open neighbourhood V_x of x in U and $\varepsilon > 0$, such that $\exp t\xi$ is defined on V_x for $t < \varepsilon$ and belongs to $\overline{\mathscr{H}}$. The pseudogroup $\overline{\mathscr{H}}$ acts by automorphisms on the sheaf $\mathfrak{h}$: for a local section ξ of $\mathfrak{h}$, the action of an element h of $\overline{\mathscr{H}}$ is defined by

$$h^*(\xi) = d/dt(h \circ \exp t\xi \circ h^{-1})|_{t=0}.$$

This sheaf is locally constant; its stalks are finite dimensional Lie algebras of Killing vector fields, and the elements of $\overline{\mathscr{H}}$ close to the identity can be obtained by integrating local sections of $\mathfrak{h}$ which are close to zero (cf. [Sal-1]).

When the space W of orbit closures of $\mathscr{H}$ is connected, the stalks of $\mathfrak{h}$ are all isomorphic to a finite dimensional Lie algebra $\mathfrak{h}$, called *the structural Lie algebra of* $\mathscr{H}$.

Remark. Let $\mathscr{F}$ be a Riemannian foliation on a complete Riemannian manifold Y, defined by the cocycle (V_i, f_i, h_{ij}) and let $\mathscr{H}$ be the associated holonomy pseudogroup of $\mathscr{F}$. Then the different pull-backs over Y of the structural sheaf $\mathfrak{h}$ by the local submersions f_i glue together to form the *central transverse sheaf* defined by Molino in [Mol-1].

2.6. Following the terminology of Molino, a Riemannian foliation on a complete Riemannian manifold is called a *Killing foliation* if the Molino central sheaf is trivial. Accordingly, a complete pseudogroup $\mathscr{H}$ of local isometries is called a *Killing pseudogroup* if the structural sheaf $\mathfrak{h}$ of $\mathscr{H}$ has a trivialization invariant by the action of $\mathscr{H}$. This implies that the stalk $\mathfrak{h}$ of $\underline{\mathfrak{h}}$ is an abelian Lie algebra isomorphic to $\mathbf{R}^k$ [Sal-2].

Examples of Killing foliations on a complete Riemannian manifold Y are given by locally free actions of connected subgroups of the group of isometries of Y [Mol-2]. Other examples are given by Riemannian foliations on complete 1-connected Riemannian manifolds (cf. [Mol-1]). Let us recall some general properties of these foliations.

2.7. Let $\mathscr{F}$ be a Riemannian foliation on a complete 1-connected Riemannian manifold Y. The leaves with trivial holonomy, called generic leaves, form an open dense set in Y. The restriction of $\mathscr{F}$ to the closure of a generic leaf F has the following structure: there is a dense subgroup Γ of $\mathbf{R}^k$, where k is the codimension of F in $\overline{F}$, acting in a properly discontinuous way on $F \times \mathbf{R}^k$ and preserving the foliation whose leaves are the factors $F \times \{x\}$. The projection of this action on $\mathbf{R}^k$ is the action of Γ by translations. The foliation $\mathscr{F}$ restricted to $\overline{F}$ is the quotient of the foliation on $F \times \mathbf{R}^k$ by this action.

One has the corresponding properties for a 1-connected complete pseudogroup $\mathscr{H}$ of local isometries of T. Namely the structural sheaf is a constant sheaf with stalk isomorphic to the trivial Lie algebra $\mathbf{R}^k$ and it has a global trivialization invariant by $\mathscr{H}$. There is an open dense set in T which is the union of orbits of points with trivial isotropy; such orbits are called generic orbits. The closure of a generic orbit is a closed manifold of dimension k.

We end up this section with the following finiteness property.

2.8. Proposition. *Let $\mathscr{H}$ be a 1-connected complete pseudogroup of local isometries of T such that the space W of orbit closures is compact. Then the restriction of $\mathscr{H}$ to the closure of a generic orbit is equivalent to a pseudogroup generated by the action of a finitely generated dense subgroup Γ of $\mathbf{R}^k$ acting by translations on $\mathbf{R}^k$.*

Proof. As $\mathscr{H}$ is 1-connected, there is on T an orientation invariant by $\mathscr{H}$. Let $\hat{T}$ be the manifold of direct orthonormal frames on T and let $\hat{\mathscr{H}}$ be the natural extension of $\mathscr{H}$ to $\hat{T}$. Then the space of orbit closures $\hat{W}$ of $\hat{\mathscr{H}}$ is a compact manifold, and the natural projection $\pi\colon \hat{T} \to \hat{W}$ is a submersion. Moreover the restriction of $\hat{\mathscr{H}}$ to any orbit closure of $\hat{\mathscr{H}}$ is equivalent to the pseudogroup generated by the action of a dense subgroup Γ of $\mathbf{R}^k$ acting by translations on $\mathbf{R}^k$ (cf. [Sal-1]). It is also equivalent to the restriction of $\mathscr{H}$ to a generic orbit closure.

For the case $m = \dim T = 2$, one uses the explicit classification of 1-connected complete pseudogroups (cf. [Hae-Sal]) to deduce that Γ is finitely generated (Γ is either or rank 0, and then $k = 0$; or of rank 2, and then $k = 1$).

For the case $m \geq 3$, we remark that the classifying space $B\hat{\mathscr{H}}$ is a principal $SO(m)$-principal fiber bundle over $B\mathscr{H}$. So one has the homotopy exact sequence

$$\to \pi_1(SO(m)) \to \pi_1(\hat{\mathscr{H}}) \to \pi_1(\mathscr{H}) \to 1.$$

As $\pi_1(\mathscr{H}) = 1$, and $\pi_1(SO(m)) = \mathbf{Z}/2\mathbf{Z}$ for $m \geq 3$, one has $\pi_1(\hat{\mathscr{H}}) = \mathbf{Z}/2\mathbf{Z}$ or 1. One also has the exact homotopy sequence

$$\to \pi_2(\hat{W}) \to \pi_1(\hat{\mathscr{H}}_0) \to \pi_1(\hat{\mathscr{H}}) \to \pi_1(\hat{W}) \to 1$$

where $\hat{\mathscr{H}}_0$ is the restriction of $\hat{\mathscr{H}}$ to a fiber of π: $\hat{T} \to \hat{W}$ (so $\pi_1(\hat{\mathscr{H}}_0) = \Gamma$). This sequence can be obtained directly, or by constructing a realization of $B\hat{\mathscr{H}}$ which is a fiber bundle over $\hat{W}$, with fiber $B\hat{\mathscr{H}}_0$. One has that $\hat{W}$ is a compact manifold with $\pi_1(\hat{W}) = 1$ or $\mathbf{Z}/2\mathbf{Z}$. In any case $\pi_2(\hat{W})$ is finitely generated, so Γ is also finitely generated.

3. Statement of the results

We first recall some basic notions on orbifolds.

3.1. *Smooth action of a Lie group on an orbifold.* Let X be a differentiable orbifold of dimension n, and let $|X|$ be its underlying topological space. The *smooth orbifold structure* X on $|X|$ is given by an atlas made up of uniformizing charts φ: $\tilde{U} \to U$ whose targets U form an open covering of $|X|$; the $\tilde{U}$'s are smooth manifolds of dimension n with a smooth effective properly discontinuous action of a group G_U and φ induces an homeomorphism of $\tilde{U}/G_U$ on U. Two charts φ: $\tilde{U} \to U$ and ψ: $\tilde{V} \to V$ are related by local diffeomorphisms: for $u \in \tilde{U}$ and $v \in \tilde{V}$ such that $\varphi(u) = \psi(v)$, there is a local diffeomorphism h of a neighbourhood W of u on a neighbourhood of v, called a change of charts, such that $\psi \circ h = \varphi$ on W. Note that the elements of G_U are particular cases of change of charts. The pseudogroup acting on the disjoint union of the $\tilde{U}$'s and generated by all changes of charts will be denoted by $\mathscr{H}_X$. The equivalence class of $\mathscr{H}_X$ does not depend on the choice of a compatible atlas for X. The orbifold structure on X is completely characterized by the equivalence class of $\mathscr{H}_X$ and the identification of $|X|$ with the space of orbits of $\mathscr{H}_X$.

A *smooth effective action of a Lie group H on a smooth orbifold X* is an effective continuous action α: $H \times |X| \to |X|$ of H satisfying the following conditions. For each $h_0 \in H$ and $x_0 \in |X|$, there are uniformizing charts

$$\varphi: \tilde{U} \to U \quad \text{and} \quad \psi: \tilde{V} \to V$$

such that $x_0 \in U$ and $\alpha(h_0, x_0) \in V$, together with a neighbourhood A of h_0 in H and a smooth map $\tilde{\alpha}: A \times \tilde{U} \to \tilde{V}$ such that $\psi(\tilde{\alpha}(h,u)) = \alpha(h, \varphi(u))$. Moreover for each $h \in A$, the map $u \to \tilde{\alpha}(h,u)$ is a diffeomorphism of $\tilde{U}$ on an open set of $\tilde{V}$.

For instance suppose that X is the orbifold quotient of a manifold $\tilde{X}$ by a smooth, properly discontinuous, effective action of a discrete group G. A smooth effective action of a Lie group H on X is a smooth action on $\tilde{X}$ of a Lie group $\tilde{H}$ which is an extension of H by G:

$$1 \to G \to \tilde{H} \to H \to 1,$$

this action of $\tilde{H}$ extending the given action of G on $\tilde{X}$.

As a specific example, let X be the quotient of $\mathbf{R}^2$ by the cyclic group G generated by a rotation of order n; if H is the circle group acting by rotation on $\mathbf{R}^2/G$, then $\tilde{H}$ is the n-fold covering of H acting by rotation on $\mathbf{R}^2$.

The above situation occurs locally around an orbit when the group H is a connected solvable Lie group: for every point x in X, there is a neighbourhood U of the orbit Hx of x such that U, as an orbifold, is the quotient of a manifold $\tilde{U}$ by the action of a discrete group G acting in a properly discontinuous way on $\tilde{U}$. This follows from the arguments of [Hae-Qua, 2.5]. Indeed, the orbit Hx is contained in a stratum of the natural stratification of X and $\pi_2(Hx) = 0$. This is not true in general for a non solvable Lie group (see [Hae-2] for a specific example when $H = SU_2$).

3.2. *Foliation on an orbifold.* By definition, a smooth foliation $\mathcal{F}_x$ on an orbifold X is given by a smooth foliation on the disjoint union of the $\tilde{U}$'s which is invariant by the changes of charts. We can choose the $\tilde{U}$'s such that the foliation on each $\tilde{U}$ is given by a surjective submersion with connected fibers on a manifold T_U, so that the holonomy pseudogroup $\mathcal{H}$ of the foliation $\mathcal{F}_x$ is generated by the local diffeomorphisms of the disjoint union T of the T_U's which are the projections of the elements of $\mathcal{H}_X$ (i.e., the change of charts of the orbifold).

3.3. *Classifying space BX of an orbifold X.* The classifying space BX of an orbifold X is a generalized Seifert bundle with a contractible generic fiber and base space X. More specifically [Hae-1], if X is a smooth orbifold of dimension n, we choose on X a Riemannian metric (i.e. a Riemannian metric in the source of each uniformizing chart invariant by $\mathcal{H}_X$). For each $N > n$, let us consider the manifold $BX(N)$ which is the bundle associated to the principal bundle P of orthonormal n-frames on X, with fiber the Stiefel manifold $V(N,n)$ of orthonormal n-frames in $\mathbf{R}^N$. Then $BX(N)$ is a generalized Seifert fibration with base space X and generic fiber $V(N,n)$, and BX is the direct limit of the $BX(N)$. The leaves of the universal foliation on BX

are the limits of the fibers of the $BX(N)$'s. They are the fibers of the projection $\bar{p}$ of BX on X.

A *continuous map* f, in the orbifold sense, of a topological space Y in an orbifold X is given by a continuous 1-cocycle defined over an open covering of Y with value in the topological groupoid of germs of elements of $\mathscr{H}_X$. There is a 1-1 correspondence between the continuous maps of Y in X and the classes of continuous maps of Y in BX under the equivalence relation given by the homotopy along the fibers of the projection $\bar{p}$ of BX on X [Hae-1].

Let $\mathscr{F}_X$ be a foliation on X; then the pull back of $\mathscr{F}_X$ by the projection of $BX(N)$ on X is a foliation $\mathscr{F}(N)$ on $BX(N)$ whose limit in BX is a foliation $B\mathscr{F}_X$; its holonomy pseudogroup is equivalent to the holonomy pseudogroup of $\mathscr{F}_X$.

The pair $(X, \mathscr{F}_X)$ is a classifying space for the holonomy pseudogroup of $\mathscr{F}_X$ if the holonomy covering of each leaf of $\mathscr{F}_X$ is contractible (cf. [Hae-1]). This will be the case if and only if the holonomy covering of each leaf of $B\mathscr{F}_X$ is contractible, i.e., if the pair $(BX, B\mathscr{F}_X)$ is a classifying space.

3.4. THEOREM. *There are canonical bijections between the following three sets*:

(1) *The set A_1 of equivalence classes of Killing foliations $\mathscr{F}$ on complete Riemannian manifolds Y (cf. 2.6) such that the leaf closures are compact, two such foliations being equivalent if their holonomy pseudogroups are differentiably equivalent*;

(2) *The set A_2 of differentiable equivalence classes of complete Killing pseudogroups $\mathscr{H}$ of local isometries such that $\mathscr{H}$ restricted to a generic orbit closure is equivalent to the pseudogroup generated by the action of a dense finitely generated group Γ of rank N of translations of $\mathbf{R}^k$*;

(3) *The set A_3 of equivalence classes of quadruples (X, T^N, L, α) where X is a differentiable orbifold, α is a smooth effective action of a torus T^N on X, and L is a dense contractible subgroup of T^N whose action on X is locally free, two quadruples (X, T^N, L, α) and $(X', T^{N'}, L', \alpha')$ being equivalent if there is an isomorphism i of T^N on $T^{N'}$ (so $N = N'$) and a diffeomorphism h of X on X' conjugating the actions α and α'.*

The bijection $A_1 \to A_3$ associates to $\mathscr{F}$ a canonical realization $(X, \mathscr{F}_X)$ of the classifying space of its holonomy pseudogroup, where $\mathscr{F}_X$ is the foliation on X whose leaves are the orbits of L. In particular there is a differentiable map p: $Y \to X$ in the orbifold sense such that $p^*(\mathscr{F}_X) = \mathscr{F}$.

The bijection of A_1 on A_2 associates to a foliation the differentiable equivalence class of its holonomy pseudogroup.

The map from A_3 to A_2 associates to (X, T^N, L, α) the holonomy pseudogroup of the foliation $\mathscr{F}_X$. The main point is to construct an inverse for this map. Namely, one can associate to a pseudogroup $\mathcal{H}$ in A_2 a quadruple (X, T^N, L, α), where $T^N = \Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$ and the dense subgroup L is the image in the quotient $T^N = \Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$ of the kernel of the homomorphism from $\Gamma \otimes \mathbf{R}$ on $\mathbf{R}^k$ sending $\gamma \otimes r$ on $r\gamma$.

3.5. COROLLARY. *Let $\mathcal{F}$ be a Killing foliation on a compact oriented connected Riemannian manifold Y. Then there is a locally free action by isometries of a connected Lie group H on a compact manifold $\hat{X}$ such that $\mathcal{F}$ and the foliation $\hat{\mathcal{F}}$ on $\hat{X}$ whose leaves are the orbits of H have equivalent holonomy pseudogroups.*

Proof. The map of A_1 in A_3 associates to $\mathcal{F}$ a foliation $\mathcal{F}_X$ on an orbifold X. We choose a Riemannian metric on X invariant by the action of T^N and we consider the manifold $\hat{X}$ of direct orthonormal frames on X. The natural projection $\pi\colon \hat{X} \to X$ is a generalized Seifert fibration whose fibers are the orbits of the natural action of $SO(n)$ on $\hat{X}$, where $n = \dim X$. The action of L on X lifts to a locally free action of L on $\hat{X}$ commuting with the action of $SO(n)$. Hence, we get a locally free action of the group $H = L \times SO(n)$ on $\hat{X}$ defining a foliation $\hat{\mathcal{F}}$ which is the inverse image by π of $\mathcal{F}_X$. The holonomy pseudogroups of $\hat{\mathcal{F}}$ and of $\mathcal{F}_X$ are equivalent.

3.6. *Remark.* If the leaves of $\mathcal{F}$ are the orbits of an isometric flow on a compact Riemannian manifold Y whose generic orbit is non compact, then the closure of the flow in the group of isometries of Y gives an action of a torus T^N on Y. It follows from the injection of A_3 in A_1 that there is a T^N-equivariant diffeomorphism of Y on the orbifold X (which is then a manifold) mapping the flow on the action of L.

By specifying Theorem 3.4 to the compact simply connected case, we get the following statement.

3.7. THEOREM. *There are canonical bijections between the following three sets*:

(1) *The set A_1 of equivalence classes of Riemannian foliations $\mathcal{F}$ on compact* 1-*connected manifolds Y, two foliations being equivalent if their holonomy pseudogroups are differentiably equivalent*;

(2) *The set A_2 of differentiable equivalence classes of* 1-*connected complete pseudogroups $\mathcal{H}$ of local isometries whose space of orbit closures is compact*;

(3) *The set A_3 of equivalence classes of quadruples (X, T^N, L, α), where α is an effective smooth action of the torus T^N on a compact* 1-*connected orbifold X and L is a dense contractible subgroup of T^N whose action on X is locally free.*

It is clear that Theorem 3.7 is a particular case of Theorem 3.4. Indeed, a Riemannian foliation on a 1-connected compact manifold is necessarily a Killing foliation [Mol-1]. Moreover, any 1-connected complete pseudogroup

$\mathcal{H}$ of local isometries is Killing (cf. 2.6); if the space of orbits closures is compact, we can apply 2.8 to verify that the hypothesis in 3.4.2) is satisfied.

4. Local model and canonical local realization

4.1. *Local model for a Killing pseudogroup.* We assume that $\mathcal{H}$ is a connected Killing pseudogroup acting on a manifold of dimension m, and that the restriction of $\mathcal{H}$ to the closure of a generic orbit is equivalent to the pseudogroup generated by the action of a dense subgroup Γ of $\mathbf{R}^k$ of rank N, acting by translations on $\mathbf{R}^k$.

It follows from [Hae-2] and [Hae-Sal] that a model for the restriction of $\mathcal{H}$ to a tubular neighbourhood of the closure of an orbit is equivalent to a pseudogroup $\mathcal{P}(\Gamma_0, \Lambda, \rho)$ described as follows:

(i) Γ_0 is a subgroup of Γ discrete in $\mathbf{R}^k$ of rank $s \le k$;

(ii) Λ is a central extension of a finite group D by Γ/Γ_0:

$$0 \to \Gamma/\Gamma_0 \to \Lambda \to D \to 1.$$

The inclusion of Γ/Γ_0 in $\mathbf{R}^k/\Gamma_0$ induces a central extension

$$0 \to \mathbf{R}^k/\Gamma_0 \to G \to D \to 1$$

which defines a Lie group G (with Lie algebra $\mathbf{R}^k$) containing Λ as a dense subgroup. The maximal subgroup K of G is unique.

(iii) a slice representation $\rho\colon K \to O(B)$ which is injective, where B is an Euclidean ball of dimension $m - k + s$ and $O(B)$ its group of isometries.

Then $\mathcal{P}(\Gamma_0, \Lambda, \rho)$ is the pseudogroup generated by the action of Λ on the homogeneous ball bundle $G \times_K B$.

We recall that $G \times_K B$ is the quotient of $G \times B$ by the equivalence relation which identifies (g, b) and $(gk^{-1}, \rho(k)b)$ if $k \in K$. The group Λ acts on $G \times_K B$ by $\lambda[g, b] = [\lambda g, b]$.

4.2. *Local realization.* In this paragraph we construct canonically a foliation $\mathcal{F}_X$ on an orbifold X whose holonomy pseudogroup is equivalent to $\mathcal{P}(\Gamma_0, \Lambda, \rho)$. The leaves of this foliation are the orbits of a locally free action of a dense subgroup L of a torus T^N acting on X.

The group Γ/Γ_0 is a dense subgroup of $\mathbf{R}^k/\Gamma_0$, It is also identified to the discrete cocompact subgroup $\Gamma \otimes \mathbf{Z}/\Gamma_0 \otimes \mathbf{Z}$ of the Lie group $\Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{R}$ (which is isomorphic to $\mathbf{R}^{N-s} \times T^s$, where s is the rank of Γ_0). After the choice of a basis of Γ, the quotient $\Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$ is canonically isomorphic to a torus T^N, and we shall always make this identification in what follows.

The linear surjective map $\Gamma \otimes \mathbf{R} \to \mathbf{R}^k$ sending $\gamma \otimes r$ on $r\gamma$ induces an homomorphism $\Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{R} \to \mathbf{R}^k/\Gamma_0$, and we have the following commutative diagram

$$\begin{tikzcd} \Gamma/\Gamma_0 \arrow[rr] \arrow[dr] & & \Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{Z} \arrow[dl] \\ & \mathbf{R}^k/\Gamma_0 & \end{tikzcd}$$

This diagram induces a commutative diagram of central extensions

$$\begin{tikzcd} O \arrow[r] & \Gamma/\Gamma_0 \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 1 \\ O \arrow[r] & \Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{Z} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 1 \\ O \arrow[r] & \mathbf{R}^k/\Gamma_0 \arrow[r] & G \arrow[r] & D \arrow[r] & 1 \end{tikzcd}$$

defining a Lie group $\tilde{G}$ whose connected component of the identity is $\Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{Z}$.

The map of Λ in $\tilde{G}$ is injective; its image is a discrete cocompact subgroup $\tilde{\Lambda}$ of $\tilde{G}$. The quotient $\tilde{G}/\tilde{\Lambda} = \Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$ is canonically isomorphic to the torus T^N (using the basis of Γ chosen above).

The homomorphism of $\tilde{G}$ in G is a surjective submersion. Its kernel $\tilde{L}$ is a contractible subgroup of $\tilde{G}$ ($\tilde{L}$ is also the projection in $\Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{Z}$ of the kernel of $\Gamma \otimes \mathbf{R} \to \mathbf{R}^k$). The projection $\tilde{G} \to \tilde{G}/\tilde{\Lambda} = T^N$ maps $\tilde{L}$ onto a dense contractible subgroup L (note that $\tilde{L} \cap \tilde{\Lambda} = 1$).

Under the map $\tilde{G} \to G$, the maximal compact subgroup $\tilde{K}$ of $\tilde{G}$ is mapped isomorphically on the maximal compact subgroup K of G. The faithful representation ρ of K in $O(B)$ gives an effective action of $\tilde{K}$ on B.

Let us consider the homogeneous ball bundle $\tilde{X} = \tilde{G} \times_{\tilde{K}} B$. The group $\tilde{G}$ acts on $\tilde{X}$ by left translations and the subgroup $\tilde{\Lambda}$ acts properly discontinuously on $\tilde{X}$ so that the quotient $X = \tilde{\Lambda}/\tilde{X}$ is an orbifold on which the torus $\tilde{G}/\tilde{\Lambda} = T^N$ operates.

The subgroup $\tilde{L}$ of $\tilde{G}$ acts freely on $\tilde{X}$ defining a foliation invariant by the action of $\tilde{G}$ (recall that $\tilde{L}$ is in the center of $\tilde{G}$). Hence the image L of $\tilde{L}$ in $T^N = \tilde{G}/\tilde{\Lambda}$ acts locally freely on X and defines a foliation $\mathscr{F}_X$ on X.

The holonomy pseudogroup of $\mathscr{F}_X$ is equivalent to the pseudogroup $\mathscr{P}(\Gamma_0, \Lambda, \rho)$ acting on $G \times_K B$. Indeed the fibers of the natural submersion $f\colon \tilde{X} = \tilde{G} \times_{\tilde{K}} B \to G \times_K B$ are the orbits of $\tilde{L}$. Moreover f is equivariant with respect to the surjective homomorphism $\tilde{G} \to G$, hence equivariant with respect to the isomorphism $\tilde{\Lambda} \to \Lambda$.

Remarks. (1) The orbifold X is a manifold if and only if Λ is without torsion (equivalently if $\Lambda \cap K = 1$). In this case Λ is free abelian.

(2) Dim $X = N + \dim B - \dim K$.

4.3. *Local model for the action of a torus T^N on an orbifold.* Let X be a smooth orbifold of dimension n, together with a smooth effective action of a torus T^N. We identify T^N with the quotient of $\mathbf{R}^N$ by the lattice $\Gamma = \mathbf{Z}^N$.

For a point $x \in |X|$, the orbit of x is a torus $T = T^N/H$ of dimension $N - s$, where H is the stabilizer of x. There is a small invariant tubular neighbourhood E of T, which is a bundle over T with fiber an orbifold which is the quotient of a ball B (of dimension $n - N + s$) by a finite group I. As $\pi_2(T) = 0$, the orbifold E is the quotient of a differentiable 1-connected manifold $\tilde{E}$ by the action of the fundamental group $\tilde{\Lambda}$ of E acting in a properly discontinuous way on $\tilde{E}$ (see [Hae-Qua], 2.5).

Let φ be the natural projection of $\tilde{E}$ on E. We denote by $\tilde{G}$ the Lie group of diffeomorphisms of $\tilde{E}$ which projects by φ on the diffeomorphisms of E given by the action of T^N. The group $\tilde{G}$ is an extension of T^N by $\tilde{\Lambda}$ which is identified to a discrete invariant subgroup of $\tilde{G}$.

Let $\tilde{x}$ be a point of $\tilde{E}$ such that $\varphi(\tilde{x}) = x$; the stability subgroup $\tilde{K}$ of $\tilde{x}$ is a compact subgroup of $\tilde{G}$ (which is an extension of H by I). The action of $\tilde{G}$ on $\tilde{E}$ is proper. So by the Slice Theorem, we can assume that $\tilde{E}$ is isomorphic to the homogeneous ball bundle $\tilde{G} \times_{\tilde{K}} B$, where B is a slice at $\tilde{x}$ which is an Euclidean ball on which $\tilde{K}$ acts through the slice representation $\tilde{\rho}\colon \tilde{K} \to O(B)$.

As $\tilde{E}$ is 1-connected, the quotient $\tilde{G}/\tilde{K}$ is also 1-connected, so it is the universal covering of the torus $T = T^N/H$, hence it is contractible. It follows that the invariant subgroup $\tilde{K}$ is the unique maximal compact subgroup of $\tilde{G}$.

Let $\tilde{G}_0$ (resp. $\tilde{K}_0$) be the component of the identity in $\tilde{G}$ (resp. $\tilde{K}$). As $\tilde{G}/\tilde{K}$ is connected, $\tilde{G}/\tilde{G}_0 = \tilde{K}/\tilde{K}_0$ is a finite group D.

The group $\tilde{G}_0$ is a covering of T^N, so it is the quotient of $\mathbf{R}^N$ by a subgroup Γ_0 of $\Gamma = \mathbf{Z}^N$, and we have $\tilde{\Lambda} \cap \tilde{G}_0 = \Gamma/\Gamma_0$. Hence we have the central extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R}^N/\Gamma_0 & \longrightarrow & \tilde{G} & \longrightarrow & D & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Gamma/\Gamma_0 & \longrightarrow & \tilde{\Lambda} & \longrightarrow & D & \longrightarrow & 1. \end{array}$$

Conversely the last extension determines the groups $\tilde{G}$ and $\tilde{K}$. Indeed, the first central extension $\tilde{G}$ is induced from the second one by the inclusion of Γ/Γ_0 in $\mathbf{R}^N/\Gamma_0$. The group $\tilde{G}$ contains $\tilde{\Lambda}$ as an invariant subgroup, $\tilde{G}/\tilde{\Lambda} = \mathbf{R}^N/\Gamma = T^N$, and $\tilde{K}$ is the unique maximal compact subgroup of $\tilde{G}$.

Summing up, we have the following description:

4.4. Proposition. *A smooth effective action of a torus T^N on an orbifold X restricted to an invariant tubular neighbourhood E of an orbit is characterized by* *the following data*:

(i) *A subgroup* Γ_0 *of the lattice* $\Gamma = Z^N$ *in* $\mathbf{R}^N$;

(ii) *A central extension* $\tilde{\Lambda}$ *of a finite group* D *by* Γ/Γ_0

$$0 \to \Gamma/\Gamma_0 \to \tilde{\Lambda} \to D \to 1;$$

(iii) *A faithful representation* $\tilde{\rho}$: $\tilde{K} \to O(B)$, *where* $\tilde{K}$ *is the maximal compact subgroup of the Lie group* $\tilde{G}$ *which is the central extension*

$$0 \to R^N/\Gamma_0 \to \tilde{G} \to D \to 1$$

induced by the inclusion of Γ/Γ_0 *in* $\mathbf{R}^N/\Gamma_0$, *and* $O(B)$ *is the group of isometries of an Euclidean ball* B.

E is the orbifold quotient of $\tilde{G} \times_{\tilde{K}} B$ by $\tilde{\Lambda}$, with the action of $\tilde{G}/\tilde{\Lambda} = T^N$.

4.5. Moreover let L be a contractible dense subgroup of codimension k in T^N, acting locally freely on X and whose orbits are the leaves of a foliation $\mathscr{F}_X$ on X. Let L_0 be the component of the identity of the inverse image of L by the canonical projection of $\mathbf{R}^N$ on $\mathbf{R}^N/\mathbf{Z}^N = T^N$. If we identify $\mathbf{R}^N/L_0$ with $\mathbf{R}^k$, the canonical projection of $\mathbf{R}^N$ on $\mathbf{R}^N/L_0 = \mathbf{R}^k$ identifies the lattice $\mathbf{Z}^N$ with a dense subgroup Γ on $\mathbf{R}^k$ (because L is dense and contractible). Let $\tilde{L}$ be the connected Lie subgroup of $\tilde{G}$ mapped on L by the projection of $\tilde{G}$ on T^N. The hypothesis that L acts locally freely on X and that $\tilde{G}/\tilde{K}$ is simply connected implies that $\tilde{L}$ is a closed subgroup and that $\tilde{L} \cap \tilde{K} = 1$. Hence the quotient $\tilde{G}/\tilde{L}$ is a Lie group G with Lie algebra identified to $\mathbf{R}^k$. Moreover $\tilde{\Lambda}$ (resp. $\tilde{K}$) maps isomorphically on a dense subgroup Λ (resp. the maximal compact subgroup K) of G. The intersection of Λ with the connected component $G_0 = \mathbf{R}^k/\Gamma_0$ of the identity in G is canonically isomorphic to Γ/Γ_0. Hence the restriction of $\mathscr{F}_X$ to the tubular neighbourhood E is equivalent to the pseudogroup generated by the action of Λ on $G \times_K B$, where K acts on B by the representation ρ: $K \to O(B)$ deduced from $\tilde{\rho}$ by the isomorphism of $\tilde{K}$ on K.

In conclusion, we see that the action of the pair (T^N, L) on the tubular neighbourhood E of the closure of an orbit of L can be deduced from the invariants describing the holonomy pseudogroup of the restriction of the foliation $\mathscr{F}_X$ to E.

5. Proof of Theorem 3.4

We have to construct maps F_{ji}: $A_i \to A_j$ for $i, j, k \in \{1, 2, 3\}$ such that $F_{kj} \circ F_{ji} = F_{ki}$ and $F_{ii} =$ identity.

The obvious maps F_{21} and F_{23} have been defined in 3.4 and F_{13} was constructed at the end of 3.3, so that $F_{21} \circ F_{13} = F_{23}$. The main task is to construct a map F_{32}: $A_2 \to A_3$, which is a right inverse of F_{23} and prove that it is a bijection. As F_{21} is obviously injective, it will follow that all maps F_{ji} are bijective.

5.1. Let $\mathscr{H}$ be a Killing pseudogroup in A_2 acting on a manifold T. We have described in 4.1 a local model for $\mathscr{H}$ and in 4.2 its realization as the holonomy pseudogroup of a foliation on an orbifold, given by a locally free action of a dense subgroup L of a torus. We still have to glue up these pieces to get a global orbifold with an action of L. This will be done with lemmas 5.2 and 5.3. We first fix up some notations.

Let $\pi\colon T \to W$ be the projection on the space W of orbit closures. We take a covering of W by open sets U_i $(i \in I)$, such that the restriction of $\mathscr{H}$ to $\pi^{-1}(U_i)$ is equivalent to the pseudogroup $\mathscr{H}_i = \mathscr{P}(\Gamma_0^i, \Lambda_i, \rho_i)$ generated by the action of a subgroup Λ_i of G_i acting on $T_i = G_i \times_{K_i} B_i$ (as in 4.1).

We can assume, after changing $\mathscr{H}$ by an equivalence, that $\mathscr{H}$ acts on the disjoint union T of the T_i's and that the restriction of $\mathscr{H}$ to T_i is $\mathscr{H}_i$. The collection $\mathscr{H}_{ji}$ of elements of $\mathscr{H}$ with source in T_i and target in T_j is an equivalence of $\mathscr{H}$ restricted to $T_i \cap \pi^{-1}(U_i)$ on $\mathscr{H}$ restricted to $T_j \cap \pi^{-1}(U_j)$.

We introduce on the space W a differentiable structure in the following manner: a map θ from W to a manifold Z is differentiable if the map $\theta \circ \pi\colon T \to Z$ is differentiable. It is equivalent to say that, for each $i \in I$ the canonical projection of B_i on B_i/K_i composed with θ is differentiable.

From the local model $\mathscr{H}_i$, we can construct as in 4.2 an orbifold X_i together with a locally free action of a dense contractible subgroup L of the torus $T^N = \Gamma \otimes \mathbf{R}/\Gamma \otimes \mathbf{Z}$ defining on X_i a foliation $\mathscr{F}_i$ whose holonomy pseudogroup is equivalent to $\mathscr{H}_i$. We summarize the notations in the following commutative diagram:

$$\begin{tikzcd}
 & \tilde{X}_i = \tilde{G}_i \times_{K_i} B_i \arrow[dl, "f_i"'] \arrow[dr, "\varphi_i"] \arrow[dd, "\tilde{p}_i"] & \\
T_i = G_i X_{K_i} B_i \arrow[dr, "\pi_i"'] & & X_i = \tilde{X}_i/\tilde{\Lambda}_i \arrow[dl, "p_i"] \\
 & U_i &
\end{tikzcd}$$

where π_i and $\tilde{p}_i$ are the projections of T_i and $\tilde{X}_i$ on the space of orbit closures $U_i = B_i/K_i$, and φ_i is the natural projection from $\tilde{X}_i$ to X_i.

Now we have to glue up the orbifolds X_i to obtain a global orbifold X with a torus action.

5.2. Lemma. (a) *For any $i, j \in I$, there is a diffeomorphism H_{ji} of $p_i^{-1}(U_i \cap U_j) \subset X_i$ on $p_j^{-1}(U_j \cap U_i) \subset X_j$ commuting with the action of T^N, such that near each point $\tilde{x}_i$ of $\tilde{p}_i^{-1}(U_i \cap U_j)$ there is a local lifting $\tilde{H}_{ji}$ of H_{ji} which projects by f_i and f_j on an element of $\mathscr{H}$.*

(b) *If H_{ji} and H'_{ji} are two such diffeomorphisms, there is a differentiable map l_{ji} of $U_i \cap U_j$ in L such that, for $x \in p_i^{-1}(U_i \cap U_j)$,*

$$H'_{ji}(x) = l_{ji}(p_i(x)).H_{ji}(x).$$

Proof. For any $x \in U_i \cap U_j$, the restriction of $\mathscr{H}$ to a tubular neighbourhood of the orbit closure $\pi^{-1}(x)$ is equivalent to a pseudogroup $\mathscr{P} = \mathscr{P}(\Gamma_0, \Lambda, \rho)$ generated by the action of a dense subgroup Λ of a Lie group G acting on $G \times_K B$ (see 4.1). We can choose B small enough so that the projection of $G \times_K B$ in W is an open set U in $U_i \cap U_j$. As $G \times_K B$ is 1-connected, the natural equivalence of $\mathscr{P}$ with $\mathscr{H}/\pi_i^{-1}(U) = \mathscr{P}(\Gamma_0^i, \Lambda_i, \rho_i)$ is generated by a map h_i: $G \times_K B \to G_i \times_{K_i} B_i$ which is a covering of its image (the argument is the same as in [Sal-2]), well defined up to a translation by an element of Λ_i and equivariant with respect to a homomorphism α_i: $\Lambda \to \Lambda_i$. This homomorphism is the restriction of a homomorphism of Lie groups $G \to G_i$, still denoted by α_i, inducing the identity on their Lie algebras (which are isomorphic to $\mathbf{R}^k$). Moreover there is a commutative diagram of central extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma/\Gamma_0 & \longrightarrow & \Lambda & \longrightarrow & D & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha_i} & & \downarrow & & \\ 0 & \longrightarrow & \Gamma/\Gamma_0^i & \longrightarrow & \Lambda_i & \longrightarrow & D_i & \longrightarrow & 1 \end{array}$$

where the restriction of α_i to Γ/Γ_0 is induced by the identity of Γ.

As in 4.2, we construct an extension $\tilde{G}$ of D by $\Gamma \otimes \mathbf{R}/\Gamma_0 \otimes \mathbf{Z}$ containing a discrete subgroup $\tilde{\Lambda}$ isomorphic to Λ, and such that $\tilde{G}/\tilde{\Lambda} = T^N$. The homomorphism α_i lifts to a homomorphism of Lie groups $\tilde{\alpha}_i$ from $\tilde{G}$ to $\tilde{G}_i$, mapping $\tilde{\Lambda}$ in $\tilde{\Lambda}_i$, and inducing the identity from $\tilde{G}/\tilde{\Lambda} = T^N$ to $\tilde{G}_i/\tilde{\Lambda}_i = T^N$.

We now construct an $\tilde{\alpha}_i$-equivariant lifting $\tilde{H}_i$: $\tilde{G} \times_{\tilde{K}} B \to G_i \times_{\tilde{K}_i} B_i$ of h_i with respect to the submersions f and f_i. To do this, we choose an homomorphism σ_i of G_i in $\tilde{G}_i$ which is a section of the projection of $\tilde{G}_i$ on G_i. There is a unique homomorphism σ of G in $\tilde{G}$ which is a section of the projection of $\tilde{G}$ on G, and such that $\sigma_i \circ \alpha_i = \tilde{\alpha}_i \circ \sigma$. The maps σ (resp. σ_i) define sections s (resp. s_i) of f (resp. f_i), which are σ (resp. σ_i) equivariant. So we get a map of $s(G \times_K B)$ in $s_i(G_i \times_{K_i} B_i)$ which extends uniquely to an $\tilde{\alpha}_i$-equivariant map $\tilde{H}_i$. If $\tilde{H}_i'$ is another $\tilde{\alpha}_i$-equivariant lifting of h_i, then there is a differentiable map $\tilde{l}$ of B in $\tilde{L} = \operatorname{Ker}(\tilde{G} \to G)$ such that

$$\tilde{H}_i'([g,b]) = \tilde{H}_i([l(b)g,b]) = \tilde{\alpha}_i(\tilde{l}(b)).\ \tilde{H}_i([g,b]).$$

Moreover the map $\tilde{l}$ is $\tilde{K}$-invariant, so it defines a differentiable map of U in $\tilde{L}$.

Passing to the quotient by $\tilde{\Lambda}$ and $\tilde{\Lambda}_i$, the map $\tilde{H}_i$ gives a T^N-equivariant diffeomorphism H_i of $\tilde{\Lambda}\setminus(\tilde{G}\times_{\tilde{K}}B)$ on $p_i^{-1}(U)\subset X_i = \tilde{\Lambda}_i\setminus(\tilde{G}_i\times_{\tilde{K}_i}B_i)$. Another choice $\tilde{H}_i'$ for a lift of h_i gives a diffeomorphism H_i' which differs from H_i by composition with a differentiable map l of U in L.

Replacing i by j, we also have a map $h_j\colon G\times_K B\to G_j\times_{K_j}B_j$ which is a covering on its image. A local inverse of h_i, composed with h_j is an element h_{ji} of $\mathscr{H}$. As before we can construct a T^N-equivariant diffeomorphism H_j, and $H_{ji} = H_j\circ H_i^{-1}$ is a T^N-equivariant diffeomorphism from $p_i^{-1}(U)$ to $p_j^{-1}(U)$, uniquely defined up to a differentiable map of U in L.

For an open covering $\{U^r\}_{r\in R}$ of $U_i\cap U_j$, one can construct such

$$H_{ji}^r\colon p_i^{-1}(U^r)\to p_j^{-1}(U^r)$$

and patch them using a partition of unity. Indeed, let $\underline{L}$ be the sheaf over W of germs of differentiable maps of W in L. It is a fine sheaf because differentiable partitions of unity exist on W. Hence the Cech cohomology groups $H^p(\mathscr{U},\underline{L})$ with respect to any covering $\mathscr{U}$ vanish for $p>0$.

In particular, the collection of maps $l_{rs}\colon U^r\cap U^s\to L$ defined by

$$H_{ji}^r(x) = l_{rs}(u).H_{ji}^s(x)\quad\text{where } x\in p_i^{-1}(u),$$

is a 1-cocycle which is a coboundary: $l_{rs}(u) = l_r(u)^{-1}.l_s(u)$. Hence $H_{ji}(x)$ defined by $l_r(p_i(x)).H_{ji}^r(x)$ for $x\in U^r$ verifies the properties of Lemma 5.2.

5.3. Lemma. (a) *One can choose the H_{ji}'s in Lemma 5.2 such that, for every $i,j,k\in I$, $H_{ki} = H_{kj}\circ H_{ji}$.*

(b) *If the H_{ji}''s are another such choice, there are differentiable maps l_i' of U_i in L such that, for $x\in p_i^{-1}(U_i\cap U_j)$,*

$$H_{ji}'(x) = l'_j(p_j(x))^{-1}.l_i'(p_i(x)).H_{ji}(x).$$

Proof. We start with a collection of H_{ji}'s as in Lemma 5.2. We can assume that $H_{ji} = H_{ij}^{-1}$. The diffeomorphism $H_{ij}\circ H_{jk}\circ H_{ki}$ of $p_i^{-1}(U_i\cap U_j\cap U_k)$ evaluated on $p_i^{-1}(u)$ is a translation by an element $l_{ijk}(u)$ of L. The l_{ijk}'s form a 2-cocycle. Indeed, by restriction above $U_i\cap U_j\cap U_k\cap U_m$, one has

$$l_{ijk}(u).l_{ikm}(u).l_{ijm}(u)^{-1} = H_{ij}\big(l_{jkm}(u).H_{ij}^{-1}\big) = l_{jkm}(u)$$

as the map H_{ij} commutes with the action of T^N (hence of L).

As $H^2(\mathscr{U}, \underline{L}) = 0$, there are differentiable maps l'_{ij} of $U_i \cap U_j$ in L such that

$$l_{ijk}(u) = l'_{ij}(u).l'_{jk}(u).l'_{ik}(u)^{-1} \quad \text{for } u \in U_i \cap U_j \cap U_k.$$

If we define

$$H'_{ij}(x) = l'_{ij}(p_i(x))^{-1}.H_{ij}(x)$$

we have $H'_{ij} \circ H'_{jk} = H'_{ik}$. The vanishing of $H^1(\mathscr{U}, \underset{\sim}{L})$ implies part (b) of Lemma 5.3.

Lemma 5.3 implies the existence of an orbifold X together with an action of the dense contractible subgroup L of T^N acting locally freely on X and defining on X a foliation $\mathscr{F}_X$ whose holonomy pseudogroup is equivalent to the given Killing pseudogroup $\mathscr{H}$. Indeed the orbifold X may be obtained by glueing up the local realizations X_i's using the diffeomorphisms H_{ji}'s of Lemma 5.3. This glueing is compatible with the action of T^N and gives the map F_{32} of A_2 in A_3.

If X' is another orbifold with a locally free action of L defining a foliation $\mathscr{F}_X$ on X with holonomy pseudogroup equivalent to $\mathscr{H}$, then using the considerations of 4.5 and part (b) of Lemma 5.3, one can construct a T^N-equivariant isomorphism of X on X'. This shows that F_{32} is bijective.

6. On the homotopy type of the generic leaf

The results of this paragraph are independent of the previous ones.

Let $\mathscr{F}$ be a Riemannian foliation on a compact simply connected Riemannian manifold Y. The foliation $\mathscr{F}$ restricted to the closure $\overline{F}$ of a generic leaf F is a Lie foliation whose holonomy pseudogroup is equivalent to the pseudogroup generated by a dense subgroup Γ of $\mathbf{R}^k$ acting by translations on $\mathbf{R}^k$. Let N be the rank of Γ (note that k is the codimension of F in $\overline{F}$).

6.1. Theorem. *With the above assumptions, the fundamental group and the homology groups of the generic leaf F are of finite type and the fundamental group of F is abelian.*

The dimension of F is greater than or equal to $N - k$, and $H_i(F, Z) = Z$ for $i = \dim \overline{F} - N$ and $H_i(F, Z) = 0$ for $i > \dim \overline{F} - N$.

If F is 1-connected, then F has the homotopy type of a finite Poincaré complex F_0 of dimension equal to $\dim \overline{F} - N$.

If F is contractible, then the orbifold X associated to $\mathscr{F}$ in Theorem 3.4 is a smooth manifold, the map $p\colon Y \to X$ is a homotopy equivalence and $\overline{F}$ has the homotopy type of T^N.

Proof. Let $\mathscr{H}$ be the holonomy pseudogroup of $\mathscr{F}$. Let $B\mathscr{H}$ be its classifying space and denote by $B\mathscr{F}$ the universal foliation on $B\mathscr{H}$. There is a classifying map f of Y in $B\mathscr{H}$ such that $\mathscr{F} = f^*(B\mathscr{F})$. A model for the classifying space of the foliation restricted to the closure $\bar{F}$ of F is the torus T^N with a linear foliation (cf. 1.2).

Let $g\colon \bar{F} \to T^N$ be the classifying map. The theoretical homotopic fiber of f and g has the homotopy type of the generic fiber F (cf. [Hae-1]). We can replace Y and $\bar{F}$ by homotopy equivalent spaces Y' and $\bar{F}'$ to obtain a commutative diagram:

$$(*)\qquad \begin{array}{ccc} \bar{F}' & \xrightarrow{\ i\ } & Y' \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ T^N & \longrightarrow & B\mathscr{H} \end{array}$$

where f' and g' are projections of locally trivial fiber bundles with fiber homotopy equivalent to F, and where i has the homotopy type of the inclusion of $\bar{F}$ in Y.

$B\mathscr{H}$ is 1-connected because $Y' \simeq Y$ is 1-connected; hence the local systems formed by the homology groups of the fibers of the fibrations f' and g' are trivial. Hence the finiteness of the homology of $\bar{F}$ and T^N implies the finiteness of the homology of F (cf. [Ser]). The homotopy exact sequence of the fibration f' shows that $\pi_1(F)$ is a quotient of $\pi_2(B\mathscr{H})$, hence is abelian.

The E^2-term of the Serre's spectral sequence of the fibration g' is isomorphic to $H^p(T^N) \otimes H^q(F)$. As $\bar{F}$ is a closed orientable manifold of dimension equal to $\dim F + k$, it follows easily that the integral homology of F is isomorphic to $\mathbf{Z}$ in dimension equal to $\dim \bar{F} - N = \dim F + k - N$ and vanishes in higher dimensions. Hence $\dim F + k - N \geq 0$.

If F is simply connected, then F has the homotopy type of a finite complex (cf. [Wal]). It follows from results of D. Gottlieb [Got-1] that F, which is the homotopy theoretical fiber of the map g of F in T^N, has the homotopy type of a finite Poincaré complex.

Assume F contractible. Then f and g are homotopy equivalences (cf. [Hae-1]). Also the restriction of f to $f^{-1}(V)$, where V is an open set of $B\mathscr{H}$ saturated by leaves of $B\mathscr{F}$, is an homotopy equivalence on V. If the orbifold X is not a manifold, there is a T^N-invariant open set U of X which is the quotient of a contractible manifold $\tilde{U}$ by the action of a discrete group G acting in a properly discontinuous way on $\tilde{U}$ and having an element of torsion. Hence BU (constructed as in 3.3) is homotopy equivalent to the classifying space BG of G, hence has infinite cohomological dimension. As a model for $B\mathscr{H}$, we can choose BX as constructed in 3.3, and BU is identified to an open set of $B\mathscr{H}$ saturated by leaves. But BU has the same homotopy type as the open set $f^{-1}(BU)$ of Y which is of course finite dimensional, hence a contradiction.

6.2. *Remark.* D. Gottlieb [Got-2] has proved, using the existence of the diagram (*), that the generic leaf is always dominated by a finite Poincaré complex and that it has the homotopy type of a finite Poincaré complex if its fundamental group is infinite (or trivial).

6.3. *Remarks about the conjecture* 1.6. The conjecture is true in the following cases:

(a) All the leaves of $\mathcal{F}$ are compact. Then $\mathcal{F}$ is a generalized Seifert fibration. The space of leaves is naturally an orbifold X and the projection p of Y on X is a classifying map for the holonomy pseudogroup of $\mathcal{F}$, and has generic fiber the generic leaf of $\mathcal{F}$.

(b) The dimension of the leaves is one. Then $\mathcal{F}$ is an isometric flow (see [Mol-Ser]). If the generic orbit for the flow is compact, we are back to case (a). If not, we apply remark 3.6.

(c) $\dim Y \leq 4$. In that case, the only possibilities for a Riemannian foliation $\mathcal{F}$ on Y are generalized Seifert foliations or isometric flows on $Y = S^3$. Indeed, let X be the 1-connected orbifold with the action of the torus T^N associated to $\mathcal{F}$ and let W be the space of orbits. If we are not in the case of a Seifert fibration, then $0 < \dim W \leq \dim X - 2$. If $\dim W \leq 2$, the only possibilities (cf. [Hae-Sal]) are:

(i) $\dim W = 1$; then $X = S^3$ with an isometric flow.

(ii) $\dim W = 2$; then $\dim X \geq 5$.

Hence if $\dim Y = 3$, then $Y = S^3$; the case when $\dim Y = 4$ can not occur because Y would have the homotopy type of a circle bundle over S^3, which is impossible if Y is 1-connected.

Remark. If $\dim Y = 5$, the only case for which $\mathcal{F}$ is not a Seifert fibration or an isometric flow, is when $\mathcal{F}$ is the pull-back by a map $f\colon Y \to S^3$ of an isometric flow $\mathcal{F}_0$ on S^3. In that case, the conjecture would imply the existence of a fibration $p\colon Y \to S^3$ with fiber S^2 such that $\mathcal{F} = p^*(\mathcal{F}_0)$.

7. Maps transverse to a foliation

The proposition below is the generalization to pseudogroups of the following well known fact: if $p\colon T' \to T$ is an étale map of Riemannian manifolds which is locally an isometry, and if T' is complete and T connected, then p is a covering projection.

7.1. A pseudogroup $\mathcal{H}$ of local isometries of T is said to be *geodesically complete* if for any unit tangent vector ξ at a point x of T and any positive number a, there is an $\mathcal{H}$-geodesic of length a and initial vector ξ.

Such an *$\mathscr{H}$-geodesic* is given by a sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_{k+1} = a$, by geodesic arcs $c_i: [t_i, t_{i+1}] \to T$ parametrized by the arc length for $0 \leq i \leq k$ and by elements h_i of $\mathscr{H}$ defined at $c_i(t_{i+1})$ such that the image by the differential of h_i of the unit velocity vector $c_i(t_{i+1})$ is the velocity vector $\dot{c}_{i+1}(t_{i+1})$ for $i > 0$ and $\xi = \dot{c}_0(0)$. This notion is obviously invariant by differentiable equivalence of pseudogroups. As an example, the holonomy pseudogroup of a Riemannian foliation on a complete Riemannian manifold is geodesically complete.

7.2. For pseudogroups, there is a natural generalization of the notion of covering. Let $\mathscr{H}'$ and $\mathscr{H}$ be pseudogroups of local diffeomorphisms of manifolds T' and T; denote by $[\mathscr{H}']$ and $[\mathscr{H}]$ the topological groupoids of germs of elements of $\mathscr{H}$ and $\mathscr{H}'$; the spaces T' and T are identified with the subspaces of units in $[\mathscr{H}']$ and $[\mathscr{H}]$. A continuous surjective homomorphism $\varphi: [\mathscr{H}'] \to [\mathscr{H}]$ is a *covering homomorphism* if the restriction φ_0: $T' \to T$ of φ is a covering map and if the kernel of φ is T' (namely $T' = \varphi^{-1}(T)$). More generally, an étale morphism of pseudogroups (cf. [Hae-2] 1.4) is a covering if it is equivalent to a covering homomorphism in the above sense.

7.3. Proposition. *Let $\mathscr{H}$ and $\mathscr{H}'$ be complete pseudogroups of local isometries of T and T' respectively. Assume that $\mathscr{H}'$ is geodesically complete and that $\mathscr{H}$ is connected. Then any homomorphism φ: $[\mathscr{H}'] \to [\mathscr{H}]$ whose restriction φ_0: $T' \to T$ is locally an isometry is equivalent to a covering homomorphism. In particular if $\mathscr{H}$ is 1-connected and $\mathscr{H}'$ is connected, then φ_0 generates an equivalence.*

Proof. We recall that the definitions of connectedness and completeness for a pseudogroup are given respectively in 2.3 and 2.4. We first note that the completeness of a pseudogroup $\mathscr{H}$ implies the following: let c and c': $[0, a] \to T$ be $\mathscr{H}$-geodesic arcs parametrized by arc length. We suppose that there is an element of $\mathscr{H}$ defined locally around $c(0)$, mapping the velocity vector of c at 0 on the velocity vector of c' at 0. Then $c(a)$ and $c'(a)$ are in the same $\mathscr{H}$-orbit.

1st step. $\varphi_0(T')$ meets all the orbits of $\mathscr{H}$. Let T_0 be the open set of T which is the union of the $\mathscr{H}$-orbits which meet $\varphi_0(T')$ and let c: $[0, a] \to T$ be a geodesic arc with $c(0)$ in T_0. There is a point x' in T' and an element h of $\mathscr{H}$ such that $h(\varphi_0(x')) = c(0)$. Let c' be a $\mathscr{H}'$-geodesic arc of length a, whose initial vector is mapped on the tangent vector of c at 0 by the differential of $h \circ \varphi_0$. By the remark above, the extremity $c(a)$ of c is in the same orbit as the extremity of c'. This shows that T_0 is an $\mathscr{H}$-invariant open and closed set, hence is equal to T by the $\mathscr{H}$-connexity assumption.

2nd step. Up to equivalence, we can assume that $T = T'$ and φ_0 is the identity. Indeed, let $U' = \{U_i'\}_{i \in I}$ be an open covering of T' such that the restriction of φ_0 to each U_i' is an isometry on $\varphi_0(U_i') = U_i$. Let T_U be the disjoint union of the open sets U_i and denote by $\mathscr{H}_U$ the pseudogroup generated by the local isometries of T_U projecting on elements of $\mathscr{H}$ by the natural projection of T_U on T. We define similarly $\mathscr{H}'_{U'}$ and $T'_{U'}$. It is clear that the projection of T_U on T (respectively of $T'_{U'}$ on T') generates an equivalence of $\mathscr{H}_U$ on $\mathscr{H}$ (resp. of $\mathscr{H}'_{U'}$ on $\mathscr{H}'$). The map of $T'_{U'}$ on T_U whose restriction to each U_i' is φ_0 is a bijective isometry generating an injective homomorphism of $[\mathscr{H}'_{U'}]$ in $[\mathscr{H}_U]$.

Hence we can assume that $T = T'$ and that $\mathscr{H}'$ is a subpseudogroup of $\mathscr{H}$.

Third step. Let $\hat{T}$ be the quotient $[\mathscr{H}]/[\mathscr{H}']$; in particular the germs $[h_1]$ and $[h_2]$ of elements of $\mathscr{H}$ are equivalent if there is a germ $[h'] \in [\mathscr{H}']$ such that $[h_2] = [h_1]\circ[h']$. The target projection gives an étale map $p: \hat{T} \to T$. We assert that p is a covering map.

The completeness of $\mathscr{H}'$ implies that $\hat{T}$ is Hausdorff. So it is sufficient to check that each geodesic arc on T can be lifted to $\hat{T}$. If $c: [0, a] \to T$ is a geodesic arc, and $[h]$ is the germ of an element of $\mathscr{H}$ with target $c(0)$, there is a lifting $\hat{c}: [0, a] \to \hat{T}$ of c such that $\hat{c}(0)$ is the class of $[h]$ in $\hat{T}$. Consider a $\mathscr{H}'$-geodesic arc c' of length a, whose initial vector is mapped by the differential of h on the initial vector of c. The lifting $\hat{c}$ is the image in $\hat{T}$ of the $\mathscr{H}$-geodesic arc obtained by composing c' with h^{-1}.

To finish the proof, let us denote by $\hat{\mathscr{H}}$ the pseudogroup of local isometries of $\hat{T}$ given by the left action of $\mathscr{H}$ on $\hat{T}$. The natural map of $[\hat{\mathscr{H}}]$ on $[\mathscr{H}]$ induced by the projection p is a covering homomorphism. Moreover the projection p generates an equivalence of $\hat{\mathscr{H}}$ on $\mathscr{H}'$ (cf. [Hae-2] 2.5).

If $\mathscr{H}$ is 1-connected, any connected covering of $\mathscr{H}$ is an equivalence (cf. [Hae-2] 2.4), so $\mathscr{H}'$ is equivalent to $\mathscr{H}$.

7.4. COROLLARY. *Let $\mathscr{F}$ be a Riemannian foliation on a compact connected manifold Y and let f be a differentiable map of a compact connected manifold Y' in Y which is transverse to $\mathscr{F}$. Let $\mathscr{F}' = f^*(\mathscr{F})$ be the foliation on Y' which is the pull back by f of $\mathscr{F}$. Then the morphism of the holonomy pseudogroup of $\mathscr{F}'$ in the holonomy pseudogroup of $\mathscr{F}$ induced by f is equivalent to a covering homomorphism.*

Proof. We recall how the transverse map f induces a morphism of the holonomy pseudogroups. Assume that the foliation $\mathscr{F}$ on Y is given by submersions $f_i: V_i \to T_i$ like in 2.1 which are surjective and with connected fibers. Then $\mathscr{F}' = f^*(\mathscr{F})$ can be defined by the submersions $f_i \circ f: f^{-1}(V_i) \to T_i$. Those submersions in general are not surjective and do not have connected fibers, so to define the holonomy pseudogroup of $\mathscr{F}'$, we cover each $f^{-1}(V_i)$ by open sets V_i^k such that the restriction f_i^k of $f_i \circ f$ to V_i^k is a submersion with connected fibers on an open set T_i^k of T_i. The holonomy pseudogroup $\mathscr{H}'$ of $\mathscr{F}'$ can be defined using the submersions f_i^k as in 2.1, and it acts on the disjoint union T' of the T_i^k's. The obvious étale map of T' on T whose restriction to T_i^k is the inclusion in T_i generates the morphism of $\mathscr{H}'$ in $\mathscr{H}$.

As $\mathscr{F}'$ is a Riemannian foliation on a compact manifold Y', its holonomy pseudogroup is geodesically complete, hence the hypothesis of the proposition 7.3 are satisfied.

Proof of Theorem 1.5. If Y is 1-connected and Y' is connected, then the morphism induced by f on the holonomy pseudogroups is an equivalence. If the generic leaf of $\mathscr{F}$ is contractible, then Y is a classifying space for the holonomy pseudogroup of $\mathscr{F}$ and we can apply Theorem 6.1. The map f: $Y' \to Y$ is a classifying map for $\mathscr{F}'$, so $\dim Y' \geq \dim Y$. Moreover $\dim Y' = \dim Y$ iff f is an homotopy equivalence.

If $\dim \mathscr{F} = 1$ and if the generic fiber is contractible, then f is homotopic to a diffeomorphism by Remark 3.6. If the generic fiber is not contractible, then $\mathscr{F}$ is a circle Seifert fibration. Let X be the orbifold which is the space of leaves of $\mathscr{F}$. From the homotopy exact sequence of the Seifert fibration $Y \to X$ (cf. [Hae-1]), we see that the connecting homomorphism $\pi_2(X) \to \pi_1(S^1)$ is surjective.

The dimension of Y' cannot be equal to the dimension of X, because the composition of f with the projection on X would be a covering, hence an isomorphism. This would imply the existence of a section of the Seifert bundle $Y \to X$, contradicting the surjectivity of the connecting homomorphism.

If $\dim Y' = \dim Y$, then the foliation $\mathscr{F}' = f^*(\mathscr{F})$ on Y' is also a circle Seifert fibration and f maps fibers to fibers inducing an isomorphism of X' on X, where X' is the orbifold which is the space of leaves of $\mathscr{F}'$. The restriction of f to a generic fiber S^1 is a map of degree one, because we have the commutative diagram

$$\begin{array}{ccccc} \pi_2(X') & \longrightarrow & \pi_1(S^1) & \longrightarrow & 1 \\ \downarrow \wr\wr & & \downarrow & & \\ \pi_2(X) & \longrightarrow & \pi_1(S^1) & \longrightarrow & 1 \end{array}$$

The leaves of $\mathscr{F}$ and $\mathscr{F}'$ are the orbits of an effective action of SO_2; using an average process we can homotopy f to a smooth SO_2-equivariant diffeomorphism, by an homotopy along the leaves.

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