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Z_2-GRADED ALGEBRAS

BY

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1. Introduction

This paper was inspired by [2]. Recast into the language of Lie superalgebras, rather than Lie algebra square roots, one of Mackey's results reads as follows. Let L be a Lie superalgebra with even part H and odd part N. Assume that N is two-dimensional, that H is three-dimensional, and that $N^2 = H$. Then either $HN = 0$ or L is the unique five-dimensional simple Lie superalgebra (the first of the orthosymplectic series). In Theorem 1 I exhibit a generalization. Any field of characteristic $\neq 2$ is admissible in Theorem 1.

I stubbornly sought to fit in characteristic 2 as well. But in characteristic 2 the notions of Lie algebra and Lie superalgebra coincide. So it was natural to make a parallel study for Lie algebras. The result was Theorem 2 in which, however, characteristic 3 was an unexpected exception.

The final section of the paper contains a number of additional remarks.

2. Lie superalgebras

Note that in all algebras the operation is being written simply as multiplication.

THEOREM 1. *Let $L = H + N$ be a Lie superalgebra with even part H and odd part N. Infinite-dimensionality is permitted and the base field can be any field of characteristic $\neq 2$. Assume that the multiplication $N \times N \to H$ is the symmetric tensor product, i.e., is as free as possible. (The mapping is not assumed to be onto.) Then there exists a unique alternate form $(\ ,\ )$ on N such that*

$$xy.z = (y, z)x + (x, z)y \tag{1}$$

for all $x, y, z \in N$.

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Proof. The uniqueness of the form is obvious.

In the case of three odd elements the super-Jacobi-identity has the usual form $xy.z + yz.x + zx.y = 0$. For characteristic $\neq 3$ this implies $x^2x = 0$ for x odd. For characteristic 3 it is customary to add $x^2x = 0$ to the axioms for a Lie superalgebra.

We begin the proof by showing that x^2z is a scalar multiple of x for all x and z in N. We just noted that $x^2x = 0$ and so we may assume that x and z are linearly independent. Complete a basis $x, z, w, \dots$ of N. Write

$$x^2z = px + qz + rw + \cdots, z^2x = sx + tz + uw + \cdots \tag{2}$$

By the super-Jacobi-identity applied to the triple x^2, z, z we have $x^2z^2 = 2z.x^2z$. By interchanging x and z we get $z^2x^2 = 2x.z^2x$. Hence $z.x^2z = -x.z^2x$. From this and (2) we get

$$pzx + qz^2 + rzw + \cdots = -(sx^2 + txz + uxw + \cdots). \tag{3}$$

Our hypothesis asserts that all the products of two basis elements of N appearing in (3) are linearly independent. Hence all the coefficients except p and t vanish and furthermore $p = -t$. The claim that x^2z is a scalar multiple of x has thus been sustained.

We are ready to define the inner product on N. If x and z are linearly dependent we set $(x, z) = 0$. When they are linearly independent we set $(x, z) = p/2$ where p is the coefficient occurring in $x^2z = px$. The equality of p and $-t$ observed in the preceding paragraph shows that $(x, z) = -(z, x)$. Since x^2z is linear in z so is (x, z), and then $(x, z) = -(z, x)$ shows that it is likewise linear in x.

It remains to verify (1). This is done by linearization. In detail,

$$\begin{aligned} xy.z &= \left((x+y)^2 - x^2 - y^2\right)z/2 \\ &= (x+y, z)(x+y) - (x, z)x - (y, z)y \\ &= (y, z)x + (x, z)y. \end{aligned}$$

Remarks. (a) The super-Jacobi-identity shows that N^2 is closed under multiplication and that the product on N^2 is determined by the given data, i.e., from the form. Thus the algebra $N^2 + N$ is uniquely determined by the form; if $N^2 = H$, L is uniquely determined. The form in turn is characterized by an integer (its rank) when N is finite-dimensional and by two cardinal numbers when N has countable dimension. In the case of uncountable dimension there are myriads of alternate forms and therefore myriads of algebras.

(b) At one extreme the form is identically 0, from which $N^2N = 0 = N^2N^2$ follows. At the other extreme the form is nondegenerate and we get a member of the family of "extreme" orthosymplectic Lie superalgebras. (These algebras do not seem to have a name yet; the term "extreme" is meant to suggest that the algebra is almost entirely an ordinary symplectic Lie algebra. More precisely, in the usual representation by n by n matrices, the upper left $n-1$ by $n-1$ corner is a symplectic Lie algebra.) In Mackey's case where N is two-dimensional, one of the two extremes must hold.

(c) In the general case let R denote the radical of the form, i.e. all $x \in N$ with $(x, N) = 0$. From (1) we see that a product of three elements of N vanishes if two of the factors lie in R. Furthermore $RN^2 = 0$ and $RN.N \subset R$. By the super-Jacobi-identity

$$RN.RN \subset (RN.R)N + (RN.N)R \subset R^2,$$
$$RN.N^2 \subset (RN.N)N \subset RN.$$

Assume $H = N^2$. Let $I = R + RN$. We see that I is an ideal in L and $I^2 \subset R^2$, $I^3 = 0$. The quotient L/I is a Lie superalgebra of the type occurring in Theorem 1, and for it the attached alternate form is nondegenerate. In sum, very little is lost in assuming the form to be nondegenerate.

(d) Here are two ways of realizing the algebras under discussion.

(i) Construct the Weyl algebra of the form. This means that we take the free associative algebra on N and divide it by the relations $xy - yx = (x, y)1$ to get an algebra W. Since W carries a natural Z_2-grading there is a Lie superalgebra structure on W. Take the subalgebra generated by N.

(ii) Construct a super inner product space M by taking the orthogonal direct sum of N with a one-dimensional space carrying a nondegenerate symmetric form. The desired algebra is the set of all linear transformations on M which are skew relative to the inner product and have finite-dimensional range.

3. Z_2-graded ordinary Lie algebras

In Theorem 2 the case where N is one-dimensional is excluded because it is trivial and because the uniqueness statement fails in that case.

THEOREM 2. *Let $L = H + N$ be a Z_2-graded Lie algebra over a field of characteristic $\neq 3$. Assume that the multiplication $N \times N \to H$ is the exterior product and that the dimension of N exceeds 1. (As in Theorem 1, the mapping is not assumed to be onto.) Then there exists a unique symmetric form (,) such that*

$$xy.z = (y, z)x - (x, z)y \quad \text{for all } x, y, z \in N. \tag{4}$$

Proof. The uniqueness statement is again obvious.

The proof resembles the proof of Theorem 1 but there are differences in detail. A significant obstacle is that the definition of (x, x) depends on the use of an auxiliary element y, and it is necessary to show that the choice of y is irrelevant.

The first step is to prove that for x and y linearly independent in N, $xy.x$ is a linear combination of x and y. Complete x, y to a basis $x, y, z, \ldots$ of N and write

$$xy.x = px + qy + rz + \cdots, yx.y = sx + ty + uz + \cdots \tag{5}$$

The Jacobi identity applied to the triple x, y, xy yields $y(xy.x) = -x(yx.y)$. Apply this to (5). The result is

$$pyx + ryz + \cdots = -(txy + uxz + \cdots). \tag{6}$$

Hence all coefficients in (6) other than p and t vanish, and furthermore $p = t$. The coefficients q and s do not appear in (6) and there is no restriction on them.

We proceed to define the form. If either x or y is 0 we simply define (x, y) to be 0. If x and y are linearly independent, we define (x, y) to be the element p above (i.e., the coefficient of x in $xy.x$), and we note at once that $(x, y) = (y, x)$ since $p = t$. For $x \neq 0$ we choose y linearly independent of x and tentatively define (x, x) to be $-q$, where q is as above (i.e. the coefficient of y in $xy.x$). It is urgent to prove that this is independent of the choice of y.

Say we have

$$xy.x = px + qy, \tag{7}$$

$$xy'.x = p'x + q'y', \tag{8}$$

where x and y are linearly independent, and x and y' are also linearly independent. We have to prove that $q = q'$. There are two cases.

(I) Assume that x, y and y' are linearly independent. Then x and $y + y'$ are linearly independent and we write

$$x(y + y').x = p''x + q''(y + y'). \tag{9}$$

The right side of (9) equals the sum of the right sides of (7) and (8). By equating the coefficients of y we get $q = q''$; from the coefficients of y' we get $q' = q''$. Hence $q = q'$. From the coefficients of x we furthermore find $p'' = p + p'$, that is

$$(x, y + y') = (x, y) + (x, y'). \tag{10}$$

This information will be used below.

(II) The remaining case is where $y' = ax + by$. We have

$$(11)\quad xy'.x = x(ax + by).x = bxy.x = bpx + bqy$$
$$= bpx + q(y' - ax) = (bp - aq)x + qy'.$$

Again $q = q'$.

In a moment we shall need to know that $(rx, rx) = r^2(x, x)$ for any scalar r. This is immediate from a comparison of the expansions of $xy.x$ and $(rx)y.(rx) = r^2xy.x$.

To complete the definition of the inner product we still need the case of (y, z) where y and z are linearly dependent but unequal. We write $y = ax$, $z = bx$ and set $(y, z) = ab(x, x)$. If rx is used instead of x we reach the same result, for $y = ar^{-1}rx$, $z = br^{-1}rx$ and the competing definition of (y, z) is $abr^{-2}(rx, rx) = ab(x, x)$.

The form is now well defined and we proceed to the proof that it is bilinear. Homogeneity with respect to scalars is immediate (cf. the second last paragraph) and so it suffices to prove additivity. We know that the form is symmetric; therefore it suffices to prove additivity in just one of the variables, say the second one. That is, we are to prove (10). We may of course assume that x is nonzero. In I above we handled the case where x, y, and y' are linearly independent. At the other extreme, if y and y' are both scalar multiples of x the truth of (10) is immediate from the definition of the form. We may therefore assume that one of y and y', say y, is not a scalar multiple of x, and that the other is a linear combination of x and y: $y' = ax + by$. Further case distinctions are now needed.

(A) $b \neq 0, -1$. Equations (7), (8), (9), and (11) are available. We have $(x, y) = p$ and $(x, y') = p' = bp - aq$. Note that $y + y' = ax + (b + 1)y$ with $b + 1 \neq 0$. The computation (11) can be repeated with y' replaced by $y + y'$. and b replaced by $b + 1$, leading to

$$(x, y + y') = (b + 1)p - aq.$$

Thus (10) holds.

(B) $b = 0$. Then $y' = ax$, so that

$$(x, y') = (x, ax) = -aq.$$

Also $y + y' = ax + y$ and the computation (11) yields

$$(x, y + y') = p - aq.$$

Again (10) holds. (It is to be noted that this is the first time that the minus sign inserted in the definition of (x, x) plays a role. It is also needed for case C.)

(C) $b = -1$. We have $y' = ax - y$ and $(x, y') = -p - aq$, again from (11) with b set equal to -1. Next, $y + y' = ax$ so that

$$(x, y + y') = -aq.$$

This is the final verification of (10).

To summarize, we have established that $(\ , \ )$ is a symmetric bilinear form and we have

$$xy.x = (x, y)x - (x, x)y. \tag{12}$$

This is the case $z = x$ of (4) and our task now is to prove (4) in full. At the corresponding point in the proof of Theorem 1 it sufficed to linearize. Here linearization produces two terms that have to be disentangled by the Jacobi identity, a procedure that produces a factor 3 that makes characteristic 3 an exception. From (12) we get

$$(x + z)y.(x + z) = (x + z, y)(x + z) - (x + z, x + z)y. \tag{13}$$

The terms $xy.x$ and $zy.z$ on the left side of (13), when expanded by (12), cancel with the corresponding portions on the right side. The result is

$$xy.z + zy.x = (x, y)z + (z, y)x - 2(x, z)y. \tag{14}$$

Permute cyclically:

$$zx.y + yx.z = (z, x)y + (y, x)z - 2(z, y)x. \tag{15}$$

Subtract (15) from (14). The terms $zy.x - zx.y$ on the left can, by the Jacobi identity, be replaced by $xy.z$. This places $3xy.z$ on the left. On the right we find

$$3(y, z)x - 3(x, z)y.$$

In short we have precisely (4), multiplied by 3. This completes the proof of Theorem 2.

With appropriate changes, the remarks after Theorem 1 are applicable. The Weyl algebra of the alternate form on N gets replaced by the Clifford algebra of the symmetric form on N, and L is the algebra of linear transformations skew with respect to the form enlarged by one dimension. See page 231 of [1] for related material.

4. Concluding remarks

(a) *A generalization.* Here is a small generalization of Theorems 1 and 2. In Theorem 1, after it is found that x^2z is always a scalar multiple of x, we can take this as the hypothesis and reach the conclusion of Theorem 1. The only change needed in the proof is that one checks that the inner product is alternate by using $z.x^2z = -x.z^2x$ once more. Similarly, for Theorem 2 we can take as our hypothesis the statement that $xy.x$ is a linear combination of x and y, citing $y(xy.x) = -x(yx.y)$ to get the inner product to be symmetric.

(b) *Characteristic* 3. I have partially determined the facts for characteristic 3; I shall state these results without proof. If the dimension of n is five or more, Theorem 2 remains valid. For three-dimensional N, I have determined all the possibilities; none of the algebras is simple. For four-dimensional N I shall only display one example. In this example the inner product is identically 0; products (such as $xy.x$) with a repeated factor vanish; products such as $xy.z$ are invariant under cyclic permutation. Thus the following equations suffice to define an algebra:

$$yz.t = x, \quad zt.x = -y, \quad tx.y = z, \quad xy.z = -t.$$

This algebra is simple. It is 10-dimensional (the odd part is 4-dimensional and the even part 6-dimensional). I do not know whether it is new. It is possible that many simple 10-dimensional Lie algebras of characteristic 3 remain to be discovered in this way.

(c) *Other classes of algebras.* The problem being studied can be repeated for any class of algebras. I shall describe (without proof) what happens for associative algebras and Jordan algebras.

In the associative case, if we put aside the trivial case where N is one-dimensional, there is total collapse: $N^3 = 0$. The same thing happens for commutative associative algebras.

For (ordinary) Jordan algebras there is a perfect analogue of Theorem 2: Let $J = H + N$ be a Z_2-graded Jordan algebra over a field of characteristic $\neq 2$. Assume that the multiplication $N \times N \to H$ is the symmetric tensor product. Then there exists a unique symmetric form $(\ , \ )$ on N such that

$$xy.z = (y, z)x + (x, z)y + 2(x, y)z$$

for all $x, y, z \in N$.

When we turn to Jordan superalgebras we get a surprise. The expected result is

$$xy.z = (y, z)x - (x, z)y + 2(x, y)z$$

with the form alternate. In particular, this would mean that

$$xy.x = (x, y)x, \quad xy.y = (x, y)y,$$

and, for N two-dimensional, xy would act on N as a scalar. But here is a counterexample, in concrete matrix style. With the usual matrix units, take $A = e_{11} + ie_{44}$ (where $i^2 = -1$), $B = e_{12} + e_{34}$. Then

$$A(AB - BA) + (AB - BA)A = B, \quad B(AB - BA) + (AB - BA)B = 0.$$

We have a three-dimensional Jordan superalgebra with A and B spanning the odd part and $AB - BA$ spanning the even part; $AB - BA$ does not act as a scalar.

References

1. N. Jacobson, *Lie algebras*, Interscience, 1962.
2. G.W. Mackey, *Some remarks on Lie superalgebras*, Czech J. Physics B, vol. 37 (1987), pp. 373–386.

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