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OUTER AUTOMORPHISMS OF GROUPS

BY

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1. Introduction

In this paper we concern ourselves with the group of outer automorphisms of groups. It is obvious that if G is a group embedded in another group H, then any inner automorphism of G lifts to an (inner) automorphism of H. Angus Macintyre asked whether this extension property actually characterizes inner automorphisms. The answer is affirmative as recently shown by Schupp [24]. He actually proved the following result which is stronger than the one stated in [24].

THEOREM [24]. *If G is a group then there exists an extension H of G with* $\operatorname{Out} H = 1$ *(i.e., H is complete) and* $\mathfrak{n}_H(G) = G$.

$\operatorname{Out} G$ denotes the factor group $\operatorname{Aut} G/\operatorname{Inn} G$ where $\operatorname{Inn} G$ is the normal subgroup of all inner automorphisms of G.

This theorem generalizes an earlier countable version due to Miller, Schupp [17]. We want to put Macintyre's question into a more general setting which will lead quite naturally to new questions. For instance, is every group the outer automorphism group $\operatorname{Out} G$ of some group G? A positive answer to the latter will be part of our main theorem. This is in contrast with results due to Robinson [22], [23] who showed that *not* every (finite) group is the automorphism group of some group. For instance, A_n, $n \neq 2, 8$, is not isomorphic to an automorphism group. In Section 3 we will provide a completely different proof of Schupp's theorem, which has the following three advantages. First of all, it gives an answer to the extension of Macintyre's question. Secondly, we use only very elementary group theory without "elements of order 160 or 81". The small cancellation theory used in [24] clearly has developed into a beautiful theory over the last 30 years [16], [25], however, some of its combinatorial details require technical computations. Therefore it might be desirable to have a "pure" group theoretic proof. Finally, our construction is close to abelian groups. This allows our extension

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group H to be locally soluble provided G is. The extensions H in [17] and [24] are free products with amalgamation. Such H's are rarely in any "reasonable" class even if G is well behaved. Another aim of this paper is to export ideas like Ines(G) (cf. [2]) and results on rigid systems [2], [5] from the theory of *abelian* groups. We hope that this will motivate non-abelian group theorists to occasionally study some of the work done by their poor commutative relatives.

Let us begin with the extension of Macintyre's question. If G is an object in some class $\mathfrak{C}$ in a category, then an endomorphism α of G is inessential if α extends to any extension H of G with $H \in \mathfrak{C}$. Let $\mathrm{Ines}(G) = \mathrm{Ines}_{\mathfrak{C}}(G)$ be the set of all inessential endomorphisms, see [2], [6], [8]. The following problems are guidelines for many investigations:

(a) Determine Ines(G) in End(G) for natural classes $\mathfrak{C}$!
(b) Prescribe End(G) modulo Ines(G) in $\mathfrak{C}$ (as a split extension)!

The following results illustrate (a) and (b) and also show that Ines(G) is known in these special cases. If $\mathfrak{C}$ is the class of cotorsion-free abelian groups, then $\mathrm{Ines}(G) = 0$ and End(G) can be prescribed, see [2], [5], [6] and [1] for the countable case. If $\mathfrak{C}$ is the class of abelian p-groups, then Ines(G) is Pierce's ideal of small endomorphisms and $\mathrm{End}(G)/\mathrm{Ines}(G)$ can be prescribed, see [2], [4] and the literature quoted in [2]. If $\mathfrak{C}$ is the class of all torsion-free separable abelian groups, then Ines(G) is the ideal of endomorphisms with finitely generated image and (b) is answered in [7]. If $\mathfrak{C}$ is the class of fields of characteristic 0, then $\mathrm{Ines}(G) = 0$ and extension fields with prescribed endomorphism monoid can be constructed; cf. [8]. In the case of fields of characteristic $p > 0$, Ines(G) are the Frobenius homomorphisms, and extension fields G with prescribed monoid mod Ines(G) can be constructed. Finally, if $\mathfrak{C}$ is the class of *all* groups, then $\mathrm{Ines}(G) = \mathrm{Inn}(G)$ by Schupp's result [24].

How about other classes of groups and (b)? Before we give an answer to this we want to recall some well-known notations due to P. Hall [10]. If $\mathfrak{C}$ is a class of groups, then $L\mathfrak{C}$ denotes the class of all local $\mathfrak{C}$-groups, i.e. $G \in L\mathfrak{C}$ if and only if every finitely generated subgroup of G is in $\mathfrak{C}$. Moreover, $G \in E_s\mathfrak{C}$ if G is a split extension (= semidirect product) of a $\mathfrak{C}$-group by a $\mathfrak{C}$-group and $\{E_s, L\}\mathfrak{C}$ is the smallest class containing $\mathfrak{C}$ closed under E_s and L. Observe that the class of locally soluble groups is closed under E_s and L, see [21]. Our main result is the following.

THEOREM. *Let κ be a cardinal and H, B groups of cardinality $\leq \kappa$. If $\kappa^{\aleph_0} = \kappa$, then there exists a group G of cardinality κ^+ such that*:
(1) $B \subset G$, $\mathfrak{n}_G(B) = B$.
(2) $\mathrm{Aut}\, G = \mathrm{Inn}\, G \rtimes H$ *and* $H \restriction B = 1$.
Moreover $G \in \{E_s, L\}\{H, B, \mathbf{Z}\}$. ($\kappa^+$ *denotes the least cardinal larger than* κ *and* $H \restriction B$ *is the restriction of* H *to* B).

Observe that G is locally soluble (torsion-free) if H and B are locally soluble (torsion-free) and that $\operatorname{Out} G$ can be prescribed arbitrarily. If G is abelian, then the class of *finite* groups of the form $\operatorname{Aut} G = \operatorname{Out} G$ is very restricted. This follows from the work of Hirsch as corrected by Corner; cf. [5] and the literature quoted there. The construction in the proof of our theorem uses two kinds of building blocks. From abelian group theory we use the existence of rigid families $\{A_\nu | \nu < \kappa^+\}$ of abelian groups of cardinality κ; i.e., $\operatorname{Hom}(A_\nu, A_\mu) = 0$ if $\nu \neq \mu$ and $\operatorname{End}(A_\nu) = \mathbf{Z}$. The existence of such families below $2^{\aleph_0}$ follows from [1] or [9] and above $2^{\aleph_0}$ from [2], [5] or [6]. The non-abelian framework is provided by wreath products. Their use is suggested by the existence of characteristic subgroups [19], [12], [10] and the Kaloujnine-Krasner theorem [15], see [18] or [25]. (Observe that cardinal numbers are ordinals; cf. [14, p. 24]. Hence cardinals are sets, cf. [14, p. 15] and we can enumerate groups by ordinals less than a given cardinal. This is standard in set theory or model theory and we will use this convenient notation through this paper.)

2. Properties of P-adic wreath products

Products and sums

If $(A, \cdot)$ is a group and B is a set, then $\prod_{x \in B} Ae_x$ denotes the (unrestricted) cartesian product of $|B|$ copies of A, i.e. if $f \in \prod_{x \in B} Ae_x$, then

$$f = \sum_{x \in B} a_x e_x = (a_x e_x)_{x \in B} \quad \text{with } f(x) = a_x \in A \text{ for all } x \in B.$$

(We want to use Σ in the representation of f. This makes some formulas easier to read and in our setting A will be abelian.) Hence f is a vector with entry a_x at the x-coordinate. Moreover

$$\bigoplus_{x \in B} Ae_x \subseteq \prod_{x \in B} Ae_x$$

denotes the (restricted) direct sum, consisting of all

$$f \in \prod_{x \in B} Ae_x \quad \text{with } f(x) = 1 \text{ for almost all } x \in B,$$

i.e.,

$$f = \sum_{x \in B} a_x e_x \quad \text{and } a_x = 1 \text{ for almost all } x \in B.$$

Then clearly $f \cdot f' = \sum_{x \in B} a_x a'_x e_x$ with f as above and $f' = \sum_{x \in B} a'_x e_x$ makes $\prod_{x \in B} Ae_x$ and $\bigoplus_{x \in B} Ae_x$ into groups.

Wreath products

If B is a group, then $B \subseteq \mathrm{Aut}(\prod_{x \in B} Ae_x)$ via

$$f^b = \sum_{x \in B} a_x e_{xb} = \sum_{x \in B} a_{xb^{-1}} e_x .$$

Clearly, the group generated by $\prod_{x \in B} Ae_x$ and B is a semidirect product $\prod_{x \in B} Ae_x \rtimes B$ of the normal subgroup $\prod_{x \in B} Ae_x$ and B, and this group is called the unrestricted wreath product $A \,\bar{\wr}\, B$ of the basis A (or $\prod_{x \in B} Ae_x$) and B. Obviously $\bigoplus_{x \in B} Ae_x$ is B-invariant and $B \subseteq \mathrm{Aut}(\bigoplus_{x \in B} Ae_x)$ acts faithfully on the direct sum. The group generated by $\bigoplus_{x \in B} Ae_x$ and B is the semidirect product of the normal subgroup $\bigoplus_{x \in B} Ae_x$ and B, which is called the (restricted) wreath product $A \wr B$ with basis A (or $\bigoplus_{x \in B} Ae_x$) and B; see H. Neumann [18] or Serret [26].

***P*-adic (Z-adic) completions**

If $(A, +)$ is an abelian group with $\bigcap_{n \in \omega} p^n A = 0$ and p a prime (or $\bigcap_{n < \omega} n!A = 0$), then $\{p^n A \colon n \in \omega\}$ ($\{n!A \colon n \in \omega\}$) used as a basis of neighborhoods of $O \in A$ defines a Hausdorff topology p-top ($\mathbf{Z}$-top) on A so that addition is continuous. Moreover A can be embedded as a pure subgroup in its completion

$$\hat{A} = (A, p\text{-top})^{\wedge} \quad (\hat{A} = (A, \mathbf{Z}\text{-top})^{\wedge}),$$

i.e.,

$$p^n A = A \cap p^n \hat{A} \quad (n!A = A \cap n!\hat{A}).$$

We will say A is p-reduced whenever $\bigcap_{n < \omega} p^n A = 0$. We will always assume that A's in this paper are abelian, p-reduced, and not elementary abelian 2-groups, i.e., not vector spaces over $\mathbf{Z}/2\mathbf{Z}$. Under these conditions $A \subset \hat{A}$ where $\hat{A} = (A, p\text{-top})^{\wedge}$ is the p-adic completion and $\hat{A}$ is abelian and p-reduced.

P-adic wreath products

Let A be a p-reduced abelian group and B a group. Then $\overline{\bigoplus_{x\in B} Ae_x}$ is a B-invariant subgroup of $\prod_{x\in B} \hat{A}e_x$ and B acts faithfully on $\overline{\bigoplus_{x\in B} Ae_x}$. The group generated by $\overline{\bigoplus_{x\in B} Ae_x}$ and B in $\hat{A}\,\overline{\wr}\,B$ is a semidirect product of the normal subgroup $\overline{\bigoplus_{x\in B} Ae_x}$ and B and this group will be called the p-adic wreath product $A \,\hat{\wr}\, B$. Clearly

$$A \wr B \subseteq A \,\hat{\wr}\, B \subset \hat{A}\,\overline{\wr}\,B.$$

We will mainly work between $A \wr B$ and $A \,\hat{\wr}\, B$.

Properties of p-adic wreath products

Let A and B be as above. Then elements in the p-adic wreath product have the crucial property:

(2.0) if $1 \neq f \in \overline{\bigoplus_{x\in B} Ae_x} \subseteq A \,\hat{\wr}\, B$, then $\mathfrak{c}_B(f)$ is finite, where $\mathfrak{c}_B(f)$ denotes the centralizer of f in B.

This observation is established as follows.

If $f = \sum_{x\in B} f_x e_x$ and $n \in \omega$, then $h_p^{\hat{A}}(f_x) \geq n$ for almost all $x \in B$. Here $h_p^{\hat{A}}(a)$ denotes the p-height of $a \in \hat{A}$ (= maximum p-power which divides a) in $\hat{A}$. Suppose $C = \mathfrak{c}_B(f)$ is infinite; then f_x is constant on all left cosets of C. Thus each entry in f gets repeated infinitely often. Hence $f_x = 1$ for all $x \in B$, i.e., $f = 1$. □

LEMMA 2.1. *Let A be abelian, not an elementary abelian 2-group and let B be any infinite group. If F is a B-invariant group with*

$$\bigoplus_{x\in B} Ae_x \subseteq F \subseteq \overline{\bigoplus_{x\in B} Ae_x},$$

then F is the largest abelian normal subgroup of $W = F \rtimes B$.

Proof. Let N be an abelian normal subgroup of W and $hb \in N$ with $h \in F$ and $b \in B$. If $g \in F$, then

$$g^{-1}hbg = g^{-1}hg^{b^{-1}}b \in N.$$

Since N is abelian, we have

$$\left(g^{-1}hg^{b^{-1}}b\right)(hb) = g^{-1}hg^{b^{-1}}h^{b^{-1}}b^2 = (hb)\left(g^{-1}hg^{b^{-1}}b\right) = hg^{-b^{-1}}h^{b^{-1}}g^{b^{-2}}b^2$$

and since F is abelian we conclude

$$g^{-1}g^{b^{-1}} = g^{-b^{-1}}g^{b^{-2}}.$$

Let $1 \neq a$ be an element in A not of order 2 and $g = ae_1$. Then

$$a^{-1}e_1 + ae_{b^{-1}} = a^{-1}e_{b^{-1}} = ae_{b^{-2}}$$

and

$$a^{-1}e_1 + a^2e_{b^{-1}} = ae_{b^{-2}}.$$

This implies $b = 1$, since $a^2 \neq 1$. Thus $h \cdot b = h \in F$ and $N \subseteq F$. Thus F is the largest abelian normal subgroup of W.

COROLLARY 2.2. *If A, B, F are as in* (2.1), *then F is a characteristic subgroup of $W = F \rtimes B$.*

We will also need some well-known elementary facts (2.3) on wreath products; see e.g., C. H. Houghton [12] or H. Neumann [18]. If f is a map and X a subset of the domain of f, then $f \restriction X$ denotes the restriction of f to X. The following remarks are easy to prove.

REMARK 2.3. *Let $\alpha \in \operatorname{Aut} A$ and $\beta \in \operatorname{Aut} B$.*
(a) *There exists $\alpha^* \in \operatorname{Aut}(A \,\overline{\wr}\, B)$ such that $\alpha^* \restriction B = \mathrm{id}$ and*

$$\left(\sum_{x \in B} a_x e_x\right)^{\alpha^*} = \sum_{x \in B} a_x^{\alpha} e_x.$$

(b) *There exists $\beta^* \in \operatorname{Aut}(A \,\overline{\wr}\, B)$ such that $\beta^* \restriction B = \beta$ and*

$$\left(\sum_{x \in B} a_x e_x\right)^{\beta^*} = \sum_{x \in B} a_x e_{x^{\beta}}.$$

We adapt the following convention: α^*, β^*, etc. will always denote the extended automorphisms as in (2.3) and also their restriction to suitable subgroups of $A \,\overline{\wr}\, B$.

LEMMA 2.4. *Let H be a subgroup of* Aut B *and N an infinite subgroup of B satisfying*:

(a) *If $h \in H$ and $h \upharpoonright N$ is conjugation by an element in B, then $h = 1$.*

If $W = F \rtimes B \subseteq A \,\hat{\wr}\, B$ (as in (2.1)) and $H^* = \{h^*: h \in H\} \subseteq$ Aut W *(as in (2.3)), then*:

(a*) *If $h^* \in H^*$ and $h^* \upharpoonright N$ is conjugation by an element in W, then $h = 1$.*

Proof. Let $x \in N$, $fb \in W$ with $f \in F$ and $b \in B$, and suppose $h^* \in H^*$ is inner on N, that is $x^{h^*} = x^{fb}$ for all $x \in N$. Then

$$x^h = x^{h^*} = x^{fb} = b^{-1}f^{-1}xfb = f^{-b}b^{-1}xfb = f^{-b}f^{x^{-1}b}b^{-1}xb$$

is in B. Thus $f^b = f^{x^{-1}b}$ and $f = f^{x^{-1}}$ for all $x \in N$. Since N is infinite and $f \in F$ by (2.0) we conclude $f = 1$. Therefore $x^h = x^b$ for all $x \in N$. Hence $h = 1$ by hypothesis (a).

LEMMA 2.5. *Let $H = \mathbf{Z} \wr B$ with B infinite. Then*:

(a) *There exists a countable subgroup $Y \subseteq H$ such that $\mathfrak{c}_H(Y^*) = 1$ for all subgroups Y^* of finite index in Y*;

(b) $\mathfrak{n}_H(B) = B$.

Here $\mathfrak{n}_H$(B) *denotes the normalizer of* B *in* H.

Proof. (a) Let F be the basic subgroup of $\mathbf{Z} \wr B$ and let Y_0 be any infinite, but countable subgroup of B. Set

$$Y = \bigoplus_{x \in Y_0} \mathbf{Z}e_x \rtimes Y_0 \subseteq \mathbf{Z} \wr B$$

and choose any $Y^* \subseteq Y$ of finite index. If $fb \in \mathfrak{c}_H(Y^*)$ with $f \in F$ and $b \in B$, then there exists $0 \neq a \in \mathbf{Z}$, $ae_1 \in Y^*$ and $ae_1 = ae_1^{fb} = ae_1^b = ae_b$, hence $b = 1$. Thus

$$f \in \mathfrak{c}_H(Y_0^*) \quad \text{with } Y_0^* = Y_0 \cap Y^* \text{ and } Y_0^* \subseteq \mathfrak{c}_B(f).$$

Since Y_0^* is infinite, $f = 1$ follows from (2.0). Therefore $\mathfrak{c}_H(Y^*) = 1$.

(b) Let $h \in B$ and $f \in F$ with $hf \in \mathfrak{n}_H(B)$. Then $B^{hf} = B^f$ and $f \in \mathfrak{n}_F(B)$. If $x \in B$, then $f^{-1}xf = f^{-1}f^{x^{-1}}x$ and $f = f^{x^{-1}}$ for all $x \in B$. Since B is infinite, we have $f = 1$ from (2.0) and $\mathfrak{n}_H(B) = B$ follows.

LEMMA 2.6. *If B is a finite group, there is a canonical embedding*

$$B \subset B' = (\mathbf{Z} \wr B)/C$$

with

$$C = \left\{ f \in \bigoplus_{x \in B} \mathbf{Z} e_x : f_1 = f_x \text{ for all } x \in B \right\} = \mathfrak{z} B'$$

and $\mathfrak{n}_{B'}(B) = B$. *Here* $\mathfrak{z} B'$ *denotes the center of* B'.

Proof. Clearly $B \cap C = 1$ and $B \subseteq B'$ is obvious. We have to show that

$$\mathfrak{n}_W(C \cdot B) = C \cdot B \quad \text{for } W = \mathbf{Z} \wr B.$$

Let $ft \in \mathfrak{n}_W(C \cdot B)$ with $f \in F = \bigoplus_{x \in B} \mathbf{Z} e_x$, $t \in B$ and $cb \in C \cdot B$. Then

$$(cb)^{ft} = t^{-1} f^{-1} cbft = f^{-t} t^{-1} cbft = f^{-1} c^t (t^{-1} b) ft = f^{-1} c^t f^{b^{-1}t} \cdot t^{-1} bt$$

is in CB. Hence $f^{-t} f^{b^{-1}t} \in C$ and $f^{-1} f^{b^{-1}} \in C$ for all $b \in B$. We get for the components of f that

$$-f_x + f_{xb^{-1}} = d(b) \in \mathbf{Z}$$

for some $d(b)$ and all $x \in B$.

Let $x_1 \in B$ with f_{x_1} minimal among the components and let $x_2 \in B$ with f_{x_2} maximal. Then

$$-f_{x_1} + f_{x_1 b^{-1}} = d(b) \geq 0 \quad \text{and} \quad -f_{x_2} + f_{x_2 b^{-1}} = d(b) \leq 0,$$

hence $d(b) = 0$ so that $f = f^b$ for all $b \in B$, i.e. $f \in \mathfrak{z} B' = C$. □

In order to control many automorphisms simultaneously, we will need the following centralizer condition $\mathfrak{c}$.

DEFINITION 2.7. We say that a group B is in $\mathfrak{c}$ if and only if there exists a countable subgroup $B' \subseteq B$ such that $\mathfrak{c}_B(B^*) = 1$ for all subgroups B^* of finite index in B'.

LEMMA 2.8. *Let B be an infinite group and let H be in $\mathfrak{c}$. Then we can find a countable subgroup $Y \subseteq H \wr B$ such that the following holds:*
If $h \in H$ and $h^* \restriction Y^*$ is inner for some $Y^* \subseteq Y$ of finite index, then $h = 1$.

Proof. Since $H \in \mathfrak{c}$ we can find a countable subgroup $H' \subseteq H$ such that

$(*)$ $\quad \mathfrak{c}_H(H^*) = 1$ for all subgroups H^* of finite index in H'.

Let $B' \subseteq B$ be infinite but countable and set $Y = H' \wr B'$, which is a countable subgroup of $H \wr B$. If Y^* is a subgroup of finite index in Y, then $Y' = Y^* \cap H'e_1$ has finite index in $H'e_1$. Hence

$$(**) \qquad \mathfrak{c}_H(Y') = 1 \quad \text{from } (*).$$

Suppose $h \restriction Y^*$ is a conjugation with $db \in H \wr B$, where $d \in F = \bigoplus_{x \in B} He_x$ and $b \in B$.

Since H' is infinite and Y' has finite index in $H'e_1$, there exists

$$1 \neq a \in H' \quad \text{with } ae_1 \in Y' \subseteq Y^*.$$

From the definition of h^* we derive

$$(ae_1)^{db} = a^d e_b = (ae_1)^{h^*} = a^h e_1.$$

Hence $b = 1$ and $h^* \restriction B = \text{id}$ implies $x = x^{h^*} = x^d$ for all $x \in B$. We obtain $d \in \mathfrak{c}_F(Y')$. On the other hand $\mathfrak{c}_F(Y') = 1$ since Y' is infinite. Thus $d = 1$, hence $h^* \restriction Y^* = 1$; i.e.,

$$h \in \mathfrak{c}_H(Y' \cap H'e_1) = \mathfrak{c}_H(Y') = 1$$

by (**), and $h = 1$. □

LEMMA 2.9. *If H is any group, then $H' = \mathbf{Z} \wr (\mathbf{Z} \wr H) \in \mathfrak{c}$ and $\mathfrak{n}_{H'}(H) = H$.*

Proof. Because of Lemma 2.5 we only have to show that $\mathbf{Z} \wr H \in \mathfrak{c}$ for any infinite H. Let Y' be any countable, infinite subgroup of H and consider $Y = \mathbf{Z} \wr Y'$. If Y^* has finite index in Y, then $\mathfrak{c}_{\mathbf{Z} \wr H}(Y^*) = 1$. □

Next we will use a non-abelian version of P. Hill's [11] favored "axiom-3-families" for totally projective p-groups.

DEFINITION 2.10. Let G be a group and $\mathfrak{C}$ a class of groups. A family $\mathscr{F}$ of at most countable subgroups of G is called a (countable) *$\mathfrak{C}$-cover* of G if:

(i) $\mathscr{F} \subseteq \mathfrak{C}$;

(ii) If $F_i \in \mathscr{F}$ $(i \in \omega)$ is an ascending chain in G, then $\bigcup_{i \in \omega} F_i \in \mathscr{F}$;

(iii) If X is a countable subset of G, then there exists $F \in \mathscr{F}$ with $X \subseteq F$.

LEMMA 2.11. *Let A be an abelian group and G be an infinite subgroup of B. Then $\mathfrak{c}_B(G) = \mathfrak{c}_{A \wr B}(G)$.*

Proof. Suppose $g \in G$ and $h \in \widehat{\bigoplus_{x \in B} Ae_x}$, $b \in B$ with $hb \in \mathfrak{c}_{A \wr B}(G)$. Then

$$g = g^{hb} = (gh^{-g}h)^b = g^b(h^{-g}h)^b.$$

Hence $g = g^b$ and $h^{-g}h = 1$. This holds for all $g \in G$ and G is infinite. Hence $b \in \mathfrak{c}_B(G)$ and $h^g = h$ for all $g \in G$ implies $h = 1$ by (2.0). We derive $\mathfrak{c}_{A \hat{\wr} B}(G) \subseteq \mathfrak{c}_B(G)$ and (2.11) holds. □

All our set theoretic notations are standard and may be found in [14]. As usual we identify an ordinal λ with the set of all ordinals less than λ, i.e., $\lambda = \{\alpha \in On | \alpha < \lambda\}$. If $\aleph$ is a cardinal, then we identify $\aleph$ with the least ordinal κ such that $|\kappa| = \aleph$. If λ is an ordinal and G_α, $\alpha < \lambda$, are sets, we call $\{G_\alpha | \alpha < \lambda\}$ an ascending continuous chain if

(a) $G_\alpha \subseteq G_\beta$ for all $\alpha \leq \beta < \lambda$,
(b) $G_\mu = \bigcup_{\alpha < \mu} G_\alpha$ for any limit ordinal $\mu < \lambda$.

Some authors call chains like this smooth or complete. For the definition of the cofinality, cf(κ), of a cardinal κ we refer to [14, p. 26]. Note that κ is called regular if cf(κ) = κ.

If κ is a cardinal, then κ^+ denotes its successor cardinal, i.e.,

$$\langle \kappa^+ = \{\alpha \in O_n : |\alpha| \leq \kappa\}.$$

Note that κ^+ is regular [14, p. 27], i.e., cf(κ^+) = κ^+.

LEMMA 2.12. *Suppose $G = \bigcup_{\alpha < \kappa^+} G_\alpha$ is the union of an ascending, continuous chain of subgroups G_α with $|G_\alpha| < \kappa^+$ and κ is an infinite cardinal. Suppose $G_\alpha \subseteq G_{\alpha+1}$ such that*

$$A_\alpha \wr G_\alpha \subseteq G_{\alpha+1} = F_\alpha \rtimes G_\alpha \subseteq A_\alpha \hat{\wr} G_\alpha$$

for some p-reduced abelian group $A_\alpha \neq 1$.

Then G has a $\mathfrak{c}$-cover $\mathscr{F}$ and $\mathfrak{c}_G(F^*) = 1$ for all $F^* \subseteq F \in \mathscr{F}$ with $[F : F^*]$ finite.

Proof. Let $\mathscr{F}$ be the set of all countable $\mathfrak{c}$-subgroups of G. Since $\mathfrak{c}$ is closed with respect to unions of countable chains, we only have to verify (2.10) (iii) for $\mathscr{F}$ to be a $\mathfrak{c}$-cover.

Let $X \subseteq G$ be a countable subgroup. Since κ^+ is regular and uncountable, cf(κ^+) = $\kappa^+ > \omega$. We can find an ordinal $\alpha < \kappa^+$ with $x \subseteq G_\alpha$. We may assume that X is infinite. Pick any infinite but countable subgroup A of A_α and set $Y = A \wr X \subseteq G_{\alpha+1}$. We claim that $Y \in \mathfrak{c}$. Suppose Y^* is a subgroup of finite index in Y and

$$ax \in \mathfrak{c}_{G_{\alpha+1}}(Y^*) \quad \text{with } a \in \widehat{\bigoplus_{x \in B} A_\alpha e_x} \quad \text{and} \quad x \in G_\alpha .$$

We can find some $1 \neq be_1 \in Y^*$ and $be_1 = be_1^{ax} = b^a e_x$, hence $x = 1$ and $a^{Y^* \cap X} = a$ where $Y^* \cap X$ is infinite. We conclude $a = 1$ from (2.0) and $\mathfrak{c}_{G_{\alpha+1}}(Y^*) = 1$. In particular $\mathfrak{c}_Y(Y^*) = 1$ and $Y \in \mathscr{F}$.

However we get more as stated in (2.12). In order to derive $\mathfrak{c}_G(Y^*) = 1$ we apply induction on $\beta < \kappa^+$. If $\mathfrak{c}_{G_\nu}(Y^*) = 1$ for all $\nu < \beta$ and β is a limit, then clearly $\mathfrak{c}_{G_\beta}(Y^*) = 1$ by continuity. If β is discrete, we apply (2.11) to derive $\mathfrak{c}_{G_\beta}(Y^*) = \mathfrak{c}_{G_{\beta-1}}(Y^*) = 1$. □

Recall that a subset X of Y is *cofinite* in Y if the complement of X in Y, i.e., $Y - X$, is finite. If A is a subgroup of B, $[B : A]$ denotes the index of A in B.

LEMMA 2.13. *Let L be an infinite group and L_i, $1 \leq i \leq n$ subgroups of L. Assume there are elements $a_i \in L$ such that $\bigcup_{i=1}^{n} a_i L_i$ is cofinite in L. Then there is an i, $1 \leq i \leq n$, with $[L : L_i] < \infty$.*

Proof. Induct on m, the number of distinct subgroups in the list $L_1, \dots, L_n$. If $m = 1$, i.e., $\bigcup_{i=1}^{n} a_i L_1$ is cofinite in L, then obviously $[L : L_1] < \infty$. Let $m > 1$, assume that $[L : L_1]$ is infinite, and set

$$W = \{i : 1 \leq i \leq n,\ L_1 = L_i\}.$$

Since W is finite, $\bigcup_{i \in W} a_i L_1 \neq L$ and we may pick $x \in L - \bigcup_{i \in W} a_i L_1$. Then

$$x L_1 \cap a_i L_i = \phi \quad \text{for all } i \in W.$$

Thus $\bigcup_{i \notin W} (a_i L_1 \cap x L_1)$ is cofinite in $x L_1$. If $a_i L_i \cap x L_1 \neq \phi$ we fix $b_i \in a_i L_i \cap L_1$ and for any $y \in a_i L_1 \cap x L_1$ we obtain $b_i^{-1} y \in L_i \cap L_1$. Thus

$$b_i (L_i \cap L_1) = a_i L_i \cap x L_1$$

and $\bigcup_{i \notin W} x^{-1} b_i (L_i \cap L_1)$ is cofinite in L_1. Now we may use our induction hypothesis and conclude $[L_1 : L_1 \cap L_i] < \infty$ for some $i \notin W$. Since

$[L_1 : L_1 \cap L_i]$ is finite, we may write every coset $a_j L_1$, $j \in W$, as a finite union of cosets mod $L_j \cap L_1$ and this finite union is naturally contained in a union of finitely many cosets mod L_j. We now rewrite the union $\bigcup_{j=1}^{n} a_j L_j$ avoiding the use of L_1. We now apply our induction hypothesis again and obtain a k with $[L : L_k] < \infty$ and $2 \leq k \leq n$. $\square$

LEMMA 2.14. *Let A be an abelian p-reduced group and B any infinite group. If $A \wr B \subseteq W \subseteq A \,\hat{\wr}\, B$, then $\mathfrak{n}_W(B) = B$.*

Proof. Suppose $fc \in \mathfrak{n}_W(B)$ where $f \in F$, $c \in B$ with $W = F \rtimes B$. Then

$$b^{fc} = f^{-c} f^{cb^{-c}} b^c \in B,$$

hence $f^c = f^{cb^{-c}}$ and $b^{-c} \in \mathfrak{c}_B(f^c)$. The centralizer $\mathfrak{c}_B(f^c)$ must be infinite, hence $f^c = 1$ by (2.0) and $f = 1$ implies $\mathfrak{n}_W(B) = B$. $\square$

3. The construction

We call an (additive) abelian group A cotorsion-free if for any prime p the group A contains neither $\mathbf{Q}$, $\mathbf{Z}/p\mathbf{Z}$ nor $\hat{\mathbf{Z}}_p$ as subgroups where the latter is the p-adic completion of $\mathbf{Z}_p$, the integers localized at p. Thus A is cotorsion-free iff A is torsion-free, reduced and contains no subgroup $\neq 0$ which is complete in the p-adic topology for any prime p.

The following technical lemma will be our weapon to "kill" unwanted automorphisms:

LEMMA 3.1. *Let B be an infinite group and $H \subseteq \operatorname{Aut}(B)$. Let K be a countably infinite subgroup of B such that the following holds:*

(∗) *If K^* is any subgroup of K with finite index then $\mathfrak{c}_B(K^*) = 1$, and if $h \in H$ with $h \restriction K^* = y \restriction K^*$, and $y \in \operatorname{Inn} B$ then $h = 1$.*

Let $(A, +)$ be a rigid cotorsion-free abelian group of at least countable rank. Then there exists a group W with $A \wr B \subseteq W \subseteq A \,\hat{\wr}\, B$ such that for any $\eta \in \operatorname{Aut} W$ with $B^\eta = B$ we have $\eta \restriction K \in (H \cdot \operatorname{Inn} B) \restriction K$. Moreover there exists $H \cong H^* \subseteq \operatorname{Aut} W$ with $H^* \restriction B = H$.

(Recall that A is rigid if $\operatorname{End}(A) = \mathbf{Z} \cdot \mathrm{id}$.)

Proof. For $x \in K$ we fix $a_x \in A$ such that $\{a_x \colon x \in K\}$ is a linearly independent set in A. We also fix a p-adic (or $\mathbf{Z}$-adic) zero-sequence

$$\{z_x | x \in K\} \subseteq \mathbf{Z} \setminus \{0\}$$

and define

$$m = \sum_{x \in K} z_x a_x e_x \in \widehat{\bigoplus_{x \in B} Ae_x},$$

the p-adic (or $\mathbf{Z}$-adic) closure of $\bigoplus_{x \in B} Ae_x$. Let

$$F = \left\langle \bigoplus_{x \in B} Ae_x, m^{\eta t} \colon \eta \in H, t \in B \right\rangle_*$$

be the pure subgroup of $\widehat{\bigoplus_{x \in B} Ae_x}$ generated by $\bigoplus_{x \in B} Ae_x$ together with

$$\{M^{\eta t} \colon \eta \in H, t \in B\}.$$

This is the smallest (pure) subgroup F of $\widehat{\bigoplus_{x \in B} Ae_x}$ that contains $\bigoplus_{x \in B} Ae_x$ and m, is invariant under H and $\operatorname{Inn} B$ and has $F / \bigoplus_{x \in B} Ae_x$ divisible. Here we identify $h \in H$ with its extension h^* to $A \wr B$, i.e., $(ae_x)^h = ae_{x^h}$. Our desired group W is simply $W = F \rtimes B$. Clearly

$$A \wr B \subseteq W \subseteq A \,\hat{\wr}\, B.$$

Suppose $\alpha \in \operatorname{Aut} B$ lifts to some $\eta \in \operatorname{Aut} W$. Then $F^\eta = F$ by (2.2) and we set $\gamma = \eta \restriction F$. γ is determined by its action on Ae_1. Since A is rigid, there are integers γ_y, $y \in B$, and $(ae_1)^\gamma = \sum_{y \in B} a\gamma_y e_y$ for all $a \in A$.

Next we want to study representations of elements in F. Let

$$w \in F \setminus \bigoplus_{x \in B} Ae_x.$$

Then there are non-zero integers $s, s_i \in \mathbf{Z}$ and distinct pairs $(\eta_i, t_i) \in H \times B$, $1 \leq i \leq n$ with

$$sw = \sum_{i=1}^{n} s_i m^{\eta_i t_i} + u \quad \text{and} \quad u \in \bigoplus_{x \in B} Ae_x.$$

Thus

$$\begin{aligned} sw - u &= \sum_{i=1}^{n} \Big(s_i \sum_{x \in K} z_x a_x e_x \Big)^{\eta_i t_i} \\ &= \sum_{i=1}^{n} \sum_{x \in K} s_i z_x a_x e_{x^{\eta_i t_i}} \\ &= \sum_{v \in B} \Bigg(\sum_{\{i \colon t_i \in K^{\eta_i} v\}} s_i z_{(vt_i^{-1})^{\eta_i^{-1}}} a_{(vt_i^{-1})^{\eta_i^{-1}}} \Bigg) e_v . \end{aligned}$$

Notice that $(vt_i^{-1})^{\eta_i^{-1}} \in K$ since $t_i \in K^{\eta_i} v$.

Let $v \in B$ and call $T_v = \{(vt_i^{-1})^{\eta_i^{-1}}\colon t_i \in K^{\eta_i}v\}$ the v-support of $sw - u$. Note that for $t_i \in K^{\eta_i}v$, we always have $z_{(vt_i^{-1})^{\eta^{-1}_i}} \neq 0$. Fix $v \in B$ and some i and let $\bar{v} \in B$ with $(vt_i^{-1})^{\eta_i^{-1}} \in T_{\bar{v}}$. Then there is a j with $(vt_i^{-1})^{\eta_i^{-1}} = (\bar{v}t_j^{-1})^{\eta_j^{-1}}$ which implies $\bar{v} = (vt_i^{-1})^{\eta_i^{-1}\eta_j}t_j$. Hence

$$(+) \qquad \left\{\bar{v}\colon \left(vt_i^{-1}\right)^{\eta_i^{-1}} \in T_{\bar{v}}\right\} \quad \text{is finite.}$$

Let w_v be the e_v-component of $sw - u$, i.e., $sw - u = \sum_{v \in B} w_v e_v$. Then $(+)$ implies that for each $v \in B$ with $w_v \neq 0$ we have only finitely many $\bar{v} \in B$ such that $\{w_v, w_{\bar{v}}\}$ is linearly dependent in A. Since

$$(ae_1)^\gamma = \sum_{y \in B} a\gamma_y e_y \in F \quad \text{and} \quad \gamma_y \in \mathbf{Z}$$

we may conclude $\gamma_y = 0$ for all but finitely many y. Let

$$B_0 = \{y \in B\colon \gamma_y \neq 0\}.$$

Note that B_0 is finite. We are now ready to compute

$$\begin{aligned} m^\gamma &= \sum_{x \in K} (z_x a_x e_x)^\gamma \\ &= \sum_{x \in K} x^{-\alpha}(z_x a_x e_1)^\gamma x^\alpha \\ &= \sum_{x \in K} x^{-\alpha}\Bigg(\sum_{y \in B_0} z_x a_x \gamma_y e_y\Bigg)x^\alpha \\ &= \sum_{x \in K}\sum_{y \in B_0} z_x a_x \gamma_y e_{yx^\alpha} \\ &= \sum_{v \in B}\Bigg(\sum_{y \in B_0 \cap vK^\alpha} z_{(y^{-1}v)^{\alpha^{-1}}} a_{(y^{-1}v)^{\alpha^{-1}}} \gamma_y\Bigg) e_v. \end{aligned}$$

Note that for fixed v the map $y \mapsto (y^{-1}v)^{\alpha^{-1}}$ is one-to-one. Since $m^\gamma \in F$, we compare the v-components of m^γ and $sw - u$ above and obtain:

For almost all $v \in B$,

$$(++) \qquad \left\{\left(vt_i^{-1}\right)^{\eta_i^{-1}}\colon t_i \in K^{\eta_i}v\right\} \supseteq \left\{\left(y^{-1}v\right)^{\alpha^{-1}}\colon y \in B_0 \cap vK^\alpha\right\}.$$

Note that $m^\gamma \notin \bigoplus_{x \in B} Ae_x$ since $(\bigoplus_{x \in B} Ae_x)^\gamma \subseteq \bigoplus_{x \in B} Ae_x$ and γ is an automorphism. The same argument applied to γ^{-1} yields

$$\left(\bigoplus_{x \in B} Ae_x\right)^\gamma = \bigoplus_{x \in B} Ae_x.$$

Thus the set at the right hand side of $(++)$ is not empty. Fix a $y \in B_0$. For $k \in K$ we have $y \in (yk^\alpha)K^\alpha$. Hence for almost all $k \in K$, we find $i = i(k)$ and $k = ((yk^\alpha)t_i^{-1})^{\eta_i^{-1}}$. This implies

$$k^\alpha = y^{-1}k^{\eta_i}t_i.$$

Let

$$K_i = \{k \in K: k^\alpha = y^{-1}k^{\eta_i}t_i\}, \quad 1 \le i \le n.$$

Then $\bigcup_{i=1}^{n} K_i$ is cofinite in K. Let m be the minimal number of K_i's with the property that $\bigcup_{i=1}^{m} K_i$ is cofinite in K. (We may have to renumber the (η_i, t_i)'s.) Let $L_i = \{k \in K: k^\alpha = k^{\eta_i t_i}\}$. For $a, b \in K_i$ we have

$$(a^{-1}b)^\alpha = t_i^{-1}a^{-\eta_i}yy^{-1}b^{\eta_i}t_i = (a^{-1}b)^{\eta_i t_i}.$$

Thus $(a^{-1}b) \in L_i$ and it is easy to see that $K_i = a_i L_i$ for some $a_i \in K_i$.

Suppose there are $i, j \le m$ with $[K : L_i] < \infty$ and $[L_i, L_i \cap L_j] < \infty$. Since $\eta_i t_i$ and $\eta_j t_j$ coincide on $L_i \cap L_j$ and $[K: L_i \cap L_j] < \infty$ we obtain $(\eta_i, t_i) = (\eta_j, t_j)$ by $(*)$, and $i = j$ follows.

Since $\bigcup_{i=1}^{m} K_i = \bigcup_{i=1}^{m} a_i L_i$ is cofinite in K we may apply 2.13 and without loss of generality we may assume $[K : L_1] < \infty$. Suppose $m > 1$. Because of the minimality condition on m, there is some

$$x \in K - a_1 L_1 \quad \text{with } xL_1 \cap a_1 L_1 = \varnothing.$$

Thus $\bigcup_{i=2}^{m} (xL_1 \cap a_1 L_1)$ is cofinite in xL_1 and $xL_1 \cap a_i L_i = b_i(L_1 \cap L_i)$ for some b_i whenever $xL_1 \cap a_i L_i \neq \varnothing$. Now $\bigcup_{i=2}^{m} (x^{-1}b_i)(L_1 \cap L_i)$ is cofinite in L_1. We apply 2.13 once more and conclude that there is some i, $2 \le i \le m$ with $[L_1: L_1 \cap L_i] < \infty$. As we saw above, this implies $L_1 = L_i$ and $1 = i$, a contradiction. Thus $m = 1$ and $a_1 L_1$ is cofinite in K. Since all cosets of L_1 in K are infinite, $[K: L_1] = 1$ and $K = L_1$ follow. □

We are now ready to prove our main theorem.

THEOREM 3.2. *Let B and H be groups, κ a cardinal with $|B|, |H| \le \kappa$ and $\kappa^{\aleph_0} = \kappa$. Then there exists a group G of cardinality κ^+ with*

$$B \subseteq G, \qquad \mathfrak{n}_G(B) = B \quad \textit{and} \quad \operatorname{Aut} G = H \rtimes \operatorname{Inn} G.$$

Moreover $H \restriction B = 1$ and $G \in \{E_s, L\}\{H, B, \mathbf{Z}\}$.

Remark. An obvious modification of the rigid family $\{A_\nu: \nu \in \kappa^+\}$ leads to 2^{κ^+} pairwise non-isomorphic groups G as in the theorem. We use terminology from Hall [10] (see also [21]): $\{E_s, L\}\{H, B, \mathbf{Z}\}$ denotes the smallest local class of groups containing $H, B, \mathbf{Z}$ and closed under split extensions (= semidirect products).

Proof. Let $G = \bigcup_{\nu<\kappa^+} G_\nu$ be a continuous (cf. remark after 2.11) chain of sets such that $|G_\nu| = |G_{\nu+1} \setminus G_\nu| = \kappa$ and $|G| = \kappa^+$. Note that κ^+ is regular and $(\kappa^+)^{\aleph_0} = \kappa^+$ since $\kappa^{\aleph_0} = \kappa$. Let $E \subseteq \kappa^+$ be a stationary set (cf. [14, p. 58]) such that $\mathrm{cf}(\nu) = \omega$ for each $\nu \in E$. Due to a well-known result by Solovay there is a partition $E = \bigcup_{\alpha<\kappa^+} E_\alpha$ of E into disjoint stationary sets E_α, c.f. [14, Theorem 85, p. 433]. We may assume $E_\alpha \subseteq \{\nu < \kappa^+: \nu > \alpha\}$. Let $\{K_\nu: \nu < \kappa^+\}$ be a list of all countable subsets of G. We may assume $K_\nu \subseteq G_\nu$ for all $\nu < \kappa^+$. Choose a rigid system of cotorsion-free abelian groups $\{A_\nu: \nu < \kappa^+\}$ where $|A_\nu| = \kappa$ for all $\nu < \kappa^+$, cf. [2]. Inductively we will define group structures on the G_ν's such that G_ν is a subgroup of G_μ for all $\nu \leq \mu < \kappa^+$.

First we define the group G_0:

Without loss of generality we may assume B infinite since otherwise (2.6) shows the existence of a countable group $B' \supseteq B$ such that $\mathfrak{n}_{B'}(B) = B$. Moreover (2.9) shows that we may assume $B \in \mathfrak{c}$. From (2.5) $H \subseteq H' \in \mathfrak{c}$ and we set $G_0 = H' \wr B$. Note that $H \subseteq \operatorname{Aut} G_0$ and $H \restriction B = 1$ by (2.8). There exists a countable subgroup $Y \subseteq G_0$ such that for all $Y^* \subseteq Y$ of finite index, $h \restriction Y^* = y \in G_0$ implies $h = 1$. We fix this Y throughout the construction. Note that $Y \in \mathfrak{c}$.

If $\lambda < \kappa^+$ is a limit ordinal we define $G_\lambda = \bigcup_{\nu<\lambda} G_\nu$ whenever the chain of groups $\{G_\nu: \nu < \lambda\}$ is already defined. Suppose G_ν is already defined. Then we have to explain how to define $G_{\nu+1}$.

The group $G_{\nu+1}$ will be either $A_\nu \wr G_\nu$ or else will satisfy

$$A_\nu \wr G_\nu \subseteq G_{\nu+1} \subseteq A_\nu \,\hat{\wr}\, G_\nu$$

as in 3.1.

Case 1. If $\nu \notin E$, then $G_{\nu+1} = A_\nu \wr G_\nu$. If $\nu \in E_\alpha$ but K_α does not contain Y then we set $G_{\nu+1} = A_\nu \wr G_\nu$ also.

Case 2. There exists an $\alpha < \kappa^+$ with $\nu \in E_\alpha$ and K_α contains Y. Then we define $G_{\nu+1}$ to be the group W of 3.1 with A_ν, K_ν, G_ν for A, K, B resp. Thus $A_\nu \wr G_\nu \subseteq G_{\nu+1} \subseteq A_\nu \,\hat{\wr}\, G_\nu$.

This finishes the construction of G and it remains to be shown that G has the required properties.

An induction with the help of (2.3b) and 2.14 shows that

$$H \subseteq \operatorname{Aut} G, \qquad H \restriction B = 1 \quad \text{and} \quad \mathfrak{n}_G(B) = B.$$

The construction of Y shows that whenever $h \restriction Y^* = y \restriction Y^*$, $y \in \operatorname{Inn} G$ and Y^* a subgroup of finite index of Y, then $h = 1$. Let

$$\alpha \in (\operatorname{Aut} G) \setminus H \cdot \operatorname{Inn} G.$$

By 2.12 our group G admits a $\mathfrak{c}$-cover $\mathscr{F}'$ for its countable subgroups. Let

$$\mathscr{F} = \{F \in \mathscr{F}' \mid Y \subseteq F\}.$$

Since $\alpha \in \operatorname{Aut} G$ and κ^+ regular, a standard back and forth argument shows that $C = \{\nu < \kappa^+ \mid (G_\nu)^\alpha = G_\nu\}$ is a cub in κ^+ (i.e., a closed unbounded subset of κ^+, cf. [14, p. 56]). (The set C is closed since if $\lambda = \sup X$, for some $X \subseteq C$ with $\lambda < \kappa^+$ then

$$\alpha(G_\lambda) = \alpha\Big(\bigcup_{\nu \in X} G_\nu\Big) = \bigcup_{\nu \in X} \alpha(G_\nu) = \bigcup_{\nu \in X} G_\nu = G_\lambda.$$

The set C is also unbounded: if $\nu = \nu_0 < \kappa^+$ then there exists $\nu_0 \le \nu_1 < \kappa^+$ with $\alpha(G_{\nu_0}) \subseteq G_{\nu_1}$ since $|G_{\nu_0}| < \kappa^+ = \operatorname{cf}(\kappa^+)$, cf. [14, p. 26ff]. This leads to a sequence

$$\nu_0 < \nu_1 < \cdots < \nu_n < \cdots < \kappa^+ \quad \text{with } \alpha(G_{\nu_n}) \subseteq G_{\nu_{n+1}}.$$

For $\lambda = \sup\{\nu_n \mid n < \omega\}$ we have $\alpha(G_\lambda) = G_\lambda$ since $G_\lambda = \bigcup_{n=1}^{\infty} G_{\nu_n}$.

Next we show that there is an $F \in \mathscr{F}$ with $\alpha \restriction F \notin (H \cdot \operatorname{Inn} G) \restriction F$. We identify $\operatorname{Inn} G = G$ and pick any $F \in \mathscr{F}$. If $\alpha \restriction F = (h \cdot y) \restriction F$, $h \in H$, $y \in G$, we may pick $\nu \in C$ with $y \in G_\nu$ and $F \subseteq G_\nu$. Since $\alpha \notin H \cdot \operatorname{Inn} G$, $\alpha \neq h \cdot y$ and there exists an element $x \in G$ with $x^\alpha \neq x^{hy}$. Since $\mathscr{F}$ is a cover there is $F' \in \mathscr{F}$ with $F \cup \{x\} \subseteq F'$. If again $\alpha \restriction F' = h'y'$, we pick $\nu < \nu' \in C$ with $y' \in G_{\nu'}$.

Since $Y \subseteq F$ we have $hy \restriction Y = h'y' \restriction Y = \alpha \restriction Y$ and $h'^{-1}h \restriction Y = y'y^{-1} \restriction Y$ implies $h = h'$ and $y = y'$. Thus we obtain the contradiction $x^{hy} \neq x^\alpha = x^{hy}$. This shows the existence of the desired F. There is an ordinal $\mu < \kappa^+$ with $F = K_\mu$ and $C \cap E_\mu$ is again stationary in κ^+. For $\nu \in C \cap E_\mu$ we have $(G_\nu)^\alpha = G_\nu$ and α lifts to a monomorphism

$$\alpha \restriction G_{\nu+1}\colon G_{\nu+1} \to G.$$

If $\alpha(G_{\nu+1}) = G_{\nu+1}$ then 3.1 implies $\alpha \restriction G_\nu \in H \cdot \operatorname{Inn} G_\nu$.

What's left to show is the following:

Claim. If $\alpha \in \operatorname{Aut} G$ with $G_\nu^\alpha = G_\nu$ then $G_{\nu+1}^\alpha = G_{\nu+1}$.

Proof of Claim. Let $F = F_\nu$ with $G_{\nu+1} = F_\nu \rtimes G_\nu$ and set $X = F^\alpha$. Since $|X| < \kappa^+$ there exists a $\lambda < \kappa^+$ with $X \subseteq G_\lambda$. Note that X is abelian, $X^{G_\nu} = X$ and $X \cap G_\nu = 1$.

Step 1. If $X \cap (G_{\beta+1} \setminus G_\beta) \neq \emptyset$, $\beta \geq \alpha$, then $X \cap (G_{\beta+1} \setminus G_\beta)$ is infinite.

To see this suppose $x = fb \in (G_{\beta+1} \cap X) \setminus G_\beta$, $1 \neq f \in F_\beta$, and $b \in A_\beta$. Then $G_\nu \subseteq G_\beta$ and $x^{G_\nu} \subseteq f^{G_\nu} \cdot b^{G_\nu} \subseteq X$ and f^{G_ν} is infinite by (2.0). □

Step 2. If $X \cap G_\beta$ is infinite, then $X \subseteq G_\beta$.

Pick $x = fb \in X$ as above. Then $\{fb\} = \{x\} = x^{(X \cap G_\beta)} \subseteq f^{(X \cap G_\beta)} b^{(X \cap G_\beta)}$ and we conclude $f = f^{X \cap G_\beta}$. Since $X \cap G_\beta$ is infinite 2.0 implies $f = 1$. □

Step 3. There exists $\beta < \kappa^+$ and a finite normal subgroup N of G_ν such that $X \subseteq F_\beta \cdot N$ and $F_\beta X = F_\beta N$.

Because of Step 1 and Step 2 we find a $\beta < \kappa^+$ with $X \subseteq G_{\beta+1}$, and $X \not\subseteq G_\beta$. Let $x = fb \in X$ as above and $a \in G_\nu$. Then $f^a b^a \in X$. Since X is abelian we derive

$$f^a b^a fb = f^a f^{b^{-a}} b^a b = fbf^a b^a = ff^{ab^{-1}} bb^a.$$

Hence $f^a f^{b^{-a}} = ff^{ab^{-1}}$. Fix $x \in G_\beta$ with $f_x \neq 0$ and let $n = h_p^{A_\beta}(f_x)$ denote the p-height of f_x in A_β. For a cofinite subset A of G_ν we have $h_p(f_{xa}) > n$ for all $a \in A$. Again for some cofinite subset $A' \subseteq A$ we have $h_p(f_{xab^{-1}}) > n$. Since $f^a f^{b^{-a}} = ff^{ab^{-1}}$ implies

$$f_{xa} + f_{xa^{-1}b^{-1}a} = f_x + f_{xab^{-1}}$$

we obtain

$$0 \not\equiv f_x \equiv f_{xa^{-1}b^{-1}a} \bmod p^{n+1}A_\beta \quad \text{for all } a \in A'.$$

Thus $b^{A'}$ is finite and A' is cofinite in G_ν. This shows that $c_{G_\nu}(b)$ has finite index in G_ν. This implies that $b \in G_\nu$ and $X \subseteq F_\beta \cdot G_\nu$. Let N be the image of the projection of X into G_ν modulo F_β. Since $N \leq G_\nu$ and G_ν normalizes X, N is normal in G_ν. If $f_a \in X$, then $a \in N$ and $(fa)^a = f^a a \in X$. Therefore

$$faf^a a = ff^{aa^{-1}} a^2 = f^2 a^2 = f^a afa = f^a f^{a^{-1}} a^2$$

which implies $2f_x = f_{xa} + f_{xa^{-1}}$ for any x and all $a \in N$. Fix x with $f_x \neq 1$ and $n = h_p^{A_\beta}(f_x)$. If N were infinite we could come up with N', a cofinite subset of N and $f_{xa}, f_{xa^{-1}} \equiv 0 \bmod p^{n+1}A_\beta$ and the contradiction $0 \not\equiv f_x = f_{xa} + f_{xa^{-1}} \equiv 0 \bmod p^{n+1}A_\beta$ would follow. Thus N is finite. This completes the proof of step 3.

In order to prove the claim, note that $F_\nu^\alpha = X \subseteq F_\beta \cdot N$, N finite implies the existence of a non-zero homomorphism

$$A_\nu e_1 \rightarrowtail F_\nu \overset{\alpha}{\rightarrowtail} F_\beta \cdot N \overset{\cdot k}{\to} F_\beta \overset{e_x}{\to} A_\beta e_x$$

where $\cdot k$ is the multiplication with $k = |N|$ and e_x is the projection of $\widetilde{\bigoplus_{x \in B} A_\beta e_x)}$ into $\widetilde{A_\beta e_x}$. Observe that $F_\beta e_x \subseteq A_\beta e_x$. Since

$$\mathrm{Hom}(A_\nu, A_\beta) = 0 \quad \text{for } \nu \neq \beta,$$

we obtain $\alpha = \beta$ and

$$G_{\nu+1}^\alpha = F_\nu^\alpha G_\nu^\alpha \subseteq (F_\alpha \cdot N) G_\nu = G_{\nu+1}.$$

The same line of reasoning applied to α^{-1} gives us $G_{\nu+1}^{\alpha^{-1}} = G_{\nu+1}$ and we conclude $G_{\nu+1}^\alpha = G_{\nu+1}$. This finishes the proof of 3.2.

Remark. In a forthcoming paper we will give an alternative proof of the above claim avoiding the use of a rigid family. We will construct locally finite groups utilizing just *one* "rigid" abelian p-group.

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