ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 1, Spring 1991

CLASSIFICATION THEORY FOR A 1-ARY FUNCTION

BY

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1. Introduction

Let T be a countable, complete 1st order theory; assume for simplicity that T has no finite models. It is sometimes possible to assign every model M of T an invariant—like a cardinal number, or something similar—such that $M \simeq M'$ if and only if M and M' have the same invariant. For instance, if T is the theory of algebraically closed fields of some fixed characteristic, then, for every $M \models T$, the isomorphism type of M is given by its transcendence degree. Let us say that T is classifiable if this assignment of invariants can be done. The classification problem, namely the problem of characterizing classifiable theories, was (almost) thoroughly solved by S. Shelah. It would be too long to report here the development and the results of Shelah's analysis (a clear introduction can be found in [B] and [Sh]); briefly summarizing, we can say that, it one agrees to the (reasonable) assumption

> T is classifiable if and only if there is an uncountable cardinal λ such that T has $< 2^{\lambda}$ non-isomorphic models of power λ,

then Shelah's main theorem states that

> T is classifiable if and only if T is superstable, presentable, shallow and satisfies the existence property.

The classification problem involves an obvious algebraic question, namely to find, given an elementary class K of 1st order structures, under which conditions a structure $M \in K$ satisfies "$Th(M)$ classifiable". This is the question we wish to deal with for the class K of all structures M with a 1-ary (total) function. Hence Sections 2 and 3 are devoted to translating in this context some of the basic notions of classification theory (like regular types, orthogonal types, and so on); §4 contains the main theorems on classifiable 1-ary functions, while, finally in Sections 5 and 6 we will study the characterization of non-multidimensional, unidimensional and categorical 1-ary functions.

Received April 6, 1988

Main references are [CK] for basic model theory, [B] and [M] for classification theory. We shall adopt the same notation as [M]. The 1st order language for the structures we are going to study contains only one extra-logical symbol for a 1-ary function f. We will denote by L this language.

As the referee let me know, some results of this paper were independently obtained by Ryaskin: see [R] for a comparison.

2. Regular types

Let M be a structure of L. Consider the following binary relation $\sim$ on M: for all $a, b \in M$,

$$a \sim b \text{ iff there are } i, j \in \omega \text{ such that } f^i(a) = f^j(b).$$

Then $\sim$ is an equivalence relation. Every $\sim$-class of M will be called a connected component of M, and, for any $a \in M$, $\gamma(a, M)$ will denote the connected component of a in M. We need introduce a further notion. Let $a \in M$, define the tree of a in M $\tau(a, M)$ in the following way:

(1) If $f^m(a) \neq a$ for all $m \in \omega - \{0\}$, then

$$\tau(a, M) = \{b \in M : \exists\, n \in \omega \text{ such that } f^n(b) = a\};$$

(2) If there is $m \in \omega - \{0\}$ such that $f^m(a) = a$, then let m be minimal with this property and put $\tau(a, M) = \{b \in M$: either $b = a$ or $\exists\, n \in \omega - \{0\}$ such that $f^n(b) = a$ but $f^{n-1}(b) \neq f^{m-1}(a)\}$.

It is easy to see, in this second case, that, if $b \in \tau(a, M)$, $b \neq a$ and n is minimal such that $f^n(b) = a$, then, for every $p \in \omega - \{0\}$, $f^p(b) = a$ if and only if $p \equiv n \pmod m$; moreover, if $p > n$, then $f^{p-1}(b) = f^{m-1}(a)$.

Notice that, in any case, if $b \in \tau(a, M)$, then $\tau(b, M) \subseteq \tau(a, M)$. Let us come back to $\gamma(a, M)$. We can distinguish two cases.

Case 1. For all $n, m \in \omega$ with $n < m$, $f^n(a) \neq f^m(a)$. Then $\gamma(a, M)$ contains no loop, namely, for all $b \in \gamma(a, M)$ and $h \in \omega - \{0\}$, $f^h(b) \neq b$. Otherwise, let $i, j \in \omega$ be such that $f^i(a) = f^j(b)$; then we have

$$f^{h+i}(a) = f^{h+j}(b) = f^j(b) = f^i(a).$$

Secondly

$$\gamma(a, M) = \bigcup_{j \in \omega} \tau(f^j(a), M).$$

Case 2. There are $n, m \in \omega$ such that $n < m$, $f^n(a) = f^m(a)$. Let $n \in \omega$ be minimal such that there is $m > n$ satisfying $f^n(a) = f^m(a)$; and let $m > n$ be minimal such that $f^n(a) = f^m(a)$. Put $b = f^n(a)$, $l = m - n$; then $f^l(b) = b$ while $f^h(b) \neq b$ for $0 < h < l$; similarly for $f^j(b)$ when $j < l$. Then $\gamma(a, M)$ contains a loop of length l:

$$\{b, f(b), \ldots, f^{l-1}(b)\}.$$

This is the only loop in $\gamma(a, M)$; in fact, if $c \in \gamma(a, M)$, $k \in \omega - \{0\}$ and $c = f^k(c)$, then there are $i, j \in \omega$ such that $f^i(b) = f^j(c)$. Let $j = kq + r$ with $0 \leq r < k$; then

$$\begin{aligned} c &= f^{k(q+1)}(c) = f^{k+j-r}(c) = f^{k-r}f^j(c) = f^{k-r}f^i(b) \\ &= f^s(b) \end{aligned}$$

where s is the remainder of the division of $k - r + i$ by l. Hence in this case $\gamma(a, M)$ is determined by the only loop

$$\{b, \ldots, f^{l-1}(b)\}$$

it contains and, moreover, by $\{\tau(f^j(b), M)\colon j < l\}$. In fact

$$\gamma(a, M) = \bigcup_{j<l} \tau(f^j(b), M);$$

notice that, for every $i < j < l$, $\tau(f^i(b), M) \cap \tau(f^j(b), M) = \emptyset$.

Now let T be a 1st order theory of a 1-ary function f. (We tacitly assume T countable, complete, with no finite models; also we adopt the usual convention that all models of T are elementary substructures of a large model U.) We wish to classify the 1-types over models of T, or, more generally, over substructures of models of T, namely over subsets of U closed under f. Hence let A be such a subset, $p \in S_1(A)$. Then p satisfies exactly one of the following conditions.

(i) There is $a \in A$ such that $v = a \in p$.

(ii) There are $a \in A$, $n \in \omega - \{0\}$, $p_0 \in S_1(\emptyset)$ such that p contains

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b\colon b \in A\} \cup p_0.$$

Notice that these conditions completely determine p. In fact, let x, y satisfy them; we claim that $tp(x|A) = tp(y|A)$. With no loss of generality we can assume $x \neq y$. Notice that, for all $h < n$ and $m \in \omega - \{0\}$, $f^m f^h(x) \neq f^h(x)$ (otherwise $f^h(x) \in A$), and, similarly, $f^m f^h(y) \neq f^h(y)$. As $x \equiv y$, there exists an automorphism φ of U such that $\varphi(x) = y$. Therefore $\varphi(f^h(x)) =$

$f^h(y)$ for all $h \in \omega$ and, in particular, $\varphi(a) = a$. Let $h < n$ be minimal such that $f^h(x) = f^h(y)$; then

$$\tau(f^{h-1}(x), U) \cap \tau(f^{h-1}(y), U) = \emptyset.$$

Furthermore φ maps $\tau(f^{h-1}(x), U)$ onto $\tau(f^{h-1}(y), U)$. Replace φ by the function φ' defined in the following way: for all $b \in U$,

$$\varphi'(b) = \begin{cases} \varphi(b) & \text{if } b \in \tau(f^{h-1}(x), U), \\ \varphi^{-1}(b) & \text{if } b \in \tau(f^{h-1}(y), U), \\ b & \text{otherwise.} \end{cases}$$

Then φ' is an automorphism of U, $\varphi'(x) = y$ and $\varphi'(b) = b$ for all $b \in A$. It follows that $tp(x|A) = tp(y|A)$.

(iii) There is $p_0 \in S_1(\emptyset)$ such that p contains

$$\{f^n(v) \neq b: b \in A, n \in \omega\} \cup p_0.$$

Again, these conditions completely determine p. In fact, let x, y satisfy them. In particular $x \equiv y$, and hence there exists an automorphism φ of U such that $\varphi(x) = y$. As both $\tau(x, U)$ and $\tau(y, U)$ are disjoint from A, by proceeding as in (ii) we can assume $\varphi(b) = b$ for all $b \in A$. Then $tp(x|A) = tp(y|A)$.

Notice that the types satisfying (ii) or (iii) are not algebraic when A is a model of T. The following lemma will play an important role in the whole paper.

LEMMA 1. *Let $M \models T$, $A \supseteq M$, A closed under f, $p \in S_1(M)$, $p' \in S_1(A)$, $p' \supseteq p$.*

(i) *If there are $a \in M$, $n \in \omega - \{0\}$ such that p is defined by*

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b: b \in M\} \cup p \restriction_{\emptyset},$$

then p' does not fork over M if and only if p' does not represent $f^{n-1}(v) = w$.

(ii) *If p is defined by*

$$\{f^n(v) \neq a: n \in \omega, a \in M\} \cup p \restriction_{\emptyset},$$

then p' does not fork over M if and only if, for all $n \in \omega$, p' does not represent $f^n(v) = w$.

Proof. (i) If p' represents $f^{n-1}(v) = w$ and p' does not fork over M, then also p represents the same formula.

Conversely, assume that p' does not represent $f^{n-1}(v) = w$. Notice that any non-forking extension of p in $S_1(A)$ must contain

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b: b \in A\} \cup p \restriction_{\emptyset}.$$

On the other hand, there is only one type over A satisfying this property, and this type is p'. It follows that p' does not fork over M.

(ii) This can be shown in a similar way.

PROPOSITION 1. *Let T be a theory of a 1-ary function f. Then T is superstable and, for all $M \models T$ and $p \in S_1(M)$, $RU(p) \leq \omega$.*

Proof. First let us show that T is superstable. Recall that a theory T is superstable if and only if:

(3) for all $M < M' \models T$ and $p \in S_1(M)$, there is only one non-forking extension of p in $S_1(M')$ (namely T is stable);

(4) there is no sequence $p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n \subseteq \cdots (n \in \omega)$ where, for all $n \in \omega$, p_n is a 1-type over a suitable model of T and p_{n+1} is a forking extension of p_n.

In our case both these conditions follow from Lemma 1 (and its proof). Hence it remains to show that, for all $M \models T$ and $p \in S_1(M)$, $RU(p) \leq \omega$ (the definition of RU and its properties can be found, for instance, in [B]). Consider the following function r: for every M, p as above,

$$r(p) = \begin{cases} \omega & \text{if } \forall\, a \in M, \forall\, n \in \omega,\ f^n(v) \neq a \in p, \\ \min\{n \in \omega: \exists\, a \in M \text{ such that } f^n(v) = a \in p\} & \text{otherwise.} \end{cases}$$

Then r is a rank in the sense of Lascar since, for all $M_1, M_2 \models T$, $p_1 \in S_1(M_1)$, $p_2 \in S_1(M_2)$:

(5) if $p_1 \simeq p_2$, then $r(p_1) = r(p_2)$;

(6) if $M_1 < M_2$ and $p_1 \subseteq p_2$, then $r(p_1) \geq r(p_2)$, and $r(p_1) = r(p_2)$ if and only if p_2 does not fork over M_1.

As RU is the minimal Lascar rank, for every M, p as above, $RU(p) \leq r(p) \leq \omega$. □

Let us give an example of a theory T of a 1-ary function such that there are $M \models T$, $p \in S_1(M)$ satisfying RU$(p) = \omega$. Let T be the theory of a 1-ary function f such that

(7) for all a, $f^{-1}(a)$ is infinite, and

(8) for all a, and $n \in \omega - \{0\}$, $f^n(a) \neq a$.

Then T is complete and ω-stable. Moreover $|S_1(\emptyset)| = 1$. Let M be a model of T (for instance, assume that M has only one connected component). For all $a \in M$ and $n \in \omega - \{0\}$, let $p_n(a)$ be the 1-type over M containing

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b : b \in M\}$$

(as $|S_1(\emptyset)| = 1$, $p_n(a)$ is unique); let p be the 1-type over M containing

$$\{f^n(v) \neq b : n \in \omega, b \in M\}$$

(again $|S_1(\emptyset)| = 1$ implies that p is unique). Then:

(9) For all $a \in M$ and $n \in \omega - \{0\}$, $RU(p_n(a)) \geq n$. This is trivial when $n = 1$, because $p_1(a)$ is not algebraic. Let $n \geq 1$, assume our claim true for n, and consider $p_{n+1}(a)$. Let $b \models p_1(a)$, $M' = M(b)$; then there is a type $p_n(b) \in S_1(M')$ such that $p_n(b)$ is a forking extension of $p_{n+1}(a)$ and $RU(p_n(b)) \geq n$. It follows $RU(p_{n+1}(a)) \geq n + 1$.

(10) $RU(p) \geq \omega$. In fact, let $b \models p$, $M' = M(b)$ and, for all $n \in \omega - \{0\}$, consider the type $p_n(b) \in S_1(M')$; then $p_n(b)$ is a forking extension of p and $RU(p_n(b)) \geq n$. Hence $RU(p) \geq n + 1$ for all $n \in \omega$.

Consequently $RU(p) = \omega$.

PROPOSITION 2. *Let T be a theory of a 1-ary function. Then T is ω-stable if and only if $S_1(\emptyset)$ is countable in T.*

Proof. ($\Rightarrow$) is obvious. ($\Leftarrow$) follows from the analysis of non-algebraic types over a model M of T. In fact, recall that a non-algebraic $p \in S_1(M)$ is defined by

(11) An element $a \in M$, a natural number $n > 0$ and a 1-type p_0 over $\emptyset$ (with the conditions $f^n(v) = a \in p$, $f^{n-1}(v) \neq b \in p$ for all $b \in M$, $p \supseteq p_0$) or by

(12) a 1-type p_0 over $\emptyset$ (with the conditions $f^n(v) \neq a \in p$ for all $a \in M$ and $n \in \omega$, $p \supseteq p_0$).

If $|S_1(\phi)| \leq \aleph_0$, then $|S_1(M)| = |M|$. □

It may be worth exhibiting an example of a non ω-stable theory T of a 1-ary function. Then consider the L-structure M containing a unique connected component, and satisfying the following further conditions:

(13) there is a (unique) element $c \in M$ such that $f(c) = c$;

(14) $f^{-1}(c)$ is infinite while, for all $b \in M - \{c\}$, $1 \leq |f^{-1}(b)| \leq 2$;

(15) for all $n \in \omega$ and $k \in \{1,2\}^{n+1}$, there is $b \in M$ such that $f(b) = a$, b has exactly $k(0)$ preimages, each of them has exactly $k(1)$ preimages, ..., each of them has exactly $k(n)$ preimages;

(16) finally, if $b_0, b_1 \in M$ and $f^n(b_0) = f^n(b_1) \neq c$ for some $n \in \omega$, then $|f^{-1}(b_0)| = |f^{-1}(b_1)|$.

Let $T = Th(M)$. We already saw that T is superstable. On the other hand, T is not ω-stable, as, in T, $|S_1(\emptyset)| = 2^{\aleph_0}$ (it suffices to consider, for every $k \in \{1,2\}^{\omega}$, the type of an element b such that $b \neq c$, $f(b) = c$—notice that c is 0-definable—and b has exactly $k(0)$ preimages, each of them has exactly $k(1)$ preimages, ..., and so on).

Proposition 3. *Let T be a theory of a 1-ary function, $M \models T$, p a non-algebraic 1-type over M. Then p is regular and its associated geometry is degenerate.*

Proof. First let us show that, for all $M \models T$ and $p \in S_1(M)$, if p is not algebraic, then p is regular. It suffices to prove that, if $M' > M$ is an a-model of T, $p' \in S_1(M')$, $p' \supseteq p$ and $p' \neq p|M'$, then $p' \perp^a p|M'$.

Case 1. There are $a \in M$, $n \in \omega - \{0\}$ such that p contains

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b : b \in M\}.$$

Then $p|M'$ is the only type over M' containing

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b : b \in M'\} \cup p \restriction_{\emptyset},$$

while there exists $a' \in M'$ such that $f^{n-1}(v) = a' \in p'$. Let $i < n$ be minimal such that there is $a' \in M'$ satisfying $f^i(v) = a' \in p'$. Then $a' \notin M$. Let $x \models p'$, $y \models p|M'$. We have to show that $x \downarrow_{M'} y$. Let $q = tp(y|M' \cup \{f^h(x): h < i\})$; then $q \supseteq p|M'$, and hence q contains

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b : b \in M'\} \cup p \restriction_{\emptyset}.$$

Suppose there is $h < i$ such that $f^{n-1}(v) = f^h(x) \in q$. Then

$$f^{n-1+i-h}(v) = a' \in q,$$

hence

$$f^{n-1+i-h}(v) = a' \in p|M'.$$

Since $n - 1 + i - h \geq n$, it follows that $a' \in M$, a contradiction. Consequently, for all $h < i$,

$$f^{n-1}(v) \neq f^h(x) \in q,$$

and so q is a non-forking extension of $p|M'$. In particular $x \downarrow_{M'} y$.

Case 2. p contains $\{f^k(v) \neq a\colon a \in M,\ k \in \omega\}$. Then $p|M'$ is the only type over M' containing

$$\{f^k(v) \neq a\colon a \in M', K \in \omega\} \cup p \restriction_{\emptyset},$$

whereas there are $n \in \omega$, $a \in M'$ such that $f^n(v) = a \in p'$ (with no loss of generality we can assume n minimal with respect to this property). Let $x \models p'$, $y \models p|M'$, we claim $x \downarrow_{M'} y$. Let

$$q = tp\big(y|M' \cup \{f^h(x)\colon h < n\}\big);$$

then $q \supseteq p|M'$, hence, for all $k \in \omega$ and $a' \in M'$, $f^k(v) \neq a' \in q$. Suppose that there are $h < n$, $k \in \omega$ such that $f^k(v) = f^h(x) \in q$. Then

$$f^{k+n-h}(v) = a \in q$$

and so

$$f^{k+n-h}(v) = a \in p|M',$$

a contradiction. Hence, for all $h < n$ and $k \in \omega$,

$$f^k(v) \neq f^h(x) \in q,$$

so that q is a non-forking extension of $p|M'$. In particular $x \downarrow_{M'} y$. Then every non-algebraic 1-type over a model of T is regular.

Let us turn now to the second of our claims. Let $M \models T$, $p \in S_1(M)$ be non-algebraic (hence regular), and consider the pregeometry associated with p, namely the structure having

(i) domain $p(U)$,

(ii) a closure operator cl defined in the following way: for every $S \subseteq p(U)$, $\mathrm{cl}(S) = \{y \in p(U)\colon y \not\downarrow_M S\}$

(see [M], comments after D.7). This pregeometry is a geometry if, for every $x \in p(U)$, $\mathrm{cl}(x) = \{x\}$ (here $\mathrm{cl}(x)$ abbreviates, as usual, $\mathrm{cl}(\{x\})$).

Case 1. There are $a \in M$, $n \in \omega - \{0\}$ such that p contains

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b\colon b \in M\}.$$

First notice that, for every $x \in p(U)$,

$$\mathrm{cl}(x) = \{y \in p(U): f^{n-1}(y) = f^{n-1}(x)\}.$$

In fact, if $y \in p(U)$ and $f^{n-1}(y) = f^{n-1}(x)$, then the formula $f^{n-1}(v) = w$ is represented in $tp(y|M \cup \{x\})$ but it is not represented in p; hence $y \not\downarrow_M x$. Conversely, suppose $y \in p(U)$,

$$f^{n-1}(y) \neq f^{n-1}(x).$$

For every $j < n - 1$, $f^{n-1}(y) \neq f^j(x)$; otherwise

$$f^n(y) = f^{j+1}(x) \neq a \quad \text{and} \quad f^n(v) = a \notin p.$$

It follows that $y \downarrow_M x$, and hence $y \notin \mathrm{cl}(x)$. More generally, for every $S \subseteq p(U)$,

$$\mathrm{cl}(S) = \bigcup_{s \in S} \mathrm{cl}(s) = \{y \in p(U): \exists\, s \in S \text{ such that } f^{n-1}(y) = f^{n-1}(s)\}.$$

In fact, if $y \in p(U)$ and there is $s \in S$ such that

$$f^{n-1}(y) = f^{n-1}(s),$$

then $y \downarrow_M S$; namely $y \in \mathrm{cl}(S)$. Conversely, let $y \in p(U)$ be such that $f^{n-1}(y) \neq f^{n-1}(s)$ for all $s \in S$. By proceeding as above, one sees that

$$tp\big(y|M \cup \{f^j(s): s \in S, j < n\}\big)$$

cannot represent $f^{n-1}(v) = w$, so that $y \not\downarrow_M S$.

Then notice that the pregeometry associated with p is not in general a geometry. However, consider the binary relation R on $p(U)$ such that, for all $x, y \in p(U)$, xRy iff $x \not\downarrow_M y$. Then R is an equivalence relation and the structure having the following is a geometry:

(i) domain $p(U)/R$,

(ii) the closure operator defined in the following way: for all $S \subseteq p(U)$,

$$\mathrm{cl}\big(\{s|R: s \in S\}\big) = \{x|R: x \in \mathrm{cl}(S)\}$$

(it is well known that this definition is correct).

The previous remarks ensure that this geometry is degenerate. Namely the closure operator is trivial in $p(U)/R$: for all $S \subseteq p(U)$,

$$\mathrm{cl}\big(\{s|R: s \in S\}\big) = \{s|R: s \in S\}.$$

Case 2. For all $a \in M$ and $n \in \omega$, $f^n(v) \neq a \in p$. First notice that, for every $x \in p(U)$, $\mathrm{cl}(x) = p(U) \cap \gamma(x, U)$. In fact, if $y \in p(U) \cap \gamma(x, U)$, then there are $i, j \in \omega$ such that $f^j(y) = f^i(x)$; then $tp(y|M \cup \{x\})$ represents $f^j(v) = f^i(w)$, and $y \not\downarrow_M x$. Conversely, suppose $y \in p(U)$ and $f^j(y) \neq f^i(x)$ for all $i, j \in \omega$; then, for any $j \in \omega$, $tp(y|M \cup \{f^i(x)\colon i \in \omega\})$ cannot represent $f^j(v) = w$, and hence $y \downarrow_M x$.

Therefore the pregeometry associated with p is not always a geometry; however, by proceeding as before, one can easily show that its quotient geometry (defined as in case 1) is again degenerate.

3. Orthogonal types

Let T be a theory of a 1-ary function f, $M \models T$, p be a regular 1-type over M. Let us try to characterize the class of p with respect to the equivalence relation $\not\perp$. Recall that, if $p, q \in S_1(M)$, then

$$p \perp^a q \Leftrightarrow \forall\, a \models p, \quad \forall\, b = q,\ a \downarrow_M b;$$
$$p \perp q \Leftrightarrow \forall\, M' > M, \qquad p|M' \perp^a q|M'.$$

PROPOSITION 4. *Let $M \models T$, p, q be non-algebraic (hence regular) types in $S_1(M)$.*

(i) *If there are $a \in M$, $n \in \omega - \{0\}$ such that p contains*

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b\colon b \in M\},$$

and, for every $x \models p$, r_0 denotes $tp(f^{n-1}(x)|\emptyset)$, then $p \not\perp q$ if and only if there is $h \in \omega - \{0\}$ such that q contains

$$\{f^h(v) = a\} \cup \{f^{h-1}(v) \neq b\colon b \in M\},$$

and, for all $y \models q$,

$$tp(f^{h-1}(y)|\emptyset) = r_0.$$

(ii) *If for all $a \in M$ and $n \in \omega$ $f^n(v) \neq a \in p$, then $p \perp q$ if and only if, for all $a \in M$ and $h \in \omega$, $f^h(v) \neq a \in q$ and there are $n, h \in \omega$ such that, $\forall\, x \models p, \forall\, y \models q$,*

$$tp(f^n(x)|\emptyset) = tp(f^h(y)|\emptyset).$$

Proof. (i) First assume $p \not\perp q$. Then there are $a' \in M$, $h \in \omega$ such that $f^h(v) = a' \in q$. Otherwise, let $x \models p$, $y \models q$, and consider

$$tp(y|M \cup \{f^j(x)\colon j < n\}).$$

This type is a non-forking extension of q, as it includes q and, if there are $h \in \omega$, $i < n$ such that

$$f^h(v) = f^i(x) \in tp(y|M \cup \{f^j(x): j < n\}),$$

then $f^{h+n-i}(v) = a \in q$, a contradiction. It follows that $x \downarrow_M y$, so that $p \perp^a q$. Similarly for $M' > M$, $p|M'$, $q|M'$. Then $p \perp q$, a contradiction.

Hence, as q is not algebraic, there are $a' \in M$, $h \in \omega - \{0\}$ such that q contains $\{f^h(v) = a'\} \cup \{f^{h-1}(v) \neq b\colon\ b \in M\}$. We claim that $a = a'$. In fact, as $p \not\perp q$, there are $M' > M$, $x \models p|M'$, $y \models q|M'$ such that $x \not\downarrow_{M'} y$. In particular $tp(y|M' \cup \{f^j(x)\colon\ j < n\})$ forks over M'. Since this type includes $q|M'$, there exist $i < h$, $j < n$ such that $f^i(y) = f^j(x)$. Therefore

$$f^{j+h-i}(x) = a' \in M,$$

hence

$$j + h - i \geq n;$$

and

$$f^{i+n-j}(y) = a \in M,$$

hence

$$i + n - j \geq h \quad \text{and} \quad j + h - i \leq n.$$

Then $j + h - i = n$ and $a' = f^n(x) = a$.

Finally let us show that, for every $y \models q$, $tp(f^{h-1}(y)|\emptyset) = r_0$. Suppose towards a contradiction that there is $y \models q$ such that

$$tp(f^{h-1}(y)|\emptyset) = s_0 \neq r_0$$

(then, for all $y \models q$, $f^{h-1}(y) \models s_0$). Let $x \models p$, $y \models q$,

$$r = tp(f^{n-1}(x)|M), \quad s = tp(f^{h-1}(y)|M).$$

Then r is the only type over M containing

$$\{f(v) = a\} \cup \{v \neq b\colon b \in M\} \cup r_0$$

and s is the only type over M containing

$$\{f(v) = a\} \cup \{v \neq b\colon b \in M\} \cup s_0.$$

Clearly r, s are regular and $p \not\perp r$, $q \not\perp s$, so that $p \not\perp q$ implies $r \not\perp s$. Hence there are $M' > M$, $x' \models r|M'$, $y' \models s|M'$ such that $x' \not\downarrow_{M'} y'$. Then $tp(y'|M' \cup \{x'\})$ contains $v = x'$; but this contradicts

$$tp(x'|\emptyset) \neq tp(y'|\emptyset).$$

Hence, for all $y \models q$, $f^{h-1}(y) = r_0$, and this concludes the first part of the proof.

Now suppose that there is $h \in \omega - \{0\}$ such that q contains

$$\{f^h(v) = a\} \cup \{f^{h-1}(v) \neq b : b \in M\}$$

and, for all $y \models q$, $f^{h-1}(y) \models r_0$. Let $x \models p$; for any $\varphi(v) \in q \restriction_{\emptyset}$,

$$\exists w \quad \big(f^{h-1}(w) = v \wedge \varphi(w)\big) \in r_0,$$

hence

$$\models \exists w \quad \big(f^{h-1}(w) = f^{n-1}(x) \wedge \varphi(w)\big).$$

A compactness argument gives an element $y \in U$ satisfying

$$\{f^{h-1}(v) = f^{n-1}(x)\} \cup q \restriction_{\emptyset}.$$

Furthermore $y \models q$, because

$$f^h(y) = f^n(x) = a \quad \text{and} \quad f^{h-1}(y) = f^{n-1}(x) \notin M.$$

But $x \not\downarrow_M y$ as $tp(y|M \cup \{x\})$ represents $f^{h-1}(v) = f^{n-1}(w)$. Hence $p \not\perp {}^a q$ and consequently $p \not\perp q$.

(ii) First suppose $p \not\perp q$. Then, for all $a \in M$ and $h \in \omega$, $f^h(v) \neq a \in q$; otherwise (i) gives $p \perp q$. Moreover there exist $M' > M$, $x \models p|M'$, $y \models q|M'$ such that $x \not\downarrow_{M'} y$; hence

$$tp\big(y|M' \cup \{f^n(x) : n \in \omega\}\big)$$

forks over M', so that there are n, $h \in \omega$ satisfying $f^h(y) = f^n(x)$. Hence

$$tp\big(f^h(y)|\emptyset\big) = tp\big(f^n(x)|\emptyset\big),$$

and the same holds for all $x \models p$, $y \models q$.

Conversely, assume that, for all $a \in M$ and $h \in \omega$, $f^h(v) \neq a \in q$ and there exist $n, h \in \omega$ such that, for all $x \models p$ and $y \models q$,

$$tp\big(f^n(x)|\emptyset\big) = tp\big(f^h(y)|\emptyset\big) \quad (= r_0, \text{say}).$$

Let $x \models p$; for every $\varphi(v) \in q \upharpoonright_{\emptyset}$, $\exists\, w\ (f^h(w) = v \wedge \varphi(w)) \in r_0$. Then

$$\models \exists\, w \quad \left(f^h(w) = f^n(x) \wedge \varphi(w)\right),$$

and a compactness argument provides an element $y \in U$ satisfying $\{f^h(v) = f^n(x)\} \cup q \upharpoonright_{\emptyset}$. Then $y \models q$, because, if there is $k \in \omega$ such that $f^k(y) \in M$, then $f^{n+k}(x) = f^{h+k}(y) \in M$. But $y \not\downarrow_M x$ as $tp(y|M \cup \{x\})$ represents $f^h(v) = f^n(w)$. Then $p \not\perp^a q$ and $p \not\perp q$.

COROLLARY. *Let $M \models T$, $p, q \in S_1(M)$, p, q regular. Then $p \perp q$ if and only if $p \perp^a q$.*

Proof. $(\Rightarrow)$ is trivial; $(\Leftarrow)$ follows from Proposition 4 and its proof.

4. The main theorems

THEOREM 1. *Let T be a theory of a 1-ary function. Then:*
(i) *T is presentable;*
(ii) *T satisfies the existence property.*

Proof. (i) Let M_0, M_1, M_2 be a-models of T satisfying $M_0 \subseteq M_1$, $M_0 \subseteq M_2$, $M_1 \downarrow_{M_0} M_2$. We have to show that if M is an a-model a-prime over $M_1 \cup M_2$ and $p \in S_1(M)$ is not algebraic, then either $p \not\perp M_1$ or $p \not\perp M_2$.

First we claim that $M = M_1 \cup M_2$. Suppose towards a contradiction that this is not true, then there is $b \in M - M_1 \cup M_2$, moreover (see [M], B.9)

$$b \not\downarrow_{M_1} M_2, \qquad b \not\downarrow_{M_2} M_1.$$

Case 1. There are $a_1 \in M_1$, $n \in \omega - \{0\}$ such that $tp(b|M_1)$ contains

$$\{f^n(v) = a_1\} \cup \{f^{n-1}(v) \neq a \colon a \in M_1\}.$$

As $b \not\downarrow_{M_1} M_2$, there exists $a_2 \in M_2$ such that

$$f^{n-1}(v) = a_2 \in tp(b|M_1 \in M_2);$$

hence $f^{n-1}(v) = a_2 \in tp(b|M_2)$. Let $i < n$ be minimal such that there is $a_2 \in M_2$ satisfying $f^i(v) = a_2 \in tp(b|M_2)$; notice that $i > 0$ because $b \notin M_2$. As $b \not\downarrow_{M_2} M_1$, there exists $a_1' \in M_1$ such that

$$f^{i-1}(v) = a_1' \in tp(b|M_1)$$

where $i - 1 < n$, a contradiction.

Case 2. For all $a_1 \in M_1$ and $n \in \omega$, $f^n(v) \neq a_1 \in tp(b|M_1)$. As $b \not\downarrow_{M_1} M_2$, there are $a_2 \in M_2$, $n \in \omega$ ($n \neq 0$ since $b \notin M_2$) such that

$$f^n(v) = a_2 \in tp(b|M_2).$$

Without loss of generality we can assume that, for every $a \in M_2$,

$$f^{n-1}(v) \neq a \in tp(b|M_2).$$

As $b \not\downarrow_{M_2} M_1$, there is $a_1 \in M_1$ such that $f^{n-1}(v) = a_1 \in tp(b|M_1)$, a contradiction.

Then $M = M_1 \cup M_2$. Let now p be a non-algebraic 1-type over M. If there are $a \in M$, $n \in \omega - \{0\}$ such that p contains

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq c : c \in M\},$$

then we have $p \not\perp M_1$ or $p \not\perp M_2$, according to whether $a \in M_1$ or M_2. Otherwise we have both $p \not\perp M_1$ and $p \not\perp M_2$. In any case T is presentable.

(ii) Let $M_0, M_1, M_2 \models T$ with $M_0 \subseteq M_1$, $M_0 \subseteq M_2$, $M_1 \downarrow_{M_0} M_2$; we need show that there is a model M of T prime and atomic over $M_1 \cup M_2$. Obviously it suffices to show that $M_1 \cup M_2$—viewed as a structure of L—is a model of T. Build an independent diagram

$$\begin{tikzcd}
N_1 & N_0 \arrow[l] \arrow[r] & N_2 \\
M_1 \arrow[u] & M_0 \arrow[l] \arrow[r] \arrow[u, no head] & M_2 \arrow[u]
\end{tikzcd}$$

where N_0, N_1, N_2 are a-models of T (see [M] after A.13, [HM] Proposition 1.3). In particular $N_1 \not\downarrow_{N_0} N_2$. Furthermore (see [M], B.11):

(1) For every L-formula $\varphi(\bar{v}, \bar{w})$ and every $\bar{a} \in M_1 \cup M_2$, if there exists $\bar{b} \in N_1 \cup N_2$ such that $\models \varphi(\bar{b}, \bar{a})$, then there is $\bar{a}' \in M_1 \cup M_2$ such that $\models \varphi(\bar{a}', \bar{a})$.

Using this fact and recalling that $N_1 \cup N_2$ (as a structure of L) is a model of T (in fact equals the a-model a-prime over $N_1 \cup N_2$), one can easily prove that also $M_1 \cup M_2$ is a model of T. $\square$

Corollary. *Let T be a theory of a* 1-*ary function. Then T is classifiable if and only if T is shallow.*

Hence we now need to study the depth $\mathrm{Dp}\, T$ of such a theory T (we follow here the definition of Dp given in [Sa]).

DEFINITION. Let $M \models T$; for every ordinal α, we define a 1-ary relation on M,

$$\operatorname{Dp} \tau(a, M) \geq \alpha, \quad (a \in M)$$

in the following way:

(2) If $\alpha = 0$, then $\operatorname{Dp} \tau(a, M) \geq \alpha$ for any $a \in M$;

(3) If α is a limit ordinal, then

$$\begin{aligned} \operatorname{Dp} \tau(a, M) \geq \alpha &\Leftrightarrow \operatorname{Dp} \tau(a, M) \geq \alpha + 1 \\ &\Leftrightarrow \text{for all } \nu < \alpha, \quad \operatorname{Dp} \tau(a, M) \geq \nu; \end{aligned}$$

(4) If $\alpha = \nu + 1$ where $\nu = 0$ or ν is a successor ordinal, then $\operatorname{Dp} \tau(a, M) \geq \alpha$ iff there is $b \in \tau(a, M)$ such that there exist infinitely many $c \in M$ satisfying $f(c) = b$, $\operatorname{Dp} \tau(c, M) \geq \nu$ and whose trees in M are pairwise isomorphic.

(Notice that, for every M, c as above, $\tau(c, M)$ can be considered as a structure of L provided we pretend $f(c) = c$ when $f(c) \notin \tau(c, M)$; in this sense we can introduce the isomorphism type of $\tau(c, M)$). One can easily see that, if $M \models T$, $a \in M$, α, β are ordinals and $\alpha \leq \beta$, then $\operatorname{Dp} \tau(a, M) \geq \beta$ implies $\operatorname{Dp} \tau(a, M) \geq \alpha$. Then the following definition is well given.

DEFINITION. Let $M \models T$, $a \in M$. Then the depth of $\tau(a, M)$ $\operatorname{Dp} \tau(a, M)$ is the least ordinal α such that $\operatorname{Dp} \tau(a, M) \geq \alpha$ but $\operatorname{Dp} \tau(a, M) \not\geq \alpha + 1$ if such an ordinal exists, and is ∞ otherwise.

LEMMA 2. *Let $M \models T$, $a, b \in M$. If $b \in \tau(a, M)$, then* $\operatorname{Dp} \tau(a, M) \geq \operatorname{Dp} \tau(b, M)$. *If $\tau(a, M)$ and $\tau(b, M)$ are isomorphic, then* $\operatorname{Dp} \tau(a, M) = \operatorname{Dp} \tau(b, M)$.

Proof. In both cases it suffices to show that for any ordinal α, $\operatorname{Dp} \tau(b, M) \geq \alpha$ implies $\operatorname{Dp} \tau(a, M) \geq \alpha$. This can be shown by induction on α. The details are left to the reader. □

LEMMA 3. *Let $M \models T$, M $\aleph_0$-saturated, $a, b \in M$, $a \equiv b$. Then* $\operatorname{Dp} \tau(a, M) = \operatorname{Dp} \tau(b, M)$ *(in fact $\tau(a, M)$ and $\tau(b, M)$ are isomorphic).*

Proof. As $a \equiv b$ and M is $\aleph_0$-saturated (hence $\aleph_0$-homogeneous), there is an automorphism of M mapping a into b and consequently $\tau(a, M)$ onto $\tau(b, M)$. □

LEMMA 4. *Let M be an a-model of T, $a \in M$, $p \in S_1(M)$, p regular, $f^n(v) \neq a \in p$ for all $n \in \omega$, $x \models p$. Then*

$$\tau(a, M) = \tau(a, M[x]).$$

(Recall that $M[x]$ denotes the a-model of T a-prime over $M \cup \{x\}$).

Proof. Suppose towards a contradiction that there exists

$$b \in \tau(a, M[x]) - M.$$

Then $tp(b|M)$ is regular, and $tp(b|M) \not\perp p$. Moreover there are $c \in \tau(a, M)$, $k \in \omega - \{0\}$ such that $tp(b|M)$ contains

$$\{f^k(v) = c\} \cup \{f^{k-1}(v) \neq c' : c' \in M\}.$$

For Proposition 4(i) there is $h \in \omega - \{0\}$ such that $f^h(v) = c \in p$, and consequently there is $n \in \omega - \{0\}$ such that $f^n(v) = a$, a contradiction. □

For every a-model M of T and for every regular $p \in S_1(M)$, we want to compare $\mathrm{Dp}(p)$ and $\mathrm{Dp}\,\tau(x, M[x])$ where $x \models p$. First we assume that there are $a \in M$, $n \in \omega - \{0\}$ such that p contains

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq b : b \in M\}.$$

Without loss of generality $n = 1$ (it suffices to replace p with

$$q = tp(f^{n-1}(x)|M)$$

where $x \models p$; in fact $q \not\perp p$ and hence $\mathrm{Dp}(q) = \mathrm{Dp}(p)$). Let $x \models p$, $M' = M[x]$. Then notice that, for every $y \in M' - M$, $tp(y|M)$ is regular and $\not\perp p$. Hence Proposition 4(i) implies that there exists $h \in \omega - \{0\}$ such that $tp(y|M)$ contains

$$\{f^h(v) = a\} \cup \{f^{h-1}(v) \neq b : b \in M\}$$

and

$$tp(f^{h-1}(y)|\emptyset) = p \restriction_{\emptyset}.$$

Put $y' = f^{h-1}(y)$; therefore $y' \models p$ and $\tau(y', M')$ is isomorphic to $\tau(x, M')$ (this follows from Lemma 3, recalling that any a-model of T is $\aleph_0$-saturated).

PROPOSITION 5. *Let M, p, x be as above. Then*

$$\mathrm{Dp}(p) = \mathrm{Dp}\,\tau(x, M[x]).$$

Proof. It suffices to show that, for every ordinal α,

$$\mathrm{Dp}(p) \geq \alpha \quad \text{if and only if } \mathrm{Dp}\,\tau(x, M') \geq \alpha$$

(where M' abbreviates $M[x]$). We proceed by induction on α. The cases $\alpha = 0$ and $\alpha = \nu$ or $\nu + 1$ for ν a limit ordinal are trivial. Hence assume $\alpha = \nu + 1$ where $\nu = 0$ or ν is a successor ordinal.

First suppose $\mathrm{Dp}(p) \geq \alpha$, then there is $q \in S_1(M')$ such that q is regular, $q \perp M$ and $\mathrm{Dp}(q) \geq \nu$. If q contains $f^k(v) \neq y$ for all $y \in M'$ and $k \in \omega$, then $q \not\perp M$; then there exist $y \in M'$ and $k \in \omega - \{0\}$ such that q contains

$$\{f^k(v) = y\} \cup \{f^{k-1}(v) \neq b \colon b \in M'\}.$$

Without loss of generality we can assume $k = 1$. Moreover $y \notin M$, otherwise $q \not\perp M$. Let h be the least natural number such that $f^h(y) \in M$, and put $y' = f^{h-1}(y)$; recall that $y' \vDash p$ and $\tau(y', M')$ is isomorphic to $\tau(x, M')$. As M' is $\aleph_0$-saturated, for all $b_0, \ldots, b_n \in M'$ satisfying $\{f(v) = y\} \cup q \restriction_{\emptyset}$, the set of formulas

$$\{f(v) = y\} \cup \{v \neq b_i \colon i \leq n\} \cup q \restriction_{\emptyset}$$

is realized in M'; in fact this set is realized in $M'[c]$ with $c \vDash q$ and so it is finitely satisfiable in M'. Consequently there exist infinitely many elements of M' satisfying

$$\{f(v) = y\} \cup q \restriction_{\emptyset}$$

(notice that all these elements have the same type over $\emptyset$, hence Lemma 3 implies that their trees in M' are pairwise isomorphic). Now let $c \vDash q$; using Lemma 3 again, for every $z \in M'$ satisfying

$$\{f(v) = y\} \cup q \restriction_{\emptyset},$$

we have

$$\mathrm{Dp}\,\tau(z, M'[c]) = \mathrm{Dp}\,\tau(c, M'[c]).$$

As $f^n(v) \neq z \in q$ for all $n \in \omega$, $\tau(z, M') = \tau(z, M'[c])$ (Lemma 4), and hence $\mathrm{Dp}\,\tau(z, M') = \mathrm{Dp}\,\tau(z, M'[c])$. The induction hypothesis ensures $\mathrm{Dp}\,\tau(z, M'[c]) \geq \nu$ because $\mathrm{Dp}(q) \geq \nu$. Hence we can conclude that there is

$y \in \tau(y', M')$ such that y admits infinitely many preimages whose trees are isomorphic and have $\mathrm{Dp} \geq \nu$. It follows that $\mathrm{Dp}\,\tau(y', M') \geq \alpha$. Since $\tau(x, M')$ is isomorphic to $\tau(y', M')$, $\mathrm{Dp}\,\tau(x, M') \geq \alpha$.

Conversely, let $Dp\,\tau(x, M') \geq \alpha$, then there is $y \in \tau(x, M')$ ($\Rightarrow y \notin M$) such that y admits infinitely many preimages whose trees are pairwise isomorphic and have $\mathrm{Dp} \geq \nu$; we can assume that y does not occur among these preimages. Then we claim that all these preimages satisfy the same type q_0 over $\emptyset$. In fact let z, z' be two of these preimages, with $z \neq z'$; then

$$\tau(z, M') \cap \tau(z', M') = \emptyset.$$

The isomorphism φ of $\tau(z, M')$ onto $\tau(z', M')$ can be extended to get an automorphism of M' (it suffices to map $\tau(z, M')$ onto $\tau(z', M')$ by φ, and $\tau(z', M')$ onto $\tau(z, M')$ by φ^{-1}, and to complete by the identity elsewhere). Let q be the only 1-type over M' containing

$$\{f(v) = y\} \cup \{v \neq b \colon b \in M'\} \cup q_0.$$

Then q is non-algebraic, hence regular. Moreover $q \perp M$ as $y \notin M$. Finally $\mathrm{Dp}(q) \geq \nu$; in fact, if $c \models q$, then, for all z as above,

$$\mathrm{Dp}\,\tau(c, M'[c]) = \mathrm{Dp}\,\tau(z, M'[c]).$$

On the other hand, $\tau(z, M'[c]) = \tau(z, M')$ as $f^n(v) \neq z \in q$ for all $n \in \omega$ (Lemma 4). Hence $\mathrm{Dp}\,\tau(c, M'[c]) = \mathrm{Dp}\,\tau(z, M') \geq \nu$ and the induction hypothesis gives $\mathrm{Dp}(q) \geq \nu$. It follows that $\mathrm{Dp}(p) \geq \alpha$. □

We turn now to the case that $f^n(v) \neq a \in p$ for all $a \in M$ and $n \in \omega$. Let $x \models p$, and put $M' = M[x]$. We claim that, for all $y \in M' - M$, $\gamma(y, M')$ is isomorphic to $\gamma(x, M')$ and

$$\gamma(y, M') \cap M = \emptyset.$$

In fact $tp(y|M) \not\perp p$; hence, for all $a \in M$ and $n \in \omega$,

$$f^n(v) \neq a \in tp(y|M),$$

and there are $n, m \in \omega$ such that

$$tp(f^n(y)|\emptyset) = tp(f^m(x)|\emptyset).$$

Then, for every $\varphi(v) \in p \restriction_{\emptyset}$,

$$M' \models \exists\, w \quad (f^m(w) = f^n(y) \wedge \varphi(w)).$$

As M' is $\aleph_0$-saturated, there is $x' \in M'$ such that $f^m(x') = f^n(y)$ and $x' \models p \restriction_{\emptyset}$. Consequently there is an automorphism φ of M' mapping x into x', and hence $\gamma(x, M')$ onto $\gamma(x', M') = \gamma(y, M')$. Moreover $\gamma(x', M') \cap M = \emptyset$, hence $x' \models p$.

PROPOSITION 6. *Let M, p be as above, $x \models p$, $M' = M[x]$. Then $Dp(p)$ is*

$$\max\{\mathrm{Dp}\, \tau(f^i(x), M') : i \in \omega\}$$

if this set has a greatest element (in case ∞), and is

$$\sup\{\mathrm{Dp}\, \tau(f^i(x), M') : i \in \omega\} + 1$$

otherwise.

Proof. First we claim that, if $\nu = 0$ or ν is a successor ordinal, then

$$\mathrm{Dp}(p) \geq \nu + 1 \quad \text{iff } \exists\, i \in \omega \text{ such that } \mathrm{Dp}\, \tau(f^i(x), M') \geq \nu + 1.$$

In fact, assume $\mathrm{Dp}(p) \geq \nu + 1$, then there is $q \in S_1(M')$ such that q is regular, $q \perp M$ and $Dp(q) \geq \nu$. As in the previous Proposition 5, we can assume that there is $y \in M'$ such that q contains

$$\{f(v) = y\} \cup \{v \neq b : b \in M'\}.$$

Then $y \notin M$, and so there are $x' \in M'$, $i \in \omega$ such that

$$x' \models p, \quad y \in \tau(f^i(x'), M').$$

Let $z_0, \ldots, z_n \in M'$ satisfy $\{f(v) = y\} \cup q \restriction_{\emptyset}$ (notice that, if there is $l \in \omega - \{0\}$ such that $f^l(y) = y$ and l is minimal with this property, then, for all $z \models q \restriction_{\emptyset}$, $z \neq f^{l-1}(y)$). The set

$$\{f(v) = y\} \cup \{v \neq z_i : i \leq n\} \cup q \restriction_{\emptyset}$$

is realized in $M'[c]$ (where $c \models q$), and consequently, as M' is $\aleph_0$-saturated, is realized also in M'; it follows that M' contains infinitely many elements satisfying $\{f(v) = y\} \cup q \restriction_{\emptyset}$. As all these elements admit the same type over $\emptyset$, Lemma 3 implies that their trees in M' are pairwise isomorphic. Furthermore, if $c \models q$, then, for every $z \in M'$ satisfying $q \restriction_{\emptyset}$ (and $f(v) = y$), we again have

$$\mathrm{Dp}\, \tau(c, M'[c]) = \mathrm{Dp}\, \tau(z, M'[c]).$$

As $f^n(v) \neq z \in q$ for all $n \in \omega$, $\tau(z, M') = \tau(z, M'[c])$. Then, for all z as above,

$$\mathrm{Dp}\,\tau(z, M') = \mathrm{Dp}\,\tau(c, M'[c]) = \mathrm{Dp}(q) \geq \nu.$$

We can conclude that there is $y \in \tau(f^i(x'), M')$ such that y admits infinitely many preimages having isomorphic trees of depth $\geq \nu$. Then

$$\mathrm{Dp}\,\tau\big(f^i(x'), M'\big) \geq \nu + 1$$

and so

$$\mathrm{Dp}\,\tau\big(f^i(x), M'\big) \geq \nu + 1.$$

Now let

$$\mathrm{Dp}\,\tau\big(f^i(x), M'\big) \geq \nu + 1 \quad \text{for some } i \in \omega.$$

Then there is $y \in \tau(f^i(x), M')$ with infinitely many preimages having isomorphic trees of depth $\geq \nu$ (as above, if there is $l \in \omega$ such that $f^{l+1}(y) = y$ but $f^l(y) \neq y$, then there is no loss of generality in excluding $f^l(y)$ among these preimages). Let z, z' be two of these preimages, with $z \neq z'$; as in Proposition 5, we can extend the isomorphism between $\tau(z, M')$ and $\tau(z', M')$ to get an automorphism of M'. In particular $tp(z|\emptyset) = tp(z'|\emptyset)$ ($= q_0$, say). Consider the unique type $q \in S_1(M')$ containing

$$\{f(v) = y\} \cup \{v \neq b \colon b \in M'\} \cup q_0.$$

Then q is regular, and $q \perp M$ since $y \notin M$. Moreover $\mathrm{Dp}(q) \geq \nu$. In fact, let z be as above, $c \models q$; then both z and c realize q_0, hence

$$\mathrm{Dp}\,\tau(c, M'[c]) = \mathrm{Dp}\,\tau(z, M'[c]).$$

But $\tau(z, M'[c]) = \tau(z, M')$ as $f^n(v) \neq z \in q$ for all $n \in \omega$, and consequently $\mathrm{Dp}\,\tau(c, M'[c]) = \mathrm{Dp}\,\tau(z, M') \geq \nu$. As

$$\mathrm{Dp}\,\tau(c, M'[c]) = \mathrm{Dp}(q)$$

(see Proposition 5), it follows that $\mathrm{Dp}(p) \geq \nu + 1$, and this concludes the proof of our claim.

At this point we have the following:

$$\begin{aligned} (5) \qquad \mathrm{Dp}(p) = 0 &\Leftrightarrow \mathrm{Dp}(p) \not\geq 1 \\ &\Leftrightarrow \text{for all } i \in \omega, \quad \mathrm{Dp}\,\tau\big(f^i(x), M'\big) \not\geq 1 \\ &\Leftrightarrow \text{for all } i \in \omega, \quad \mathrm{Dp}\,\tau\big(f^i(x), M'\big) = 0. \end{aligned}$$

(6) If $\nu = 0$ or ν is a successor ordinal, then $\mathrm{Dp}(p) = \nu + 1 \Leftrightarrow \mathrm{Dp}(p) \geq \nu + 1$, $\mathrm{Dp}(p) \not\geq \nu + 2 \Leftrightarrow \exists\ i \in \omega$ such that $\mathrm{Dp}\, \tau(f^i(x), M') \geq \nu + 1$ but, $\forall\ i \in \omega$,

$$\mathrm{Dp}\, \tau\big(f^i(x), M'\big) \not\geq \nu + 2 \Leftrightarrow \big\{\mathrm{Dp}\, \tau\big(f^i(x), M'\big) \colon i \in \omega\big\}$$

has a greatest element, and this element is $\nu + 1$.

(7) If ν is a limit ordinal, then $\mathrm{Dp}(p) = \nu + 1 \Leftrightarrow \forall\ \mu < \nu$, μ successor, $\mathrm{Dp}(p) \geq \mu + 1$ but $\mathrm{Dp}(p) \not\geq \nu + 2 \Leftrightarrow$ for every $\mu < \nu$, μ successor, there is $i \in \omega$ such that $\mathrm{Dp}\, \tau(f^i(x), M') \geq \mu + 1$ but, for all $i \in \omega$, $\mathrm{Dp}\, \tau(f^i(x), M') \not\geq \nu + 2$; then either $\{\mathrm{Dp}\, \tau(f^i(x), M')\colon i \in \omega\}$ has a greatest element, and this element equals $\nu + 1$, or $\{\mathrm{Dp}\, \tau(f^i(x), M')\colon i \in \omega\}$ does not have a greatest element, but in this case $\sup\{\mathrm{Dp}\, \tau(f^i(x), M')\colon i \in \omega\} = \nu$.

(8) Finally, let $\mathrm{Dp}(p) = \infty$; if there is $i \in \omega$ such that $\mathrm{Dp}\, \tau(f^i(x), M') = \infty$, then we are done. Otherwise, for all $i \in \omega$, $\mathrm{Dp}\, \tau(f^i(x), M')$ is an ordinal (α_i, say); let α be a successor ordinal such that $\alpha \geq \alpha_i$ for all $i \in \omega$. Then $\mathrm{Dp}(p) \geq \alpha + 1$, and hence there exists $i \in \omega$ such that $\mathrm{Dp}\, \tau(f^i(x), M') \geq \alpha + 1 > \alpha_i$, a contradiction. □

It is easy to give an example of a deep theory of a 1-ary function (see [Sa]). With a good deal of patience, one could find, for every successor ordinal $\alpha < \omega_1$, a shallow theory of a 1-ary function with depth α. We prefer to omit here the corresponding list of examples, and to concentrate our attention on the theories T such that $\mathrm{Dp}\, T = 1$ (namely, the so-called non-multidimensional theories).

5. The non-multidimensional case

PROPOSITION 7. *Let T be a theory of a 1-ary function. Then* $\mathrm{Dp}\, T = 1$ *if and only if* $\{a\colon f^{-1}(a)$ *is infinite*$\}$ *is* 0*-definable and finite.*

Proof. ($\Leftarrow$) Let M be an a-model of T, and let $p \in S_1(M)$ satisfy $\mathrm{Dp}(p) \geq 1$. Then there is $q \in S_1(M[x])$ (for $x \models p$) such that q is regular and $q \perp M$. Hence there are $y \in M[x] - M$ and $n \in \omega - \{0\}$ such that q contains

$$\{f^n(v) = y\} \cup \{f^{n-1}(v) \neq b\colon b \in M[x]\}.$$

With no loss of generality $n = 1$. As q is not algebraic,

$$\{z \in M[x]\colon f(z) = y\}$$

is infinite, and this contradicts our hypothesis since $y \notin M$.

($\Rightarrow$) Let M be an a-model of T; first assume that $\{a \in M\colon f^{-1}(a)$ is infinite$\}$ is not finite. We distinguish two cases.

Case 1. For all $a \in M$ and $n \in \omega$, there are at most finitely many elements $b \in M$ satisfying $f^n(b) = a$ and $f^{-1}(b)$ infinite.

As $\{b \in M\colon f^{-1}(b)$ is infinite$\}$ is not finite, the set

$$\{f^n(v) \neq a\colon a \in M, n \in \omega\} \cup \{\exists^{>n} w(f(w) = v)\colon n \in \omega\}$$

is finitely satisfiable in M, and hence can be extended to a (non-algebraic) type $p \in S_1(M)$. Let $x \models p$, and consider $M[x]$. The set

$$\{f(v) = x\} \cup \{v \neq b\colon b \in M[x]\}$$

is finitely satisfiable in $M[x]$; let q be a 1-type over $M[x]$ extending this set. Then q is not algebraic and hence regular; $q \perp M$ as $x \notin M$. Then $\mathrm{Dp}(p) \geq 1$ and $\mathrm{Dp}\, T \geq 2$; but this contradicts the hypothesis.

Case 2. There are $a \in M$ and $n \in \omega$ such that there exist infinitely many elements $b \in M$ satisfying $f^n(b) = a$ and $f^{-1}(b)$ infinite.

Choose a, n such that n is minimal ($n > 0$, of course), and consider the set

$$\{f^n(v) = a\} \cup \{f^{n-1}(v) \neq d\colon d \in M\} \cup \{\exists^{>k} w\, (f(w) = v)\colon k \in \omega\}.$$

This set is finitely satisfiable in M; in fact, for every $d \in M$ such that $f(d) = a$, the minimality of n entails that there are only finitely many $b \in M$ satisfying $f^{n-1}(b) = d$, $f^{-1}(b)$ infinite. Consequently, for all $h \in \omega$, $d_0, \ldots, d_h \in M$ with $f(d_0) = \cdots = f(d_h) = a$, there is $b \in M$ such that $f^n(b) = a$, $f^{n-1}(b) \neq d_0, \ldots, d_h$, $f^{-1}(b)$ infinite.

Let p be a 1-type over M extending the foregoing set, and put $x \models p$. Consider

$$\{f(v) = x\} \cup \{v \neq b\colon b \in M[x]\}$$

in $M[x]$. This set is finitely satisfiable in $M[x]$, and hence can be extended to get a 1-type $q \in S_1(M[x])$. q is not algebraic (and consequently is regular); $q \perp M$ as $x \notin M$. Then we again have $\mathrm{Dp}(p) \geq 1$, hence $\mathrm{Dp}\, T \geq 2$, a contradiction.

It follows that, for any a-model M of T, $\{a \in M\colon f^{-1}(a)$ is infinite$\}$ is finite. Fix M and let $a_0, \ldots, a_k$ be the elements of M having infinitely many preimages. A compactness argument produces a natural number N such that

$$\models \forall v \quad (\exists^{>N} w\, (f(w) = v) \to \exists^{\infty} w \quad (f(w) = v)),$$

otherwise

$$\{v \neq a_0, \ldots, a_k\} \cup \{\exists^{>n} w\, (f(w) = v) : n \in \omega\}$$

would be satisfiable. Then $\{a$: $f^{-1}(a)$ is infinite$\}$ can be defined without parameters, for instance by the formula

$$\exists^{>N} w \quad (f(w) = v). \qquad \square$$

We point out that in the proof of Proposition 7, we have implicitly shown that if $\mathrm{Dp}\, T = 1$, then there is N such that, for every a, either $|f^{-1}(a)| \leq N$ or $f^{-1}(a)$ is infinite.

COROLLARY. *If f is injective (more generally if there is $m \in \omega$ such that, for every a, $|f^{-1}(a)| \leq m$), then* $\mathrm{Dp}\, T = 1$.

Proof. $\{a$: $f^{-1}(a)$ is infinite$\} = \emptyset$. $\square$

6. The unidimensional case

Let us take care now of the unidimensional case. We recall that T is unidimensional if, for any $M \models T$, there is a unique $\not\perp$ -class of regular types over M; in particular, T is non-multidimensional. Hence, in the case of a 1-ary function, $\{a$: $f^{-1}(a)$ is infinite$\}$ is 0-definable and finite, moreover there is $R \in \omega$ such that, for every a, either $|f^{-1}(a)| \leq R$ or $f^{-1}(a)$ is infinite. But in the unidimensional case a stronger result holds.

LEMMA 5. *Let T be a unidimensional theory of a 1-ary function f. Then*

$$\left|\{a : f^{-1}(a) \textit{ is infinite}\}\right| \leq 1.$$

Proof. Let $M \models T$, $a_0, a_1 \in M$ be such that both $f^{-1}(a_0)$ and $f^{-1}(a_1)$ are infinite. Then, for every $i \leq 1$,

$$\{f(v) = a_i\} \cup \{v \neq b : b \in M\}$$

can be extended in at least one way to a nonalgebraic, hence regular, type $p_i \in S_1(M)$. If $a_0 \neq a_1$, then $p_0 \perp p_1$ (Proposition 4.(i)). It follows that $a_0 = a_1$. $\square$

Notice that in general every $\aleph_1$-categorical theory T is unidimensional; the converse is also true for T ω-stable. However there exist non-multidimensional theories T of a 1-ary function which are not ω-stable. An example is just provided by the theory T we exhibit after Proposition 2; in fact T is not

ω-stable, but $\operatorname{Dp} T = 1$, as $\{a\colon f^{-1}(a) \text{ is infinite}\} = \{c\}$ is 0-definable and finite.

PROPOSITION 8. *Let T be the theory of a 1-ary function. Then T is unidimensional if and only if T is $\aleph_1$-categorical.*

Proof. On account of the previous remarks, it suffices to show that, if T is unidimensional, then T is ω-stable. Hence assume T unidimensional.

Case 1. There exists a (unique) element $a \in U$ such that $f^{-1}(a)$ is infinite.

Let $M \models T$; then $a \in M$ and the set

$$\{f(v) = a\} \cup \{v \neq b\colon b \in M\}$$

can be enlarged to a regular type $p \in S_1(M)$. This type is uniquely determined, as, if $q \in S_1(M)$ contains

$$\{f(v) = a\} \cup \{v \neq b\colon b \in M\},$$

then $q \not\perp p$, and hence $q \restriction_{\emptyset} = p \restriction_{\emptyset}$ (see Proposition 4.(i)), so that $p = q$. Denote by p^M the unique 1-type over M extending $\{f(v) = a\} \cup \{v \neq b\colon b \in M\}$. As T is unidimensional, for all $M \models T$, every regular type over M is not orthogonal to p^M.

We claim that:

(i) There is $N \in \omega$ such that, for all $b \in U$, there are $n, m \in \omega$ such that $n < m \leq N$ and $f^n(b) = f^m(b)$.

Otherwise $\{f^n(v) \neq f^m(v)\colon n, m \in \omega,\ n < m\}$ is finitely satisfiable in any model of T. Hence there are $M \models T$, $b \in M$ such that, for every $n, m \in \omega$ with $n < m$, $f^n(b) \neq f^m(b)$. But in this case

$$\{f^n(v) \neq c\colon n \in \omega, c \in M\}$$

is finitely satisfiable. (In fact, let $n_0, \ldots, n_k \in \omega$, $c_0, \ldots, c_k \in M$, consider

$$\{f^{n_i}(v) \neq c_i\colon i \leq k\}$$

and choose $h \in \omega$ such that, for all $i \geq h$, $f^i(b)$ does not occur among $c_0, c_1, \ldots, c_k$; then $f^{n_i} f^h(b) \neq c_0, \ldots, c_k$.) Then there is a type $q \in S_1(M)$ extending $\{f^n(v) \neq c\colon n \in \omega, c \in M\}$. q is regular, and $q \perp p^M$, a contradiction. Then (i) holds.

Recall:

(ii) There is $R \in \omega$ such that, for all $b \in U$ with $b \neq a$, $|f^{-1}(b)| \leq R$.

(i) and (ii) imply that, for every $M \models T$ and $x, y \models p^M$, $\tau(x, U)$ and $\tau(y, U)$ are finite and isomorphic.

Now consider $M \models T$ and a regular $q \in S_1(M)$. Let $z \models q$. As $q \not\perp p^M$, there is $n \leq N$ such that $f^n(z) = a$ and $f^{n-1}(z) \models p^M$. Then there are only finitely many non-algebraic 1-types over M. Hence T is ω-stable.

Case 2. For all $a \in U$, $|f^{-1}(a)| \leq R$. We claim that, in this case, for all non-algebraic $p_0, q_0 \in S_1(\emptyset)$, $M \models T$, $a \in p_0(M)$, there is $b \in q_0(M)$ such that $a \sim b$. Notice that, if this is true, then the proof is accomplished as it follows that $S_1(\emptyset)$ is countable and consequently, owing to Proposition 2, T is ω-stable.

Suppose towards a contradiction that there are non-algebraic $p_0, q_0 \in S_1(\emptyset)$, $M \models T$ and $a \in p_0(M)$ such that $q_0(M) \cap \gamma(a, M) = \emptyset$. For every $n, m \in \omega$, $\{b \in M\colon f^m(b) = f^n(a)\}$ is finite, hence there is a formula $\varphi_{nm}(v) \in q_0$ such that, if $b \in M$ and $f^m(b) = f^n(a)$, then $\models \neg \varphi_{nm}(b)$. It follows that, for every $n, m \in \omega$,

$$\forall w \quad (f^m(w) = f^n(v) \rightarrow \neg \varphi_{nm}(w)) \in p_0.$$

Consequently, for every $M \models T$, $a \in p_0(M)$, we have

$$q_0(M) \cap \gamma(a, M) = \emptyset.$$

Let now $M \models T$. The sets

$$p_0 \cup \{v \neq b\colon b \in M\}, \quad q_0 \cup \{v \neq b\colon b \in M\}$$

are finitely satisfiable in M because p_0, q_0 are not algebraic; let $p, q \in S_1(M)$ extend

$$p_0 \cup \{v \neq b\colon b \in M\}, \quad q_0 \cup \{v \neq b\colon b \in M\}$$

respectively; notice that p, q are uniquely determined (in fact, if $x \neq b$ for all $b \in M$, then $f^n(x) \neq b$ for all $b \in M$, $n \in \omega$). Then $p \not\perp q$, and hence (see Proposition 4.(ii) and its proof) there are $a \models p$, $b \models q$ such that $a \sim b$, a contradiction.

In [Si] a characterization of $\aleph_1$-categorical theories of a 1-ary function is given; of course this works also for unidimensional theories. We recall that [Si] includes a characterization of $\aleph_0$-categorical theories of a 1-ary function, too.

References

[B] J. Baldwin, *Fundamentals of stability theory*, Perspectives in Mathematical Logic, Springer, New York, 1988.

[CK] C.C. Chang and H.J. Keisler, *Model theory*, Studies in Logic 73, North Holland, Amsterdam, 1973.

[HM] L. Harrington and M. Makkai, *An exposition of Shelah's "Main Gap": counting uncountable models of ω-stable and superstable theories*, Notre Dame J. Formal Logic, vol. 26 (1985), pp. 139–178.

[M] M. Makkai, *A survey of basic stability theory, with particular emphasis on orthogonality and regular types*, Israel J. Math., vol. 49 (1984), pp. 181–238.

[R] A. Ryaskin, "The number of models of complete theories of unars" in *Model theory and its applications*, Tr. Inst. Mat., vol. 8 (1988), pp. 162–182, Novosibirsk (Russian).

[Sa] J. Saffe, *The number of uncountable models of ω-stable theories*, Ann. Pure Appl. Logic, vol. 24 (1983), pp. 231–261.

[Sh] S. Shelah, *Classification of first order theories which have a structure theorem*, Bull. Amer. Math. Soc., vol. 12 (1985), pp. 227–232.

[Si] Y. Shishmarev, *Categorical theories of a function*, Math. Zametki, Akad. Nauk SSSR, vol. 11 (1972), pp. 89–98 (Russian); English translation in Math. Notes, Acad. Sci. USSR, vol. 11 (1972), pp. 58–63.

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