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AN ELEMENTARY NONSTANDARD PROOF OF STONE'S REPRESENTATION THEOREM

BY

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Summary

A neat nonstandard proof of Stone's representation theorem is given. Improving on previous proofs (Loeb [5], Brunet [2]), it uses the remarkably simple fact that infinitesimal members of a filter on X, in any enlargement, are always compact for a natural topology on $^*X$.

1. Preliminaries

Let $\mathscr{E}$ be an enlargement containing a given set X. Recall the following: The standard subsets of $^*X$ form a base of open (and therefore closed) sets for a compact topology τ on $^*X$ (this topology is called "S-topology" by Luxemburg [6]). Moreover, given any filter $\mathscr{F}$ on X, its monad $\mu(\mathscr{F})$ is defined as the intersection of all $^*F$ where $F \in \mathscr{F}$, and there exists an internal subset I of $^*X$ such that $I \in {}^*\mathscr{F}$ and $I \subseteq \mu(\mathscr{F})$. Any such I is called an *infinitesimal member* of the filter $\mathscr{F}$ (see Machover and Hirschfeld [7], also Haddad [4]).

THEOREM 1. *Any infinitesimal member of a filter on X is τ-compact.*

Proof. Let I be an infinitesimal member of a filter $\mathscr{F}$ on X. Let $\mathscr{G}$ be any filter on X and suppose that, for any $G \in \mathscr{G}$, the subset $^*G$ meets I. It suffices to prove that the monad $\mu(\mathscr{G})$ meets I. Since any element of $\mathscr{G}$ meets every element of $\mathscr{F}$, there exists a filter $\mathscr{H}$ on X which is finer than both $\mathscr{F}$ and $\mathscr{G}$. Let J be an infinitesimal member of $\mathscr{H}$. Clearly, $J \subseteq \mu(\mathscr{H}) \subseteq \mu(\mathscr{G})$. Since $I \in {}^*\mathscr{F}$, it is an element of $^*\mathscr{H}$ and therefore meets J, so that $\mu(\mathscr{G})$ meets I.

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2. Stone's representation theorem

The spaces $\mathscr{L}^\infty$ and L^∞. Let $(X, \mathscr{M})$ be a measurable space and let $\mathscr{N}$ be a σ-ideal of $\mathscr{M}$, which means $\mathscr{N}$ is a subset of $\mathscr{M}$ such that:

(i) $\varnothing \in \mathscr{N}$ and $X \notin \mathscr{N}$;
(ii) for each $(A, B) \in \mathscr{M} \times \mathscr{N}$, if $A \subseteq B$, then $A \in \mathscr{N}$;
(iii) any countable union of elements of $\mathscr{N}$ is again an element of $\mathscr{N}$.

A mapping from X to $\mathbf{R}$ is said to belong to $\mathscr{L}^\infty$ whenever it is $\mathscr{M}$-measurable and is bounded on the complement N^C of some $N \in \mathscr{N}$.

For every $f \in \mathscr{L}^\infty$, let

$$\|f\| = \inf\{t \in \mathbf{R}_+ : \{x \in X : |f(x)| > t\} \in \mathscr{N}\}.$$

Call L^∞ the quotient space of $\mathscr{L}^\infty$ modulo the equivalence relation

$$f \equiv g \text{ whenever } \{x \in X : f(x) \neq g(x)\} \in \mathscr{N}.$$

Notice that the space L^∞ thus defined is complete metric.

THEOREM 2 (Stone). *There is a compact Hausdorff space $(\hat{X}, \hat{T})$ such that L^∞ is isometric to the space $C(\hat{X})$ of real continuous functions on $\hat{X}$ endowed with the uniform metric.*

In the following, let $\mathscr{E}$ be a given enlargement containing both $(X, \mathscr{M})$ and $\mathbf{R}$. Let $\mathscr{F} = \{N^C : N \in \mathscr{N}\}$. Clearly $\mathscr{F}$ is a filter on X.

Let Y be an infinitesimal member of that filter. Theorem 1 asserts that Y is τ-compact. Moreover, since $Y \subseteq \mu(\mathscr{F})$, and since $\{x \in X : |f(x)| \leq \|f\|\} \in \mathscr{F}$ for every $f \in \mathscr{L}^\infty$, we must have $|{}^*f(y)| \leq {}^*\|f\|$ for every $y \in Y$, so that $|{}^\circ({}^*f(y))| \leq \|f\|$.

For every $f \in \mathscr{L}^\infty$, let $\bar{f}$ denote the real-valued bounded function defined on Y such that $\bar{f}(y) = {}^\circ({}^*f(y))$ for each $y \in Y$. Consider the coarsest topology on Y for which all functions $\bar{f}$ are continuous, and call this topology $\bar{T}$. Then consider the quotient space $\hat{X}$ obtained from the space $(Y, \bar{T})$ and the equivalence $x \equiv y$ whenever $\bar{f}(x) = \bar{f}(y)$ for every $f \in \mathscr{L}^\infty$.

Let p denote the quotient map from Y onto $\hat{X}$, and let $\hat{y} = p(y)$ for $y \in Y$. For each $f \in \mathscr{L}^\infty$, define $\hat{f}$ such that $\hat{f}(\hat{y}) = \bar{f}(y)$ for $y \in Y$. Then let $\hat{T}$ denote the coarsest topology on $\hat{X}$ for which every $\hat{f}$ is continuous.

REMARK 1. Referring to Cutland's discussion [3, pp. 548–549] of Loeb's Ω [5, p. 77], we have to admit that our $\hat{X}$ is essentially the same as Loeb's Ω.

Note that Anderson's construction of $\hat{X}$ is also essentially the same as that of Loeb as he himself says [1, p. 676].

REMARK 2. Let $\|\hat{f}\| = \sup_{y \in Y} |\bar{f}(y)|$. For every $f \in \mathscr{L}^\infty$, we have $\|f\| = \|\hat{f}\|$. Indeed, on one hand, $|\bar{f}(y)| \leq \|f\|$ for every $y \in Y$ as has been noticed earlier. On the other hand, ${}^*\{x \in X\colon |f(x)| > t\} \subseteq Y^C$ for every real $t > \|\hat{f}\|$, so that $\|f\| \leq \|\hat{f}\|$.

PROPOSITION 2.1. *The space $(\hat{X}, \hat{T})$ is compact Hausdorff.*

Proof. The proof is a series of lemmas.

LEMMA 1. *For each element $f \in \mathscr{L}^\infty$, the function $\bar{f}$ is continuous in the τ-topology.*

Let $a < b$ in $\mathbf{R}$; then

$$\begin{aligned}
\bar{f}^{-1}(]a, b[) &= {}^*f^{-1}(st^{-1}(]a, b[)) \cap Y \\
&= \bigcup_{n \in \mathbf{N}} {}^*f^{-1}\left({}^*\left[a + \frac{1}{n}, b - \frac{1}{n}\right]\right) \cap Y \\
&= \bigcup_{n \in \mathbf{N}} {}^*\left(f^{-1}\left(\left[a + \frac{1}{n}, b - \frac{1}{n}\right]\right)\right) \cap Y
\end{aligned}$$

which is a τ-open subset of Y.

LEMMA 2. *The space $(Y, \overline{T})$ is compact.*

It follows from Lemma 1 that the topology $\overline{T}$ is coarser than the τ-topology on Y which is compact, hence the result.

LEMMA 3. *The space $(\hat{X}, \hat{T})$ is compact.*

Indeed, for each $f \in \mathscr{L}^\infty$, the map $\hat{f} \circ p = \bar{f}$ is a continuous real-valued function on $(Y, \overline{T})$. From the definition of the topology $\hat{T}$, it follows that p is continuous from $(Y, \overline{T})$ onto $(\hat{X}, \hat{T})$. The result follows from Lemma 2.

LEMMA 4. *The space $(\hat{X}, \hat{T})$ is Hausdorff.*

Let $\hat{x} \neq \hat{y} \in \hat{X}$; then by definition there is $f \in \mathscr{L}^\infty$ with $\hat{f}(\hat{x}) \neq \hat{f}(\hat{y})$. Hence $\{\hat{f}\colon f \in \mathscr{L}^\infty\}$ separates the points in $\hat{X}$, and since $\mathbf{R}$ is Hausdorff, so is $(\hat{X}, \hat{T})$.

REMARK 3. Let $f \in \mathscr{L}^\infty$ and $g \in \mathscr{L}^\infty$. If $f \equiv g$, then $\hat{f} = \hat{g}$.
Indeed, let

$$A = \{x \in X: f(x) = g(x)\}.$$

Since $A \in \mathscr{F}$, we know that $Y \subseteq {}^*A$, so that ${}^*f(y) = {}^*g(y)$ for each $y \in Y$. Thus we may regard $\hat{}$ as a mapping from L^∞ to $C(\hat{X})$. Let $\hat{}(L^\infty)$ be the image of L^∞ under this mapping.

PROPOSITION 2.2. *The spaces L^∞ and $C(\hat{X})$ are isometric.*

Proof. By construction (see Remark 2), the mapping $\hat{}$ is an isometry between L^∞ and $\hat{}(L^\infty)$. Since L^∞ is a complete metric space, clearly $\hat{}(L^\infty)$ is a closed subset in $C(\hat{X})$. Moreover, $\hat{}(L^\infty)$ is separating and contains the constants. So by the Stone-Weierstrass Theorem, $C(\hat{X})$ is equal to $\hat{}(L^\infty)$. This ends the proof of Proposition 2.2, whence the proof of Theorem 2.

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