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REPRESENTING MEASURES ON MULTIPLY CONNECTED PLANAR DOMAINS

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The linear functional $f \to f(a)$ of evaluation of an analytic function f at a point a in a g holed bounded planar domain admits representation in the form $f(a) = \int_{\partial D} f\,dm$, where the non-negative measure m supported on the boundary ∂D of D belongs to the g dimensional compact convex set M_a of representing measures for a. This convex set M_a of representing measures is a subset of the vector space $M_{\mathbf{R}}(\partial D)$ of real Borel measures on ∂D. By fixing a natural basis, the convex set M_a can be affinely identified with a convex set C_a in $\mathbf{R}^g$. Throughout this paper it will be assumed that the positively oriented boundary of D is the union

$$\partial D = b_0 \cup b_1 \cup \cdots \cup b_g$$

of the disjoint simple closed analytic curves $b_0, b_1, \dots, b_g$ with $b_1, \dots, b_g$ the boundaries of the holes and b_0 the boundary of the unbounded component of the complement.

It will be shown that the convex set C_a has the smooth parametrization π_a: $\mathbf{R}^g \to C_a$ given by

$$\pi_a(x) = \frac{1}{2\pi}\vec{\nabla}\left\{\log \frac{\theta(x)}{\theta(x+\omega_a)}\right\}, \tag{0.1}$$

where θ is the Riemann theta function associated with the Schottky double X of D. The vector constant ω_a appearing in (0.1) is $\omega_a = (\omega_1(a), \dots, \omega_g(a))$, where $\omega_j(a)$ is the harmonic measure of $b_j (j = 1, \dots, g)$ based at a. Since the θ function is $\mathbf{Z}^g$ periodic, then π_a provides a covering of C_a by the real g dimensional torus $\mathbf{T}_0 = \mathbf{R}^g/\mathbf{Z}^g$.

The parametrization (0.1) can be explained in the following manner. Let

$$\operatorname{Jac}(X) = \mathbf{C}^g/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$$

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be the Jacobian variety of the marked double, where τ is the "B-period" matrix. Using a translate Φ_a: $X^{(g)} \to \mathrm{Jac}(X)$ of the classical Abel-Jacobi map one can pull back (in a biholomorphic manner) the real torus $\mathbf{T}_0$ in $\mathrm{Jac}(X)$ to a real g dimensional variety V_a in the g fold symmetric product $X^{(g)}$. The torus V_a is a natural covering σ: $V_a \to B_a$ of the collection $B_a \subset X^{(g)}$ of critical divisors of elements in M_a. Note that each element dm in M_a is the restriction to ∂D of a symmetric meromorphic one-form dw on X. The g points (counting multiplicity) in the closure $\overline{D}$ of D where dw/dz vanishes constitute the critical divisor $\mathscr{D}_m$ of m. The elements in the fiber $\sigma^{-1}(\mathscr{D}_m)$, when $\mathscr{D}_m$ has k distinct points $p_1, \ldots, p_k$ with multiplicities $n_1, n_2, \ldots, n_k$ in D, consist of the $(n_1+1)(n_2+1)\cdots(n_k+1)$ divisors $\mathscr{D}$ in $X^{(g)}$ providing the decomposition of the zero divisor of dw in the form $(dw)_0 = \mathscr{D} + J\mathscr{D}$.

One nice feature is that we have a commutative diagram

$$\begin{array}{ccccc} V_a & & \xrightarrow{\quad \Phi_a \quad} & & \mathbf{T}_0 \\ \sigma\big\downarrow & & & & \big\downarrow \pi_a \\ B_a & \cong & M_a & \cong & C_a \end{array} \tag{0.2}$$

where the identifications "$\cong$" are canonical. The work of John D. Fay [5] is essential to the above results. First, the identification of V_a with $\mathbf{T}_0$ using Φ_a is simply a translation of Fay's characterization [5, p. 118] of the divisors of symmetric definite meromorphic differentials. Second, the explicit form of π_a uses a non-trivial theta function representation of meromorphic differentials by Fay [5, p. 25].

There are two tori of Hardy spaces which are closely related to the torus parametrization π_a: $\mathbf{T}_0 \to C_a$. Given $\mathscr{D}$ in V_a such that $\sigma(\mathscr{D}) = \mathscr{D}_m$ let $H^2_{\mathscr{D}^+}(dm)$ be the closure in $L^2(dm)$ of the meromorphic functions on $\overline{D}$ having at most poles at the restriction $\mathscr{D}^+$ of $\mathscr{D}$ to $\overline{D}$. The orthogonal complement $K^{2,a}_{\mathscr{D}^-}(dm)$ of $H^2_{\mathscr{D}^+}(dm)$ is the closure in $L^2(dm)$ of the meromorphic functions on $J\overline{D}$ vanishing at Ja having at most poles at the restriction $\mathscr{D}^-$ of $\mathscr{D}$ to $J\overline{D}$. Thus V_a or, equivalently, T_0 parametrizes a torus of Hardy space decompositions of $L^2(dm)$, $m \in M_a$. This torus of Hardy spaces provides the complete set of pure $C(\partial D)$-subnormal models for a completely contractive unital (c.c.u.) representation r_a of the closure $R = R(\overline{D})$ in $C(\partial D)$ of the rational functions with poles off $\overline{D}$. This representation r_a associates with f in R the operator on the one-dimensional Hilbert space $\mathbf{C}$ of multiplication by the complex number $f(a)$. The existence of this covering of M_a by a torus of single valued Hardy spaces follows from the work of Vern Paulsen [10].

The second torus of Hardy space models for the c.c.u. representation r_a of $R(\overline{D})$ was developed by Abrahamse and Douglas [1]. These models are the spaces $H_u^2(dm_a)$, $u = (u_1, \dots, u_g)$ in $\mathbf{T}^g$ ($\mathbf{T}$ is the unit circle in $\mathbf{C}$) consisting of the closure in L^2 of harmonic measure m_a based at a, of the multiplicative holomorphic functions on $\overline{D}$ whose continuation along b_j produces a change in the germ by the multiplicative factor u_j ($j = 1, \dots, g$).

The explicit correspondence between the torus V_a of single valued Hardy space models and the multiplicative Hardy space models $H_u^2(dm_a)$, $u \in \mathbf{T}^g$, is given. Indeed, if $\mathscr{D}$ is in V_a, there is an explicit unitary map $U_{\mathscr{D}}$ from $H^2_{\mathscr{D}^+}(dm)$ to $H_u^2(dm_a)$ intertwining the operator of multiplication by z on the spaces precisely when $\mathscr{D}$ and u are related by

$$u = \exp\left(-2\pi i\left[\Phi_a(\mathscr{D}) + \Phi(\mathscr{D}_{m_a}) + \omega_a\right]\right).$$

In essence, the translate of the Abel-Jacobi map linearizes the correspondence between the single valued and multiplicative tori of Hardy space models for the representation r_a of $R(\overline{D})$.

The structure of the remainder of this paper is as follows. Section 1 analyses the torus of divisors of representing measures. Section 2 describes the theta function parametrization of the convex set of representing measures. Section 3 establishes the explicit connections between two tori of Hilbert space models for the representation r_a of $\mathrm{Rat}(\overline{D})$ given by evaluation at a. Section 4 describes an example.

1. The divisors of representing measures

The double X of the g holed bounded planar domain is a compact Riemann surface $X = D \cup \partial D \cup D'$ of genus g where D' is a second copy of D glued to the bordered Riemann surface $D \cup \partial D$ along ∂D. The conformal structure on D' is the conjugate of the conformal structure on D. Thus the involution J: $X \to X$ which fixes ∂D and interchanges points in D with their twins in D' is anticonformal. We mark the double by completing the cycles $b_1, \dots, b_g$ to a canonical homology basis as follows. Fix p_0 in b_0. Let α_j be a crosscut in D from p_0 to a point on b_j and set a_j to be the cycle $a_j = \alpha_j \cup -J\alpha_j$, $j = 1, \dots, g$. Then $a_1, \dots, a_g$; $b_1, \dots, b_g$ is a canonical homology basis having the requisite intersection properties. From now on X refers to the double with this marked homology basis.

Let $G(z) = G(z, a)$ denote the Green's function for D with pole at a. The meromorphic differential

$$dw_a = \frac{1}{\pi i}\partial G dz \quad \text{on } \overline{D}$$

can be reflected to D' by setting $dw_a = \overline{J^* dw_a}$ on D'. The restriction dm_a of

dw_a to ∂D is harmonic measure based at a. That is

$$dm_a = -\frac{1}{2\pi}\frac{\partial G}{\partial \eta}\,ds = -\frac{1}{\pi i}\partial G dz|_{\partial D},$$

where $\partial/\partial\eta$ is the outward normal derivative and ds is arclength measure on ∂D. Thus the representing measure dm_a for evaluation of analytic functions at a is the restriction to ∂D of an element dw_a in the space of meromorphic differentials $\mathscr{M}^{(1)}(X)$ which is symmetric ($J^*dw_a = \overline{dw_a}$) and non-negative ($dw_a/ds \geq 0$) on ∂D. We next observe that these properties of dm_a are shared by all representing measures.

Let $\omega_j = \omega_j(z)$ be harmonic measure of b_j based at z in D and let dw_j be the reflection of the holomorphic differentials $\partial\omega_j\, dz$ to holomorphic differentials on X. The measures $dm_j = i\, dw_j|_{\partial D} j = 1, \ldots, g$ are real and form a basis for $R^\perp$ in $M_{\mathbf{R}}(\partial D)$. Consequently, any element m in M_a has a unique representation in the form

$$m = m_a + \sum_{j=1}^{g} c_j m_j,$$

where $c_m = (c_1, \ldots, c_g)$ is in $\mathbf{R}^g$. Thus dm is the restriction to ∂D of the element

$$dw = dw_a + i\sum_{j=1}^{g} c_j\, dw_j$$

in $\mathscr{M}^{(1)}(X)$. As a result we have identified M_a with the collection of elements dw in $\mathscr{M}^{(1)}(X)$ which are symmetric ($J^*\, dw = \overline{dw}$), non-negative ($dw/ds \geq 0$) having only simple poles at a, Ja with $2\pi i\,\text{Residue}[dw] = 1$ at a. The fact that every such meromorphic differential corresponds to a representing measure follows from the residue theorem.

For the remainder of this paper it will be assumed that the basis $m_1, \ldots, m_g$ of $R^\perp$ is fixed as above. Using this basis the coefficient map W_a: $M_a \to \mathbf{R}^g$ defined by $W_a(m) = c_m$ provides a linear affine bijection between M_a and the compact convex body $C_a = W_a(M_a)$ in $\mathbf{R}^g$. The fact that $\partial G/\partial\eta > 0$ on ∂D (see, Tsuji [15, p. 15]) insures that C_a contains a neighborhood of the origin in $\mathbf{R}^g$ and, consequently, C_a is g dimensional.

As noted above the mapping W_a from M_a to C_a is a bijection. In the sequel we will use the notation $m(c)$ for the element $W_a^{-1}(c)$, where $c =$

$(c_1, \ldots, c_g)$ is in M_a. Note

$$m(c) = m_a + \sum_{j=1}^{g} c_j m_j.$$

The divisor group $\operatorname{Div}(X)$ of X will be written additively. Consequently, the typical divisor $\mathscr{D}$ is a formal finite sum

$$\mathscr{D} = \sum_{p \in X} n_p p, \quad n_p \in \mathbf{Z},$$

and addition and comparison are done pointwise. The collection of non-negative divisors $\mathscr{D} = p_1 + \cdots + p_d$ of degree $d \geq 1$ will be identified with the d fold symmetric product $X^{(d)} = X^d / S_d$ where S_d is the symmetric group on d letters. Recall $X^{(d)}$ has the structure of a compact d dimensional complex space.

The pole-zero divisor of an element in $\mathscr{M}^{(1)}(X)$ has degree $2g - 2$. Consequently if dw in $\mathscr{M}^{(1)}(X)$ restricts on ∂D to a representing measure m in M_a, then

$$(dw) = \mathscr{D} + J\mathscr{D} - a - Ja \tag{1.1}$$

where $\mathscr{D}$ is in $X^{(g)}$. It is very important for our purposes to note that the presentation of (dw) in the form (1.1) is not unique. There is one $\mathscr{D}$ providing the representation (1.1) which is supported on $\overline{D}$. This divisor is denoted $\mathscr{D}_m$ and consists of the points in $\overline{D}$ where $dw/dz = 0$. Consequently, $\mathscr{D}_m$ is referred to as the critical divisor of $\mathscr{D}_m$. Note that since $dw/ds \geq 0$, zeros of dw/dz on ∂D are of even order. These boundary critical values of m are only counted in $\mathscr{D}_m$ with half order.

Suppose the critical divisor $\mathscr{D}_m$ restricted to D (not $\overline{D}$) has the form $n_1 p_1 + \cdots + n_s p_s$, where $p_1, \ldots, p_s$ in D are distinct. Then there are precisely $(n_1 + 1)(n_2 + 1) \cdots (n_s + 1)$ choices of $\mathscr{D}$ in $X^{(g)}$ providing the representation (1.1). Generically, $\mathscr{D}_m$ has g distinct points in D and in this case there will be 2^g ways of providing the representation (1.1) for some $\mathscr{D}$ in $X^{(g)}$. The divisors $\mathscr{D}$ satisfying (1.1), where $dw|_{\partial D} = dm$, can be conveniently viewed as the set of reflections of the critical divisor $\mathscr{D}_m$.

The collection of critical divisors $\{\mathscr{D}_m : m \in M_a\} \subset \overline{D}^{(g)}$ will be denoted by B_a. There is a natural bijection between B_a and M_a which associates a representing measure with its critical divisor. The notation V_a will be used for the collection of all divisors $\mathscr{D}$ in $X^{(g)}$ which provide the representation (1.1) for some dw with $dw|_{\partial D}$ in M_a. The usual retraction $r\colon X^{(g)} \to \overline{D}^{(g)}$ restricts to a "covering" map $\sigma\colon V_a \to B_a$ which is "branched" over $\mathscr{D}_m$ in B_a which have either critical values on ∂D or multiple critical values in D.

The Abel-Jacobi map allows us to identify V_a with the real torus $\mathbf{T}^g = \mathbf{R}^g / \mathbf{Z}^g$. We first recall the essentials of the Abel-Jacobi map. The holomorphic one-forms $dw_1, \ldots, dw_g$ introduced above form a basis for the space

$\Omega(X)$ of holomorphic differentials dual to the homology basis $a_1, \ldots, a_g$; $b_1, \ldots, b_g$ which we have used to mark X. This means the following. Let

$$d\vec{w} = (dw_1, \ldots, dw_g)^t$$

be the column vector constructed from $dw_1, \ldots, dw_g$. The $g \times 2g$ Riemann period matrix has the form

$$\left[\int_{a_1} d\vec{w} \cdots \int_{a_g} d\vec{w} \colon \int_{b_1} d\vec{w} \cdots \int_{b_g} d\vec{w}\right] = [I \colon \tau],$$

where I is the $g \times g$ identity matrix. It follows from Riemann's bilinear relation that the $g \times g$ symmetric complex B-period matrix τ has positive definitive imaginary part. Further, from the explicit form of $dw_1, \ldots, dw_g$ it is clear that for our marked double τ is purely imaginary. It follows from the general properties of τ mentioned above that $\tau = iP$ with P a real symmetric positive matrix. The complex torus $\mathrm{Jac}(X) = \mathbf{C}^g/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$ is called the Jacobian variety of the marked Riemann surface X. Note that because we are working with the double of a planar domain the anticonformal map $J[z] = -[\bar{z}]$ is well defined on $\mathrm{Jac}(X)$, where $[z]$ denotes the class of z in $\mathbf{C}^g$ modulo the period lattice $\mathbf{Z}^g + \tau\mathbf{Z}^g$.

The Abel-Jacobi map based at p_0 in X is the holomorphic map ζ_0: $X \to \mathrm{Jac}(X)$ defined by

$$\zeta_0(p) = \int_{p_0}^{p} d\vec{w} \bmod(\mathbf{Z}^g + \tau\mathbf{Z}^g).$$

This map extends linearly to $\mathrm{Div}(X)$. Jacobi's theorem states that the holomorphic map ζ_0: $X^{(d)} \to \mathrm{Jac}(X)$ is surjective for $d \geq g$. Abel's theorem establishes that $\zeta_0(\mathscr{D}_1) = \zeta_0(\mathscr{D}_2)$ for divisors $\mathscr{D}_1$, $\mathscr{D}_2$ of the same degree if and only if they are equivalent modulo principal divisors, that is, $\mathscr{D}_1 = \mathscr{D}_2 + (f)$ for some f in the algebra $\mathscr{M}(X)$ of meromorphic functions on X. Here we will always assume that the base point p_0 of the Abel-Jacobi map is in b_0. This leads to the symmetry $\zeta_0 \circ J = J \circ \zeta_0$.

It is necessary to work with a translate of the Abel-Jacobi map. Let Δ_0 be the classical Riemann constant based at p_0 in b_0. The explicit form of Δ_0 is

$$\Delta_0 = \left[-\sum_{k=1}^{g}\left\{\int_{a_k} \vec{w}_0(p)\, dw_k - \frac{1}{2}\tau_{kk}e_k\right\}\right], \tag{1.2}$$

where $\vec{w}_0(p) = \int_{p_0}^{p} d\vec{w}$ and $e_1, \ldots, e_g$ is the standard basis in $\mathbf{C}^g$. The constant Δ_0 plays a significant role in Riemann's study of the zero locus and singularities of the theta function. We will return to such matters below. For now we note that $-2\Delta_0 = K_X^0$, where $K_X^0 = \zeta_0((dw))$ for any dw in $\mathscr{M}^{(1)}(X)$. With the normalizations in effect here, we have $J\Delta_0 = \Delta_0$.

Define Φ_a: $X^{(g)} \to \mathrm{Jac}(X)$ by

$$\Phi_a(\mathscr{D}) = \zeta_0(\mathscr{D}) - \zeta_0(a) + \Delta_0.$$

The mapping Φ_a is independent of the base point p_0. It is trivial to check that for $\mathscr{D}$ in V_a satisfying (1.1)

$$\begin{aligned} J\Phi_a(\mathscr{D}) &= \zeta_0(J\mathscr{D}) - \zeta_0(Ja) + \Delta_0 \\ &= \zeta_0((dw)) - \Phi_a(\mathscr{D}) + 2\Delta_0 \\ &= -\Phi_a(\mathscr{D}). \end{aligned}$$

In other words Φ_a maps V_a into the subvariety

$$T = \{t \in \mathrm{Jac}(X): Jt = -t\}$$

of Jac(X). This subvariety T is the union over $\nu \in \mathbf{Z}^g/2\mathbf{Z}^g$ of the 2^g real g dimensional tori $T_\nu = \{\mu + i\nu: \mu \in \mathbf{R}^g/\mathbf{Z}^g\}$. Fay [5, p. 118] has characterized the subvariety T in the following manner. The torus $T_\nu, \nu = (\nu_1, \ldots, \nu_g)$, $\nu_i \in \mathbf{Z}/2\mathbf{Z}$, is precisely the set of points $\zeta_0(\mathscr{D}) + \Delta_0$ where $\mathscr{D} + J\mathscr{D}$ is the divisor of a symmetric meromorphic differential dw for which the sign of dw/ds is $(-1)^{\nu_k}$ on $b_k, k = 1, \ldots, g$, and $dw/ds \geq 0$ on b_0. In particular, $T_0 = \mathbf{T}^g = \mathbf{R}^g/\mathbf{Z}^g$ is the image in Jac(X) under $\mathscr{D} \to \zeta_0(\mathscr{D}) + \Delta_0$ of those divisors $\mathscr{D}$ of degree $g-1$ with $\mathscr{D} + J\mathscr{D}$ the divisor of a symmetric positive semidefinite ($dw/ds \geq 0$ on ∂D) meromorphic differential. The following result is just a translation of this result of Fay.

PROPOSITION 1.1. *Let Φ_a be the mapping from $X^{(g)}$ to* Jac(X) *defined by $\Phi_a(\mathscr{D}) = \zeta_0(\mathscr{D}) - \zeta_0(a) + \Delta_0$. Then Φ_a maps V_a bijectively onto the real torus $\mathbf{T}^g = \mathbf{R}^g/\mathbf{Z}^g$ in* Jac(X). *In fact, Φ_a maps a neighborhood of V_a biholomorphically onto a neighborhood of $\mathbf{T}^g$ in* Jac(X).

Proof. As indicated above the first assertion of the proposition follows from Fay [5, p. 118]. In order to see the biholomorphic nature of Φ_a: $V_a \to \mathbf{T}^g$ one argues as follows. For a divisor $\mathscr{D}$ let

$$i(\mathscr{D}) = \dim_{\mathbf{C}}\{dw \in \mathscr{M}^{(1)}(X): (dw) \geq \mathscr{D}\}$$

be the usual index. Let $X_1^{(g)}$ be the set of divisors $\mathscr{D}$ in $X^{(g)}$ with $i(\mathscr{D}) \geq 1$. It is known that

$$\zeta_0: X^{(g)} \sim X_1^{(g)} \to \mathrm{Jac}(X)$$

is biholomorphic onto its image (see, e.g., Farkas and Kra [4, p. 141]). Since Φ_a is a translate of ζ_0, we need only observe that $\mathscr{D}$ in V_a has index zero. Note that $i(\mathscr{D} - a) = i(\mathscr{D})$. If $i(\mathscr{D} - a) \geq 1$, then by the Riemann-Roch Theorem [4, p. 126] there would be a non-constant meromorphic f such that $\mathscr{D} - a + (f) \geq 0$. Let dw be a positive semidefinite symmetric meromorphic differential with $(dw) = \mathscr{D} - a + J(\mathscr{D} - a)$. Set $g = \overline{f \circ J}$. Then $gf\,dw$ would be holomorphic on X and non-negative on ∂D. This contradicts Cauchy's Theorem. It follows that $i(\mathscr{D} - a) = i(\mathscr{D}) = 0$ and the proof is complete.

Remark. The anticonformal involution J leaves V_a invariant. This involution transforms via Φ_a to the involution on $\mathbf{T}^g$ of reflection in the point $[-\omega_a/2]$. More specifically, it is easily verified that $\Phi_a J = R_a \Phi_a$, where $R_a([t]) = [-t - \omega_a]$, for $[t]$ in $\mathbf{T}^g$.

2. A theta function parametrization of the set of representing measures

In this section it will be shown that the mapping π_a defined by (0.1) completes the commutative diagram (0.2). The proof that π_a provides this parametrization of M_a involves a non-trivial representation of elements in $\mathscr{M}^{(1)}(X)$ in terms of the Klein prime form. This representation appears in the work of J. Fay [5, p. 25]. We will recall in as brief a manner as possible the relevant material dealing with theta functions. Our notations and normalizations are closely aligned with those in Mumford [7].

Associated with a symmetric $g \times g$ complex matrix τ which has positive definite imaginary part (i.e., τ is an element in the Siegel upper half-space) is the classical theta function

$$\theta(z, \tau) = \sum_{n \in \mathbf{Z}^g} \exp\left\{2\pi i\left(\frac{1}{2} n^t \tau n + n^t z\right)\right\}, \quad z \in \mathbf{C}^g.$$

This even entire function is quasi-periodic with respect to the period lattice $L_\tau = \mathbf{Z}^g + \tau\mathbf{Z}^g$ in the sense that for m, $n \in \mathbf{Z}^g$ and $z \in \mathbf{C}^g$,

$$\theta(z + m + n\tau, \tau) = \exp 2\pi i\left(-\frac{1}{2} n^t \tau n - n^t z\right)\theta(z, \tau).$$

In particular, θ is $\mathbf{Z}^g$ periodic.

The quasi-periodicity of θ implies that the subset Θ_r of the complex torus $\mathbf{C}^g/L_\tau$, where all derivatives of θ of order less than or equal to r vanish is well defined. Moreover, it is obvious that for c, d fixed in $\mathbf{C}^g$ the ratio

$$f(z) = \frac{\theta(z-c)}{\theta(z-d)}$$

provides an example of a multiplicative meromorphic function on the torus $\mathbf{C}^g/L_\tau$. In fact, the germs of this function f transform according to the rule

$$f(z+n) = f(z);\ f(z+\tau n) = \exp(2\pi i n^t \cdot (c-d))f(z)$$

for n in $\mathbf{Z}^g$. Thus the logarithmic derivatives

$$\frac{d}{dz_j}\log\left\{\frac{\theta(z-c)}{\theta(z-d)}\right\}, \qquad j = 1, \ldots, g,$$

are meromorphic on $\mathbf{C}^g/L_\tau$. Note that the component functions of our parametrization π_a given by (0.1) are of this latter form.

In the case where τ is the B-period matrix of a marked Riemann surface, theorems of Riemann describe Θ_0 and Θ_1. Suppose X is a compact Riemann surface with fixed canonical homology basis $a_1, \ldots, a_g$; $b_1, \ldots, b_g$. Let $d\vec{w} = (dw_1, \ldots, dw_g)^t$, where $dw_1, \ldots, dw_g$ is the normalized dual basis of $\Omega(X)$, $\tau = [\int_{b_j} dw_i]$ the B-period matrix and ζ_0: $X \to \operatorname{Jac}(X)$ the Abel-Jacobi map based at p_0 in X. Riemann has established the following two results.

1^0. There is an absolute constant Δ_0 given by (1.2) such that for e in $\mathbf{C}^g$ either

$$\theta\left(\int_{p_0}^{p} d\vec{w} - e\right)$$

vanishes identically or has precisely g zeros $p_1, \ldots, p_g$ such that $\zeta_0(p_1 + \cdots + p_g) + \Delta_0 = [e]$, where $[e]$ denotes the class of e in $\mathbf{C}^g/L_\tau$.

2^0. For $r \geq 0$, let $X_r^{(g-1)}$ be the subset of $X^{(g-1)}$ consisting of divisors $\mathscr{D}$ with $i(\mathscr{D}) \geq r+1$. Let W_r^{g-1} be the image of $X_r^{(g-1)}$ under the Abel-Jacobi map ζ_0: $X^{(g-1)} \to \operatorname{Jac}(X)$. Then

$$\Theta_r = W_r^{g-1} + \Delta_0.$$

Remark. The subset $X_1^{(g)}$ consisting of the points $\mathscr{D}$ in $X^{(g)}$, where $i(\mathscr{D}) \geq 1$ was used in the proof of Proposition 1.1. Let $W_1^g \subset \operatorname{Jac}(X)$ be the image of this subset under ζ_0. Then $W_1^g + \Delta_0$ is precisely the subset Θ^0 of those $[e]$ in $\operatorname{Jac}(X)$, where $\theta(\int_{p_0}^p d\vec{w} - e)$ vanishes identically. Consequently, the map $\eta_0(\mathscr{D}) = \zeta_0(\mathscr{D}) + \Delta_0$ maps $X^{(g)} \sim X_1^{(g)}$ biholomorphically onto $\operatorname{Jac}(X) \sim \Theta^0$.

Let us trace out the significance of the above remark for our situation of the marked double. Note first that for the case of a double

$$\theta(Jz, \tau) = \overline{\theta(z, \tau)},$$

where $Jz = -\bar{z}$, $z \in \mathbf{C}^g$. In particular, $\theta(z, \tau)$ is real valued and periodic on $\mathbf{R}^g$. We have noted that for x in $\mathbf{R}^g$, $[x] = \zeta_0(\mathscr{D}) + \Delta_0$, with $\mathscr{D} + J\mathscr{D}$ the divisor of a symmetric meromorphic differential with $i(\mathscr{D}) = 0$. Thus by the result of Riemann in 2^0, $\theta(x, \tau) \neq 0$ for all x in $\mathbf{R}^g$. Since $\theta(0, \tau) > 0$, then $\theta(x) = \theta(x, \tau) > 0$ for x in $\mathbf{R}^g$. It follows that $\pi_a\colon \mathbf{T}^g \to \mathbf{R}^g$ defined as in (0.1) by

$$\pi_a([x]) = \frac{1}{2\pi} \vec{\nabla} \log \frac{\theta(x)}{\theta(x + \omega_a)}$$

is a real analytic map of $\mathbf{T}^g$ to $\mathbf{R}^g$.

In order to introduce the Klein prime form it is convenient to work with theta functions having characteristics. Given e in $\mathbf{C}^g$, we can write $e = b + \tau a$ for unique a, b in $\mathbf{R}^g$. The (first order) theta function with characteristics a, b is defined by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \exp\left\{2\pi i\left(\frac{1}{2}a^t\tau a + a^t(z + b)\right)\right\}\theta(z + b + \tau a).$$

Obviously, $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is simple multiple of θ with argument translated by $e = b + \tau a$. The quasi-periodicy of θ implies, for m, n in $\mathbf{Z}^g$,

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z + m + \tau n, \tau)$$
$$= \exp\left\{2\pi i\left(a^t m - b^t n - \frac{1}{2}n^t\tau n - n^t z\right)\right\}\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau).$$

In particular, we can write down a multiplicative meromorphic function with arbitrary character as a ratio of theta functions with characteristics.

The 2^{2g} points of the form $\frac{1}{2}\mu + \frac{1}{2}\tau\nu$, μ, $\nu \in \mathbf{Z}^g/2\mathbf{Z}^g$ in $\mathbf{C}^g/L_\tau$ are called half-periods. These half-periods are called even or odd according to whether $\mu^t\nu$ is even or odd. The theta functions with even (odd) half integer characteristics are even (odd). In particular, the theta function $\theta(z, \tau)$ vanishes at the 2^{g-1} $(2^g - 1)$ odd half-periods. As is shown in Mumford [7, p. 208] there is a non-singular odd half-period. Fix such a point

$$[e_0] = \left[\tfrac{1}{2}\mu_0 + \tfrac{1}{2}\tau\nu_0\right] \qquad (\mu_0, \nu_0 \in \mathbf{Z}^g/2\mathbf{Z}^g)$$

in $\Theta_0 \sim \Theta_1$.

In case τ is the B-period matrix of a marked compact Riemann surface of genus g, the multiple valued holomorphic function defined on $X \times X$ by

$$F(p,q) = \theta[e_0]\left(\int_p^q d\vec{w}\right),$$

where $\theta[e_0]$ denotes the theta function with characteristics $\frac{1}{2}\nu_0$, $\frac{1}{2}\mu_0$, has the nice property that for p (respectively, q) fixed the multiple valued holomorphic function $F(p,\cdot\,)$ (respectively, $F(\cdot, q)$) has zero divisor $p_1 + \cdots + p_{g-1} + p$ (respectively, $p_1 + \cdots + p_{g-1} + q$) where the divisor $\mathcal{E} = p_1 + \cdots + p_{g-1}$ is independent of p (respectively, q) and satisfies

$$\zeta_0(\mathcal{E}) + \Delta_0 = [e_0].$$

In this last equation the Abel-Jacobi map and Riemann constant Δ_0 are computed relative to some fixed point p_0 in X. The above remark follows easily from the results 1^0 and 2^0 of Riemann.

Since $\zeta_0(2\mathcal{E}) = -2\Delta_0$, then there must be an element dw_{e_0} in $\Omega(X)$ with $(dw_{e_0}) = 2\mathcal{E}$. Indeed, this holomorphic one form is given by

$$dw_{e_0} = \sum_{j=1}^{g} \frac{d\theta}{dz_j}[e_0](0)\, dw_j.$$

The holomorphic line bundle $L_{\mathcal{E}}$ over X determined by the divisor class of $\mathcal{E}$ has the property that $L_{\mathcal{E}} \otimes L_{\mathcal{E}}$ is equivalent to the canonical bundle. Choose a holomorphic section $\sqrt{dw_{e_0}}$ of $L_{\mathcal{E}}$ with $(\sqrt{dw_e})^2 = dw_e$.

The prime form E is defined by

$$E(p,q) = \frac{\theta[e_0]\left(\int_p^q d\vec{w}\right)}{\sqrt{dw_{e_0}(p)}\,\sqrt{dw_{e_0}(q)}}.$$

This form $E(p,q)$ can be considered as a holomorphic form of weight $-\frac{1}{2}$, $-\frac{1}{2}$ on $\tilde{X} \times \tilde{X}$, where $\tilde{X}$ is the universal cover of X.

The prime form is a building block for the construction of differentials and functions on X. In the work below, only sectional interpretation of this form will be important. Further, the half-order differential $\sqrt{dw_{e_0}}$ can be conveniently cancelled in most of the formulae below.

Fix q_0. Then $E(p, q_0)$ is a multiple valued holomorphic differential of weight $-\frac{1}{2}$ in the variable p. The multiple valued nature of $E(p, q_0)$ (which arises from the function $\theta[e_0]$ appearing in the form $E(p,q)$) can be described as follows. Fix coordinate charts at p_0, q_0. Beginning and ending at

p_0 continue $E(p, q_0)$ along a cycle c which is homologous to $\Sigma(n_j a_j + m_j b_j)$. When $E(p, q_0)$ is computed near p_0, q_0 in the same coordinate charts, then this continuation produces the differential $E(p, q_0)$ multiplied by

$$\exp \pi i(\nu_0 m - \mu_0 n - m\tau\mu_0)\exp\left(2\pi \operatorname{im} \int_p^{q_0} d\vec{w}\right).$$

The most important feature is that the divisor of $E(p, q_0)$ is well defined and equals q_0.

Let $c \neq d$ be two points on the marked compact Riemann surface X. The notation $d\lambda_{c-d}$ will be used for the unique element in $\mathscr{M}^{(1)}(X)$ with simple poles at c, d and normalized so that

$$\int_{a_j} d\lambda_{c-d} = 0, \quad j = 1, \ldots, g; \ \operatorname*{Res}_{p=c} d\lambda_{c-d} = 1.$$

The following representation of $d\lambda_{c-d}$ is given as Formula (1) of Mumford [7, p. 3.224] and was first established by Fay [5, Prop. 3.10].

For z in $\mathbf{C}^g$ and $c \neq d$ in X

$$d\lambda_{c-d}(p) = \frac{E(c,d)}{E(c,p)E(p,d)} \frac{\theta\left(\int_p^d d\vec{w} + z\right)\theta\left(\int_c^p d\vec{w} + z\right)}{\theta\left(\int_c^d d\vec{w} + z\right)\theta(z)}$$
$$- \sum_{j=1}^{g} \frac{d}{dz_j} \log\left\{\frac{\theta\left(z + \int_c^d d\vec{w}\right)}{\theta(z)}\right\} dw_j(p). \tag{2.1}$$

We now return to the case where X is the marked double of a planar domain D. In this case, for a fixed in D, the normalized differential $d\lambda_{Ja-a}$ agrees with the meromorphic differential $d\Omega_{Ja-a}$ which is normalized to have the real parts of all periods zero with $(d\Omega_{Ja-a}) \geq -a - Ja$ and $\operatorname{Res}_{p=a} d\Omega_{Ja-a} = -1$. In fact, note that

$$J^* \, d\Omega_{Ja-a} = -\overline{d\Omega}_{Ja-a}.$$

Thus

$$\operatorname{Re} \int_{\alpha_k} d\Omega_{Ja-a} = \frac{1}{2}\int_{\alpha_k} (d\Omega_{Ja-a} - J^* \, d\Omega_{Ja-a})$$
$$= \frac{1}{2}\int_{a_k} d\Omega_{Ja-a}.$$

This shows that $d\Omega_{Ja-a} = d\lambda_{Ja-a}$.

There is a simple connection between $d\Omega_{Ja-a}$ and the element dw_a in $\mathscr{M}^{(1)}(X)$ which restricts to harmonic measure dm_0 on ∂D. This connection is

$$dw_a = \frac{1}{2\pi i}\, d\Omega_{a-Ja} = \frac{1}{2\pi i}\, d\lambda_{a-Ja}.$$

It follows from Fay's formula (2.1) that for any x in $\mathbf{R}^g$

$$\begin{aligned} dw_a(p) &+ \frac{1}{2\pi}\sum_{j=1}^{g}\frac{d}{dx_j}\log\left\{\frac{\theta(x)}{\theta(x+\omega_a)}\right\} d\eta_j(p) \\ &= \frac{1}{2\pi i}\,\frac{E(a,Ja)}{E(a,p)E(p,Ja)} \\ &\times \frac{\theta\left(\int_{p_0}^{p} d\vec{w} - \left(\int_{p_0}^{Ja} d\vec{w} - x\right)\right)\theta\left(\int_{p_0}^{p} d\vec{w} - \left(x + \int_{p_0}^{a} d\vec{w}\right)\right)}{\theta(x+\omega_a)\theta(x)} \end{aligned} \tag{2.2}$$

where $d\eta_j = i\,dw_j$ restricts to our basis of $R^{\perp}$ in $M_{\mathbf{R}}(\partial D)$. To obtain (2.2) from (2.1) one must let $z = -x$ in (2.1).

An examination of the right side of (2.2) shows that it is a meromorphic differential of the form $\gamma g_0\, dw_{e_0}$, where γ is a constant and g_0 is the meromorphic function

$$g_0(p) = \frac{\theta\left(\int_{p_0}^{p} d\vec{w} - \left(\int_{p_0}^{Ja} d\vec{w} - x\right)\right)\theta\left(\int_{p_0}^{p} d\vec{w} - \left(x + \int_{p_0}^{a} d\vec{w}\right)\right)}{\theta[e_0]\left(\int_{a}^{p} d\vec{w}\right)\theta[e_0]\left(\int_{a}^{p} d\vec{w}\right)\theta(x)\theta(x+\omega_a)}.$$

A careful computation of the multiplicative nature of g_0 shows how to choose the integral paths in order to make g_0 single valued.

Riemann's theorem described above shows the divisor of the differential on the right side of (2.2) has the form

$$p_1 + \cdots + p_g - a + J(p_1 + \cdots + p_g - a),$$

where, with $\mathscr{D} = p_1 + \cdots + p_g$,

$$\Phi_a(\mathscr{D}) = \zeta_0(\mathscr{D}) - \zeta_0(a) + \Delta_0 = [x].$$

Since $2\pi i\,\mathrm{Res}_{p=a}\, dw_a = 1$ and $\mathscr{D}$ is in V_a, we conclude the differential in

(2.2) restricts on ∂D to the element $m(c)$ in M_a, where c in C_a is given by

$$c = \frac{1}{2\pi}\vec{\nabla} \log \frac{\theta(x)}{\theta(x+\omega_a)}.$$

The above discussion completes the proof of the following:

THEOREM. *Let $[x]$ be in $\mathbf{T}^g = \mathbf{R}^g/\mathbf{Z}^g \subset \mathrm{Jac}(X)$, where X is the marked double of the planar domain D and a fixed in D. Then*

$$dm_a + \frac{1}{2\pi}\sum_{j=1}^{g} \frac{d}{dx_j} \log \frac{\theta(x)}{\theta(x+\omega_a)}\, dn_j \tag{2.3}$$

is a representing measure for evaluation at a. The critical divisor $\mathscr{D}$ of the representing measure (2.3) *is the unique point in B_a satisfying*

$$\Phi_a(\mathscr{D}) = [x].$$

Further, the mapping π_a: $\mathbf{T}^g \to \mathbf{R}^g$ defined by

$$\pi_a([x]) = \frac{1}{2\pi}\vec{\nabla} \log \frac{\theta(x)}{\theta(x+\omega_a)},$$

completes the commutative diagram (0.2).

Remarks. 1^0. A priori there is no reason to expect that the range of π_a: $\mathbf{T}^g \to \mathbf{R}^g$ is convex. The author would like to see an explanation of this convexity which uses only the properties of theta functions.

2^0. It is easily verified that π_a commutes with the involution on $\mathbf{T}^g$ of reflection in the point $[-\frac{1}{2}\omega_a]$.

3^0. The identity (2.2) can be viewed as a presentation of the representing measure (2.3) in the form $f\,d\lambda$, where

$$d\lambda(p) = \frac{1}{2\pi i}\frac{E(a, Ja)}{E(a,p)E(p,Ja)}$$

is a multiple valued differential and

$$f(p) = \frac{\theta\left(\int_a^p d\vec{w} - x\right)\theta\left(\int_a^p d\vec{w} + x + \omega_a\right)}{\theta(x)\theta(x+\omega_a)}$$

is a multiple valued holomorphic function on X with continuation independent of x in $\mathbf{R}^g$. The divisor of f is the sum $\mathscr{D}_1 + \mathscr{D}_2$, where $\mathscr{D}_1$, $\mathscr{D}_2$ are in V_a with $\Phi_a(\mathscr{D}_1) = [x]$ and $\Phi_a(\mathscr{D}_2) = [-x - \omega_a]$.

4^0. It follows from the commutative diagram (0.2) that $\pi_a \colon T_0 \to C_a$ is generically 2^g to 1. In fact, π_a is $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$ to 1 over $\pi_a \circ \phi_a(\mathscr{D})$, when $\mathscr{D}$ in V_a has k distinct points $p_1, \ldots, p_k$ in D with respective multiplicities $n_1, \ldots, n_k$.

5^0. It is possible to use Riemann's addition formula

$$\theta(u + v, \tau)\theta(u - v, \tau) = 2^{-g} \sum_{\eta \in \frac{1}{2}\mathbf{Z}^g/\mathbf{Z}^g} \theta\begin{bmatrix} 0 \\ \eta \end{bmatrix}\left(u, \frac{1}{2}\tau\right)\theta\begin{bmatrix} 0 \\ \eta \end{bmatrix}\left(v, \frac{1}{2}\tau\right)$$

combined with the representation in 3^0 to embed M_a into $\mathbf{R}^{2^g}$ as the range of $\rho_a \colon T_0 \to \mathbf{R}^{2^g}$ defined by

$$\rho_a(x) = \left(\ldots, \frac{\theta\begin{bmatrix} 0 \\ \eta \end{bmatrix}\left(x + \frac{1}{2}\omega_a, \frac{1}{2}\tau\right)}{\theta(x)\theta(x + \omega_a)}, \ldots \right)_{\eta \in \frac{1}{2}\mathbf{Z}^g/\mathbf{Z}^g}.$$

The mapping ρ_a also can be viewed as a theta function parametrization of M_a; however, ρ_a places M_a in a higher dimensional Euclidean space and does not appear to aid in the study of the convex geometry of M_a.

The author would like to thank Werner Kleinert for suggesting the possibility of this embedding into the "Kummer variety".

3. Hardy space models for representations of $R(\overline{D})$

In this section it will be shown how each point in the torus $\mathbf{T}^g$ corresponds to a natural Hardy space decomposition of $L^2(dm)$, where m is a representing measure for evaluation at a. The torus of Hardy spaces provides a complete set of models for the one-dimensional representation $f \to f(a)$ of $R(\overline{D})$ as an algebra of operators on the one-dimensional Hilbert space $\mathbf{C}$. This torus of models is explicitly related to another torus of models for this representation which was described by Abrahamse and Douglas [1].

The notations in this section are consistent with those given earlier. From (0.2) we have the commutative diagram

$$\begin{array}{ccc} V_a & \xrightarrow{\Phi_a} & \mathbf{T}^g \\ & {\scriptstyle \xi_a}\searrow \quad \swarrow{\scriptstyle \Psi_a} & \\ & M_a & \end{array}$$

where $\Psi_a = W_a^{-1}\pi_a$ and $\xi_a = \delta_a^{-1} \cdot \sigma$, where $\delta_a \colon M_a \to B_a$ is the natural bijection associating with a representing measure m in M_a its critical divisor $\delta_a(m) = \mathscr{D}_m$ in B_a. It will be shown how each point in the fiber $\xi_a^{-1}(m)$ leads to a natural orthogonal decomposition of $L^2(dm)$. Equivalently, each point $[x]$ in the real torus $\mathbf{T}^g = \mathbf{R}^g/\mathbf{Z}^g$ corresponds to a Hardy space decomposition of $L^2(dm)$, where $m = \Psi_a([x])$.

We begin with the following definition. Given a divisor $\mathscr{D}$ supported on $\overline{D}$, we let $L(\overline{D}\colon \mathscr{D})$ denote the collection of f in the space $\mathscr{M}(\overline{D})$ of meromorphic functions on $\overline{D}$ satisfying $(f) + \mathscr{D} \geq 0$. Given a representing measure m in M_a, let $H^2_{\mathscr{D}}(dm)$ be the closure of $L(\overline{D}\colon \mathscr{D}) \cap L^2(dm)$ in $L^2(dm)$. Similarly, for $\mathscr{D}$ supported on $J\overline{D}$, the space $L(J\overline{D}\colon \mathscr{D})$ has an analogous interpretation and the closure of $L(J\overline{D}\colon \mathscr{D}) \cap L^2(dm)$ in $L^2(dm)$ will be denoted $K^2_{\mathscr{D}}(dm)$. It is obvious that $K^2_{\mathscr{D}}(dm) = \overline{H^2_{J\mathscr{D}}(dm)}$. The notation $K^{2,a}_{\mathscr{D}}(dm)$ will denote the subspace $K^2_{\mathscr{D}}(dm)$ obtained as the closure in $L^2(dm)$ of the subspace of those f in $L(J\overline{D}\colon \mathscr{D})$ vanishing at Ja. We are particularly interested in these spaces when the divisor $\mathscr{D}$ arises from an element in V_a. In this case the following result holds.

THEOREM 3.1. *Let $\mathscr{D}$ be in V_a with $\xi_a(\mathscr{D}) = m$. Set $\mathscr{D}^+ = \mathscr{D}|\overline{D}$ and $\mathscr{D}^- = \mathscr{D}|J\overline{D}$. Then*

$$L^2(dm) = H^2_{\mathscr{D}^+}(dm) \oplus K^{2,a}_{\mathscr{D}^-}(dm). \tag{3.1}$$

Proof. Note first that the restrictions of $L(\overline{D}\colon \mathscr{D}^+)$ and $L(J\overline{D}\colon \mathscr{D}^-)$ belong to $L^2(dm)$. Further, if f is in $L(\overline{D}\colon \mathscr{D}^+)$ and g is in $L(J\overline{D}\colon \mathscr{D}^-)$ with $g(Ja) = 0$, then $f\bar{g}$ is the restriction to ∂D of an element k meromorphic on $\overline{D}$ with $(k) + \mathscr{D}_m - a \geq 0$. Thus $f\bar{g}\,dm = h(z)\,dz$ on ∂D, where h is holomorphic in a neighborhood of $\overline{D}$. As a consequence $\int f\bar{g}\,dm = 0$ and this shows that $H^2_{\mathscr{D}^+}(dm)$ is orthogonal to $K^{2,a}_{\mathscr{D}^-}(dm)$.

It remains to show that $L(\overline{D}\colon \mathscr{D}^+) + L(J\overline{D}\colon \mathscr{D}^-)$ is dense in $L^2(dm)$. Suppose $k\,dm$, $k \neq 0$ in $L^2(dm)$, annihilates $L(\overline{D}\colon \mathscr{D}^+) + L(J\overline{D}\colon \mathscr{D}^-)$. Then $k\,dm$ annihilates $\mathrm{Rat}(\overline{D})$ and $\overline{\mathrm{Rat}(\overline{D})}$ and, consequently, $k\,dm$ is the restriction to ∂D of an element $d\eta$ in $\Omega(X)$. Note it is now clear that k is meromorphically extendable to a neighborhood of $\overline{D}$. Since k annihilates $L(\overline{D}\colon \mathscr{D}^+) + L(J\overline{D}\colon \mathscr{D}^-)$, then $(d\eta) \geq \mathscr{D}|_{X-\partial D}$. Further, $d\eta$ must have a zero at each of the points in $\mathscr{D}|_{\partial D}$ (counting multiplicity). Indeed, if this were not the case, $k = d\eta/dm$ would have a double pole in ∂D at some point in the support of $\mathscr{D}$. This would imply $\int |k|^2\,dm = +\infty$, which is impossible. We conclude $(d\eta) \geq \mathscr{D}$ which yields $i(\mathscr{D} - a) \geq 1$ which we know is not true. Thus $k = 0$ and the proof is complete.

Remark. Roughly the above result is saying the following. Let m in M_a have critical divisor

$$\mathscr{D}_m = n_1P_1 + \cdots + n_sP_s + R_1 + \cdots + R_t$$

where $P_1, \dots, P_s$ are distinct points in D and $R_1, \dots, R_t$ are on ∂D. Corresponding to any of the $(n_1 + 1) \cdots (n_s + 1)$ divisors $\mathscr{D}$ in V_a,

$$\mathscr{D} = m_1 P_1 + \cdots + m_s P_s + m_1^1 JP_1 + \cdots + m_s^1 JP_s + R_1 + \cdots + R_t,$$

where m_j; m_j^1 are non-negative integers satisfying $m_j + m_j^1 = n_j$ ($j = 1, \dots, s$), there is an orthogonal decomposition (3.1) where the elements in $H^2_{\mathscr{D}^+}(dm)$ are meromorphic on D with poles allowed at $m_1 P_1 + \cdots + m_s P_s + R_1 + \cdots + R_t$ and the elements in $K^{2,a}_{\mathscr{D}^-}(dm)$ are meromorphic on $J\overline{D}$, vanish at J_a and are allowed poles at $m_1^1 JP_1 + \cdots + m_s^1 JP_s + R_1 + \cdots + R_t$.

The torus of Hardy spaces $H^2_{\mathscr{D}^+}(dm)$, $\mathscr{D} \in V_a$ with $\xi_a(\mathscr{D}) = m$ can be used to model dilations of one-dimensional representations of $\mathrm{Rat}(\overline{D})$ as an algebra of operators on Hilbert space. There is a second torus of multiplicative Hardy spaces which served the same purpose. The multiplicative Hardy space models were studied by Abrahamse and Douglas [1]. Below we want to describe the use of $H^2_{\mathscr{D}^+}(dm)$ as models and to give explicit unitary maps between the single valued and multiplicative models.

Much of the discussion below is due to Vern Paulsen. Indeed, Paulsen [10] worked out the theory of the torus of Hardy spaces $H^2_{\mathscr{D}^+}(dm)$ when D was doubly connected. The author's contribution here was to use the torus of J. Fay as described above to realize the torus of Hardy space models $H^2_{\mathscr{D}^+}(dm)$ in the higher genus. In addition, the explicit unitary maps between the single valued and multiplicative models appear here for the first time.

It is worth spending the extra effort to describe the dilation of representations for subalgebras of C^*-algebras. To this end let $\mathscr{A}$ be a subalgebra (with unit) of the C^*-algebra $\mathscr{B}$ and r_0: $\mathscr{A} \to \mathscr{L}(\mathscr{H})$ a unital representation of $\mathscr{A}$ as an algebra of operators on the Hilbert space $\mathscr{H}$. By a $\mathscr{B}$-dilation of r_0, we mean a representation π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$ of $\mathscr{B}$ as an algebra of operators on the superspace $\mathscr{K} \supset \mathscr{H}$ such that the restriction to $\mathscr{A}$ of the compression of π_0 to $\mathscr{H}$ is r_0. This means

$$r_0(a) = P_{\mathscr{H}} \pi_0(a)|_{\mathscr{H}}, \quad a \in \mathscr{A},$$

where $P_{\mathscr{H}}$: $\mathscr{K} \to \mathscr{H}$ is the orthogonal projection. It is only necessary to consider dilations which are minimal in the sense that $\pi_0(\mathscr{B})\mathscr{H}$ is dense in $\mathscr{K}$. Further, two $\mathscr{B}$-dilations π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$ and $\tilde{\pi}_0$: $\mathscr{B} \to \mathscr{L}(\tilde{\mathscr{K}})$ are said to be unitarily equivalent in case there is a unitary U: $\mathscr{K} \to \tilde{\mathscr{K}}$ with $Uh = h$, $h \in \mathscr{H}$, such that $U\pi_0(b) = \tilde{\pi}_0(b)U$, $b \in \mathscr{B}$.

Not every representation r_0: $\mathscr{A} \to \mathscr{L}(\mathscr{H})$ has a $\mathscr{B}$-dilation, however, completely contractive unital (c.c.u.) representations have $\mathscr{B}$-dilations. A unital representation r_0: $\mathscr{A} \to \mathscr{L}(\mathscr{H})$ is called a c.c.u. representation in case $r_0 \otimes I$: $\mathscr{A} \otimes M_n \to \mathscr{L}(\mathscr{H}) \otimes M_n$ is contractive for all n, where M_n denotes the algebra of complex $n \times n$-matrices. In this case the $\mathscr{B}$-dilations are constructed in two steps. The first step is to consider the completely positive map ϕ_0: $\mathscr{A} + \mathscr{A}^* \to \mathscr{L}(\mathscr{H})$ defined by

$$\phi_0(a_1 + a_2^*) = \phi_0(a_1) + (\phi_0(a_2))^*$$

and to use Arveson's extension theorem [2] to extend ϕ_0 to a completely positive map $\tilde{\phi}_0$: $\mathscr{B} \to \mathscr{L}(\mathscr{H})$. The second step in constructing the $\mathscr{B}$-dilations is to use Stinespring's theorem [14] to obtain a minimal representation π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$, where $\mathscr{H} \subset \mathscr{K}$ is such that $\tilde{\phi}_0 = P_{\mathscr{H}}\pi_0$. Obviously, π_0 obtained in this way is a $\mathscr{B}$-dilation. Indeed, every $\mathscr{B}$ dilation is obtained in this manner. As a consequence the $\mathscr{B}$-dilations are parametrized by the completely positive extensions $\tilde{\phi}_0$: $\mathscr{B} \to \mathscr{L}(\mathscr{H})$ of the completely positive map ϕ_0: $\mathscr{A} + \mathscr{A}^* \to \mathscr{L}(\mathscr{H})$. Again we emphasize that we speak only of minimal $\mathscr{B}$-dilations.

Let r_0: $\mathscr{A} \to \mathscr{L}(\mathscr{H})$ be a unital representation and π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{H})$ a $\mathscr{B}$-dilation of r_0. Then the multiplicative nature of r_0 forces $P_{\mathscr{H}}\pi_0(a_1a_2)|_{\mathscr{H}} = P_{\mathscr{H}}\pi_0(a_1)P_{\mathscr{H}}\pi_0(a_2)|_{\mathscr{H}}$, for elements a_1, a_2 in $\mathscr{A}$. That is, the map from $\pi_0(\mathscr{A})$ to $\mathscr{L}(\mathscr{H})$ sending $\pi_0(a)$ to its compression $P_{\mathscr{H}}\pi_0(a)|_{\mathscr{H}}$ is an algebra homomorphism. In this case one says that $\mathscr{H}$ is a semi-invariant subspace for $\pi_0(\mathscr{A})$. Sarason [13] has shown that semi-invariant subspaces are differences of invariant subspaces. This means that there is a nested pair $\mathscr{M} \subset \mathscr{N}$ of subspaces $\mathscr{M}$, $\mathscr{N}$ invariant under $\pi_0(\mathscr{A})$ such that $\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$. In general, the decomposition of $\mathscr{H}$ as the difference of invariant subspaces is not unique. Paulsen [10] has made a detailed study of the decomposition of a semi-invariant subspace as a difference of invariant subspaces (see, the example below). Paulsen introduces the concept of a $\mathscr{B}$-subnormal model for a representation r_0: $\mathscr{A} \to \mathscr{L}(\mathscr{H})$ as a triple $(\pi_0, \mathscr{M}, \mathscr{N})$, where π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$ is a $\mathscr{B}$-dilation of r_0 and $\mathscr{M}$, $\mathscr{N}$ are a nested pair $(\mathscr{M} \subset \mathscr{N})$ of $\pi_0(\mathscr{A})$ invariant subspaces such that $\mathscr{N} \ominus \mathscr{M} = \mathscr{H}$. An equivalence relation is given on the family of $\mathscr{B}$-subnormal models for r_0: $\mathscr{A} \to \mathscr{L}(\mathscr{H})$ by identifying $(\pi_0, \mathscr{M}, \mathscr{N})$ with $(\pi_0', \mathscr{M}', \mathscr{N}')$ in case the $\mathscr{B}$-dilations π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$ and π_0': $\mathscr{B} \to \mathscr{L}(\mathscr{K}')$ are unitarily equivalent as described above under the unitary U mapping $\mathscr{K}$ to $\mathscr{K}'$. The collection of unitary equivalence classes of $\mathscr{B}$-subnormal models can be considered as a fibration over the unitary equivalence classes of $\mathscr{B}$-dilations, where for π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$ a $\mathscr{B}$-dilation the fiber over this point consists of the unitary equivalence classes of $(\pi_0, \mathscr{M}, \mathscr{N})$ over all nested pairs $(\mathscr{M}, \mathscr{N})$ of $\pi_0(\mathscr{A})$ invariant subspaces with $\mathscr{N} \ominus \mathscr{M} = \mathscr{H}$. A complete set of representatives of the $\mathscr{B}$-subnormal models can be obtained by allowing π_0: $\mathscr{B} \to \mathscr{L}(\mathscr{K})$ to be the (minimal) Stinespring extensions of completely positive extensions to $\mathscr{B}$ of ϕ_0: $\mathscr{A} + \mathscr{A}^* \to \mathscr{L}(\mathscr{H})$ and allowing $(\mathscr{M}, \mathscr{N})$ to vary over all nested pairs $(\mathscr{M}, \mathscr{N})$ of $\pi_0(\mathscr{A})$ invariant subspaces satisfying $\mathscr{N} \ominus \mathscr{M} = \mathscr{H}$. These models are referred to as canonical subnormal models for r_0.

The relevant example here is the simplest representation of the algebra $\mathscr{A} = R(\overline{D})$, where $R(\overline{D})$ denotes the closure of $\mathrm{Rat}(\overline{D})$ in the C^*-algebra

$\mathscr{B} = C(\partial D)$ of continuous complex valued functions on the boundary of the domain D. This simplest representation is the homomorphism r_a: $R(\overline{D}) \to \mathscr{L}(\mathbf{C})$ which associates with each rational function f the operator on $\mathbf{C}$ of multiplication by $f(a)$. It is not difficult to see that r_a is a c.c.u. representation. Each Arveson extension ϕ_0: $C(\partial D) \to \mathscr{L}(\mathbf{C})$ is given by $\phi_0(f) = \int f\,dm$, for m in M_a. Thus the $C(\partial D)$-dilations of r_a are parametrized by M_a, where each m in M_a corresponds to the usual representation π_m: $C(\partial D) \to \mathscr{L}(L^2(dm))$ of $C(\partial D)$ as the algebra of multiplication operators on $L^2(dm)$.

If we fix m, then the canonical subnormal models that go with π_m are the triples $(\pi_m, \mathscr{M}, \mathscr{N})$, where $\mathscr{M} \subset \mathscr{N}$ are $\mathrm{Rat}(\overline{D})$ invariant subspaces of $L^2(dm)$ such that $\mathscr{N} \ominus \mathscr{M} = \mathbf{C}$, where $\mathbf{C}$ denotes the complex numbers identified as the constant functions in $L^2(dm)$. It turns out that the pairs $(\mathscr{M}, \mathscr{N})$ are of the form $(H^{2,a}_{\mathscr{D}^+}(dm), H^2_{\mathscr{D}^+}(dm))$, where $\mathscr{D}$ is in V_a and satisfies $\xi_a(\mathscr{D}) = m$ and $H^{2,a}_{\mathscr{D}^+}(dm)$ is the closure of $L(\overline{D}\colon \mathscr{D}_+ - a)$ in $L^2(dm)$.

We do not give a direct proof of the above description of the subnormal models. Such a direct proof based on the work of Paulsen [10] is possible. The method we use here is to write down explicit unitary maps between $H^2_{\mathscr{D}^+}(dm)$ and the multiplicative Hardy space models of Abrahamse and Douglas [1] and Sarason [11]. We must first describe these multiplicative Hardy spaces.

The discussion here is limited to the case of scalar valued functions. Suppose $u = (u_1, \ldots, u_g)$, $|u_1| = \cdots = |u_g| = 1$, is a point in the g-torus $\mathbf{T}^g$. Setting $\chi_u(b_j) = u_j$, $j = 1, \ldots, g$, defines a homology character on $\overline{D}$. Suppose f is a multiple valued meromorphic function on $\overline{D}$ which admits continuation along any path in $\overline{D}$. Then f is said to be a multiplicative meromorphic function belonging to the character χ_u in case continuation of the germ f_0 along a closed path γ produces the germ $f_1 = \chi_u(\gamma) f_0$. Note that because we consider only unimodular characters, then these multiplicative meromorphic functions have the property that $|f|$ is automorphic with respect to the group of deck transformations on the universal cover of D. For this reason these multiplicative meromorphic functions are called modulus automorphic. Two other natural interpretations of the multiplicative meromorphic functions are possible. One interpretation (see, e.g., Ball [3]) is obtained by taking a system $\alpha_1, \ldots, \alpha_g$ of crosscuts from $b_1, \ldots, b_g$ to b_0 and consider meromorphic functions on $D - \bigcup_{j=1}^{g} \alpha_j$ with multiplicative jump by a factor u_j across α_j $(j = 1, \ldots, g)$. Alternatively, one can use the character χ_u to define a flat unitary line bundle $\mathscr{E}_u$ over D and consider the multiplicative meromorphic functions as sections of $\mathscr{E}_u$ [1].

Now fix a in D and continue to let m_a denote harmonic measure on ∂D based at a. For χ_u, u in $\mathbf{T}^g$, a character, let $H^2_u = H^2_u(D, m_a)$ denote the closure in $L^2(dm_a)$ of holomorphic functions on $\overline{D}$ belonging to the character χ_u. The operator S_u defined on f in H^2_u by $S_u f(z) = z f(z)$ is called a bundle shift. The operator S_u is clearly the restriction to H^2_u of the normal operator $Nf(z) = zf(z)$ acting on $L^2(dm_a)$. Further the operator S_u is pure in the sense that it has no reducing subspaces on which it acts as a normal operator. Consequently, S_u is a pure subnormal operator. The space H_u^2 is a functional Hilbert space. This means that given p in D there is an e_p^u in H_u^2 such that $\langle f, e_p^u \rangle = f(p)$, for all f in H_u^2. In particular $H_u^{2,a} = \{f \in H_u^2: f(a) = 0\}$ is a closed subspace of H_u^2, which indeed is $H_u^2 \ominus \operatorname{span}(e_a^u)$. The remarks appearing in this paragraph are in Abrahamse and Douglas [1].

We are now in a position to show how to obtain a subnormal model for r_a: $R(\overline{D}) \to \mathscr{L}(\mathbf{C})$ using each multiplicative Hardy space H_u^2, $u \in \mathbf{T}^g$. First we impact $\mathbf{C}$ onto the subspace $\mathscr{H}_0$ spanned by e_a^u using the isometry $\lambda \to \lambda \|e_a^u\|^{-1} e_a^u$. The representation r_a: $R(\overline{D}) \to \mathscr{L}(\mathbf{C})$ transplants to $r_a(f)h = f(a)h$, $h \in \mathscr{H}_0$, which is a representation of $R(\overline{D})$ on $\mathscr{L}(\mathscr{H}_0)$. Obviously, π_u: $C(\partial D) \to L^2(dm_a)$ sending f in $C(\partial D)$ to the operator on $L^2(dm_a)$ of multiplication by f is a $C(\partial D)$-dilation of r_a: $R(\overline{D}) \to \mathscr{L}(\mathscr{H}_0)$.

The triples $(\pi_u, H_u^{2,a}, H_u^2)$, $u \in \mathbf{T}^g$, represent all pure $C(\partial D)$-subnormal models for r_a. This last result is due to Abrahamse and Douglas [1]. The next theorem sets up an explicit correspondence between the multiplicative and canonical $C(\partial D)$-subnormal models.

THEOREM 3.2. *Let $\mathscr{D}$ be in V_a with $\xi_a(\mathscr{D}) = m$. Suppose $\Phi_a(\mathscr{D})$ is the point t in the torus $T_0 = \mathbf{R}^g/\mathbf{Z}^g$. Let u in $\mathbf{T}^g$ be defined by*

$$u = \exp\Big(-2\pi i\big(t\Phi_a(\mathscr{D}_{m_a}) + w_a\big)\Big), \tag{3.2}$$

where exponentiation is done componentwise. There is a unitary transformation $U_{\mathscr{D}}$: $H^2_{\mathscr{D}^+}(dm) \to H_u^2$ such that $U_{\mathscr{D}} S_{\mathscr{D}} = S_u U_{\mathscr{D}}$, where $S_{\mathscr{D}}$ is the subnormal operator of multiplication by z on $H^2_{\mathscr{D}^+}(dm)$. As a consequence the pure subnormal models

$$\big(\pi_{\mathscr{D}}, H^{2,a}_{\mathscr{D}^+}(dm), H^2_{\mathscr{D}^+}(dm)\big) \quad \textit{and} \quad \big(\pi_u, H_u^{2,a}, H_u^2\big)$$

are equivalent if and only if (3.2) holds. Moreover, the operator $U_{\mathscr{D}}$ is multiplication by the restriction to ∂D of an explicit multiplicative meromorphic function on X.

Before beginning the proof of the above theorem we make the following remarks:

(i) The unitary $U_{\mathscr{D}}$ is constructed using the theta functions associated with X. This is not too surprising since the theta function is a natural tool providing explicit correspondence between line bundles and divisors. The argument here is complicated by the fact that the correspondence between the spaces $H_u^2(D\colon dm_a)$ and $H^2_{\mathscr{D}^+}(dm)$ is required to be unitary and the measure dm is, in general, different than dm_a.

(ii) The big picture is now clear. We have the three equivalent tori coverings: (1) $\sigma\colon V_a \to B_a$, (2) $\pi_a\colon \mathbf{R}^g/\mathbf{Z}^g \to C_a$, (3) the tori

$$\{(\pi_{\mathscr{D}}, H^{2,a}_{\mathscr{D}^+}(dm), H^2_{\mathscr{D}^+}(dm)) \colon \mathscr{D} \in V_a\}$$

of single-valued pure subnormal models of r_a covering the completely positive extensions M_a of r_a. Moreover, there are explicit fiber preserving maps relating these coverings. In addition there is the fourth torus $\{(\pi_u, H^{2,\alpha}_u, H^2_u)\colon u \in \mathbf{T}^g\}$ of "multiplicative" pure subnormal models of r_a. The unitarily equivalent single-valued and multiplicative pure subnormal operators are determined by Theorem 3.2.

The proof of Theorem 3.2 will require some preliminaries. Recall that for P, Q $(P \neq 0)$ on the marked double, the notation $d\Omega_{P-Q}$ is used for the unique element in $\mathscr{M}^{(1)}(X)$ having real parts of all periods equal to zero and such that

$$(d\Omega_{P-Q}) \geq -P - Q \quad \text{with } \operatorname*{Res}_{z=Q} d\Omega_{P-Q} = -1.$$

Given a divisor $\mathscr{D} = \sum_{k=1}^r (P_k - Q_k)$ of degree zero the function

$$V_{\mathscr{D}} = \exp\left(\sum_{k=1}^{r} \int_{P_0}^{p} d\Omega_{P_k - Q_k} \right)$$

is a multiplicative meromorphic function with divisor $(V_{\mathscr{D}}) = \mathscr{D}$ which belongs to a unimodular character $\chi_{\mathscr{D}}\colon \pi_1(X) \to \mathbf{T}$.

For a divisor $\mathscr{D}$ of degree zero there is only one (up to a constant multiple) multiplicative meromorphic function having divisor $\mathscr{D}$ belonging to a unimodular character. In other words, $V_{\mathscr{D}}$ and $\chi_{\mathscr{D}}$ associated with a divisor $\mathscr{D}$ of degree zero are unique. The function $V_{\mathscr{D}}$ can be given in terms of theta functions (or the prime form).

Note that in the case where the divisor $\mathscr{D}$ has the form

$$\mathscr{D} = \mathscr{D}_1 - J\mathscr{D}_1, \mathscr{D}_1 = \sum_{k=1}^{r} P_k, P_1, \ldots, P_r \text{ in } \overline{D},$$

then the multiplicative function $V_{\mathscr{D}}$ having divisor $\mathscr{D}$ belongs to the normalized character $\chi_{\mathscr{D}}$ given by

$$\chi_{\mathscr{D}}(a_j) = 1; \chi_{\mathscr{D}}(b_j) = \exp\left(-2\pi i \sum_{k=1}^{r} \omega_j(P_k) \right), \quad j = 1, \ldots, g.$$

Further for $\mathscr{D}$ of the form $\mathscr{D} = \mathscr{D}_1 - J\mathscr{D}_1$ the function $V_{\mathscr{D}}$ has modulus 1 on ∂D.

Proof of Theorem 3.2. Let $\mathscr{D}$ be in V_a with $\xi_a(\mathscr{D}) = m$. Let

$$\mathscr{D}_1 = \mathscr{D}_m - 2\mathscr{D}^+ - J(\mathscr{D}_m - 2\mathscr{D}^+), \quad \mathscr{D}_a = \mathscr{D}_{m_a} - J\mathscr{D}_{m_a}$$

and set

$$h = \frac{V_{\mathscr{D}_a}}{V_{\mathscr{D}_1}} \frac{dm}{dm_a}.$$

The function h belongs to a normalized character and divisor

$$(h) = 2J\mathscr{D}_m - 2J\mathscr{D}_{m_a} + 2\mathscr{D}^+ - 2J\mathscr{D}^+.$$

Further, $|h| = dm/dm_a$ on ∂D. Find a multiplicative meromorphic function V belonging to a normalized character χ_0 with divisor

$$(V) = J\mathscr{D}_m - J\mathscr{D}_{m_a} + \mathscr{D}^+ - J\mathscr{D}^+$$

satisfying $V^2 = h$. This V can be given in the explicit form

$$V = C_0 \exp\left\{ - \sum_{k=1}^{g} \int_{P_0}^{p} d\Omega_{JP_k - JQ_k} + \sum_{j=1}^{s} \int_{P_0}^{p} d\Omega_{Q_{ij} - JQ_{ij}} \right\},$$

where

$$\mathscr{D}_{m_a} = P_1 + \cdots + P_g, \mathscr{D}_m = Q_1 + \cdots + Q_g, \mathscr{D}^+ = Q_{i_1} + \cdots + Q_{i_s}$$

and C_0 is a constant.

A short computation establishes that the character χ of V satisfies $\chi(b_j) = u_j = \exp 2\pi i t_j + (\Phi_a(\mathscr{D}_{m_a}))_j + \omega_j(a)$, $j = 1, \ldots, g$. As a consequence, the unitary mapping

$$U_{\mathscr{D}} f = Vf, \quad f \in H^2_{\mathscr{D}^+}(dm)$$

carries $H^2_{\mathscr{D}^+}(dm)$ onto H^2_u with $U_{\mathscr{D}} S_{\mathscr{D}} = S_u U_{\mathscr{D}}$. This completes the proof of the theorem.

4. Examples

The examples discussed here were introduced by Nash [8] and Sarason [12] in investigations of the convex geometry of the space of representing measures.

The earlier notations and conventions remain in effect. Let a be fixed in D and δ_a: $M_a \to B_a$ the map which associates the critical divisor with a representing measure. A measure m in M_a is in the boundary of M_a if and only if the support of $\mathscr{D}_m$ intersects ∂D [8, p. 131]. This last remark can be seen by considering the homeomorphism $\delta_a^{-1} = W_a^{-1}\pi_a\Phi_a$: $B_a \to M_a$. This homeomorphism extends to a homeomorphism of a neighborhood of B_a in $X^{(g)}$ to a neighborhood of M_a in $M_{\mathbf{R}}(\partial D)$. It is clear that the annihilator $R^\perp$ of $\operatorname{Rat}(\overline{D})$ in $M_{\mathbf{R}}(\partial D)$ can be identified with the set of restrictions to ∂D of the space $\Omega_s(X)$ of symmetric holomorphic differentials on X.

The following results of Sarason and Nash give information about the convex geometry of M_a in terms of critical divisors.

1^0. Sarason [12]. (See also Lemma 3.2 of [8].) A measure m in M_a fails to be an extreme point of M_a if and only if there is an ν element in $R^\perp$ such that $(d\nu|\partial D) \geq (dm|\partial D)$. In particular, since elements in $R^\perp$ have divisors of degree $2g-2$, then any m in M_a with $\mathscr{D}_m$ supported on ∂D is an extreme point of M_a.

2^0. Nash [8]. The measure m in ∂M_a is an isolated extreme point if and only if $\mathscr{D}_m$ is supported on ∂D. Further, if ∂M_a contains an isolated extreme point, then D is conformally equivalent to the complex sphere minus a finite number of closed slits in the real axis.

Note that if the critical divisor is supported on ∂D, then $2\Phi_a(\mathscr{D}_m) = -[\omega_a]$. Thus $\Phi_a(\mathscr{D}_m)$ is one of the 2^g possibilities

$$\left[-\frac{1}{2}\omega_a + \frac{1}{2}n\right], \quad n \in \mathbf{Z}^g/2\mathbf{Z}^g.$$

In particular, M_a has at most 2^g isolated extreme points. (In general, there are 2^g solutions of $2\Phi_a(\mathscr{D}) = -[\omega_a]$ in V_a; however, these solutions will not be supported on ∂D.) We will examine a class of examples where the maximum number of isolated extreme points occurs.

Let D be a domain obtained from the open unit disc $\{z\colon |z| < 1\}$ by removing g disjoint closed discs $\overline{D}_1, \dots, \overline{D}_g$ centered at points $d_1, \dots, d_g$ in the real axis. For definiteness it can be assumed that $d_1 < d_2 < \cdots < d_g$. The intersection of D with the real axis is the union of the $g+1$ open subintervals

$$I_0 = (s_0, s_1), I_1 = (s_2, s_3), \dots, I_g = (s_{2g}, s_{2g+1}),$$

where $-1 = s_0 < s_1 < \cdots < s_{2g} < s_{2g+1} = 1$. The closure of I_j is denoted $\bar{I}_j$, $j = 0, \dots, g$.

On the double X of D the interval $\bar{I}_j$ forms the lower half of the "circle" $\mathbf{T}_j = \bar{I}_j \cup J\bar{I}_j$, $j = 0, \dots, g$. In addition, to the anticonformal symmetry J: $X \to X$ the double X has the extra anticonformal symmetry Q: $X \to X$ of reflection in the real axis. The holomorphic involution QJ is such that X/QJ

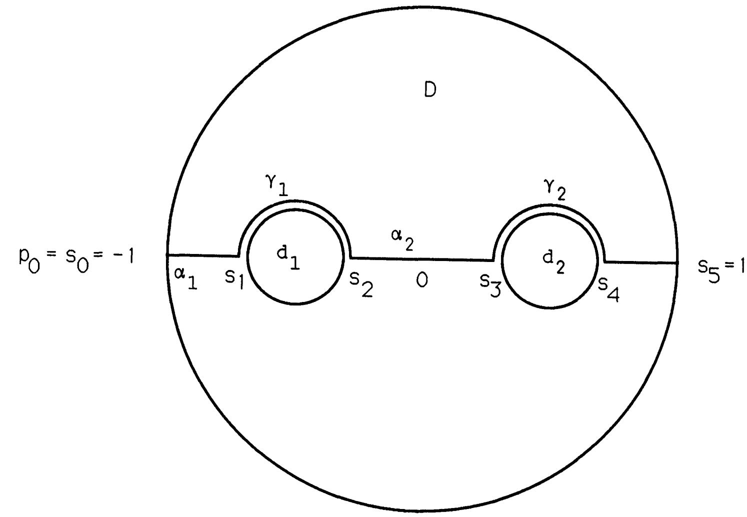

FIG. 1

is the complex sphere and, consequently, these doubles are hyperelliptic. Fix the base point $p_0 = s_0 = -1$. A convenient choice for the crosscuts $\alpha_1, \ldots, \alpha_g$ is to let α_j join p_0 to s_{2j-1} by running along the real axis and the top halves $\gamma_k (k = 1, \ldots, j-1)$ of the boundaries b_k of D_k. See Fig. 1 for $g = 2$.

Let $dw_1, \ldots, dw_g$ be our usual basis of $\Omega(X)$ dual to the canonical homology basis $a_1, \ldots, a_g; b_1, \ldots, b_g$, where as above $a_k = \alpha_k \cup (-J\alpha_k)$, $k = 1, \ldots, g$. It is obvious from the symmetry that

$$\int_{\gamma_i} dw_j = \tfrac{1}{2}\tau_{ij},$$

where $\tau = [\tau_{ij}]$ is the B-period matrix of the marked double.

On the real axis $dw_j = \frac{1}{2}\partial_x \omega_j \, dx$ and, consequently, the divisor of dw_j is of the form $\mathscr{D}_j + J\mathscr{D}_j$, where $\mathscr{D}_j \geq 0$ is of degree $g - 1$ and consists of one point from each of the subintervals $I_0, I_1, \ldots, I_g$ which do not abut b_j. Moreover, the differentials dw_j satisfy

$$Q^* \, dw_j = \overline{dw_j} \quad \text{and} \quad J^*(dw_j) = -\overline{dw_j}, \quad j = 1, \ldots, g.$$

For real a the Green's differential

$$dw_a = \frac{1}{2\pi i} \, d\Omega_{a-Ja}$$

also possesses the symmetries

$$Q^* \, dw_a = \overline{dw_a} \quad \text{and} \quad J^* \, dw_a = -\overline{dw_a}.$$

Thus any differential dw in $M^{(1)}(X)$ whose restriction to ∂D is in M_a also possesses these symmetries. In particular, for real a the critical divisor $\mathscr{D}_m$ of m in M_a is supported on the union $\bar{I}_0 \cup \bar{I}_1 \cup \cdots \cup \bar{I}_g$.

The Riemann constant Δ_0 based at $p_0 = -1$ is a half-period. By a direct computation using our basis of $\Omega(X)$ one can conclude that

$$\Delta_0 = \left[\frac{1}{2}\vec{1} + \frac{1}{2}\vec{g}\right], \tag{4.1}$$

where $\vec{1} = (1, 1, \dots, 1)^t$ and $\vec{g} = (g, g-1, \dots, 2, 1)^t$. The direct derivation of (4.1) is a bit bothersome. (See, also Mumford [7, 3.81].)

Fix a real in D. The critical divisor $\mathscr{D}_m$ of m in M_a consists of exactly one point from each of the g closed subintervals $\bar{I}_0, \bar{I}_1, \dots, \bar{I}_g$ which do not contain a. This can be established as in [8, Lemma 3.13] or by using the explicit form of Δ_0. In fact, it is a simple exercise to show that for $\mathscr{D}$ a divisor of degree g supported on $\bar{D}$ intersected with the real axis

$$\zeta_0(\mathscr{D}) - \zeta_0(a) + \Delta_0$$

is in $\mathbf{R}^g/\mathbf{Z}^g$ if and only if $\mathscr{D}$ consists of one point from each of the g closed subintervals $\bar{I}_0, \bar{I}_1, \dots, \bar{I}_g$ which do not contain a.

PROPOSITION. *Let D be a domain obtained from the unit disc by removing g disjoint closed discs centered at points on the real axis. Let $I_0, I_1, \dots, I_g$ be the open intervals forming the intersection of D with the real axis and $\mathbf{T}_j = \bar{I}_j \cup J\bar{I}_j$ the "circles" obtained by reflecting the closure $\bar{I}_j$ of I_j into the double X of D, $j = 0, 1, \dots, g$.*

Fix a real in D. The collection B_a in $X^{(g)}$ of critical divisors of representing measures in M_a is

$$B_a = J_1 \times \cdots \times J_g \subset X^{(g)},$$

where $J_1, \dots, J_g$ are those intervals $\bar{I}_0, \dots, \bar{I}_g$ not containing a. The torus of Hardy spaces V_a is

$$V_a = \mathbf{T}_1' x \cdots x \mathbf{T}_g' \subset X^{(g)},$$

where $\mathbf{T}_1', \dots, \mathbf{T}_g'$ are those circles $\mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_g$ not containing a.

Further given $\mathscr{D}_1 = \mathrm{p}_1 + \cdots + \mathrm{p}_g$, where $\mathrm{p}_j \in \mathbf{T}_j'$, $\mathrm{j} = 1, \dots, \mathrm{g}$, then $\mathscr{D}_1$ is the critical divisor of the representing measure m given as in (2.3) with

$[x] = \Phi_a(\mathscr{D}_1)$. Any divisor $\mathscr{D}$ in V_a over $\mathscr{D}_1$ corresponds to the orthogonal decomposition (3.1) of $L^2(dm)$.

Remark. The above proposition gives very explicit information concerning M_a for a real. For example, it is clear that M_a has 2^g extreme points. This answers a question of Nash [8, p. 134]. The case where a is not real is not so simple. An example in Sarason [12, p. 376] shows that, in general, V_a is not a product of circles from X.

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