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SOME BOUNDEDNESS RESULTS FOR ZERO-CYCLES ON SURFACES

BY

JAMES D. LEWIS[1]

Introduction

Let X be a smooth, complex projective algebraic surface (which will be assumed throughout the rest of this paper). For any smooth variety V of dimension n, we will denote by $CH^k(V)$ the corresponding Chow group of algebraic cycles of codimension k in V (modulo *rational* equivalence), and write $CH_{n-k}(V) = CH^k(V)$. Our main focus of attention is on the subgroup $A_0(X)$ of zero-cycles of *degree* 0 in $CH_0(X)$, and more particularly on $T(X) =$ kernel of the *Albanese* map $\mathring{a}$: $A_0(X) \to Alb(X)$. Before stating the main theorem ((0.3)), we introduce the following terminology.

(0.1) DEFINITION. Let $B_0(X)$ be a subgroup of $A_0(X)$. We say that $A_0(X)/B_0(X)$ is finite dimensional if there exists a (possibly reducible) smooth curve E, a cycle z in $CH^2(E \times X)$ such that the composite

$$J(E) \xrightarrow{z_*} A_0(X) \longrightarrow A_0(X)/B_0(X)$$

is surjective.

Example. We can write $T(X) = A_0(X)/B_0(X)$ where $B_0(X)$ is defined as follows. By Poincaré's complete reducibility theorem, there exists an abelian variety B and a homomorphism f such that the composite

$$B \xrightarrow{f} A_0(X) \xrightarrow{\mathring{a}} Alb(X)$$

is an isogeny (see [8: (1.2)]). Clearly $T(X) + f(B) = A_0(X)$, moreover using $T(X)$ torsionless [7] it follows that $f(B) \cap T(X) = 0$. Now set $B_0(X) = f(B)$.

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We remark that in general $T(X)$ is *not* finite dimensional (see (0.4)(ii) below).

Let $Pic^0(X)$ be the Picard variety of X (with Lie algebra $H^{0,1}(X)$), and consider the following schema setting (see also [1(1): p. 1.11]):

$$
\begin{array}{ccccccc}
H^{0,1}(X)\otimes_{\mathbf{C}} H^{0,1}(X) & \xrightarrow{\wedge} & H^{0,2}(X) & \longrightarrow & V^{0,2} & \longrightarrow & 0\\
\updownarrow & & \updownarrow & & \updownarrow & & \\
Pic^0(X)\otimes_{\mathbf{Z}} Pic^0(X) & \xrightarrow{\cap} & T(X) & \longrightarrow & \text{Coker} & \longrightarrow & 0
\end{array}
\tag{0.2}
$$

where $V^{0,2}$ and Coker are the respective cokernels. We prove:

(0.3) THEOREM. (1) *Suppose $V^{0,2} \neq 0$. Then Coker is infinite dimensional.*
(2) *Conversely, if Kodaira dimension $\mathscr{K}(X) \leq 1$, then Coker $= 0$ if $V^{0,2} = 0$.*

(0.4) *Remarks.* (i) It is reasonable to conjecture that for all smooth surfaces $X, V^{0,2} = 0$ iff Coker $= 0$.

Some examples in support of this conjecture are the following. If X is the Fano surface of lines on a smooth cubic threefold (see [1(1): Ex. 1.7)] or [3]), or $X =$ abelian surface (see [1(2): (A.2)]), or say $X = E \times C$ where E, C are smooth curves (cf. below), then $V^{0,2} = 0$ and Coker $= 0$.

(ii) It is a theorem of Mumford ([6]) that $H^{0,2}(X) \neq 0$ implies $T(X)$ infinite dimensional, moreover there is a conjecture of Bloch (converse result) that states "$H^{0,2}(X) = 0$ implies $T(X) = 0$ (equivalently $A_0(X)$ finite dimensional by [7(2); §4])", which has been verified in the case $\mathscr{K}(X) \leq 1$ ([1(2)]). Now suppose $\mathscr{K}(X) = 2$ and $H^{0,2}(X) = 0$. Then by [2: (5.1), p. 395, i.e. $\mathscr{X}(O_X) \geq 1$] it follows that $\dim H^{0,1}(X) = q \leq Pg = \dim H^{0,2}(X) = 0$. In particular the conjecture in (0.4)(i) above implies Bloch's conjecture.

(iii) Let Y be an abelian surface and $X \approx Y/\pm 1$ the corresponding Kummer counterpart with *rational* map β: $Y \to X$. Then $H^{0,1}(X) = 0$, $H^{0,2}(X) \cong \mathbf{C}$; however we do have

$$H^{0,1}(Y) \otimes H^{0,1}(Y) \xrightarrow{\wedge} \!\!\!\!\rightarrow H^{0,2}(Y) \xrightarrow{\beta_*}\!\!\!\!\rightarrow H^{0,2}(X);$$

moreover

$$Pic^0(Y)^{\otimes_{\mathbf{Z}} 2} \xrightarrow{\beta_* \circ \cap} T(X)$$

is surjective. One expects a similar phenomena to hold for all smooth surfaces X. Specifically, for a surface X and filtration of

$$CH_0(X) = CH^2(X)\colon F^0CH^2(X) = CH^2(X) \supset F^1CH^2(X)$$
$$= A_0(X) \supset F^2CH^2(X) = T(X) \supset 0,$$

one hopes for the situation on the *graded* piece $Gr^2(CH^2(X))$ below:

$$Gr^0(CH^2(X)) = \mathbf{Z}\ \text{(discrete part)};$$
$$Gr^1(CH^2(X)) = A^{\otimes_{\mathbf{Z}}1} = A;$$
$$Gr^2(CH^2(X)) = T(X) \twoheadleftarrow B^{\otimes_{\mathbf{Z}}2}$$

where $A = Alb(X)$ and B are abelian varieties. A general boundedness conjecture for $CH^k(V)$ will appear in a future paper.

Acknowledgement. Shortly after this paper was completed, we discovered a thesis by T. Fatemi (cf. [3]) cited in [1], which overlaps with some of the results of this paper. To be specific, Theorem (0.3) *part (2)* above is proven in [3; §1], and the example of the Fano surface of lines in (0.4)(i) above is established in [3; §2]. In regard to the conjecture in (0.4)(i) above, we should remark that *the surjectivity of the cup product* (i.e., $V^{0,2} = 0$) *should imply* Coker $= 0$ is cited in [3; p. 1] as a conjecture of Bloch. Finally our proof of (0.3)(2) (given in Section 2 of this paper) is different in character to that given in [3; §1]. Our desire to include a proof of (0.3)(2) is based on our understanding that [3] doesn't appear to be published in the literature, and that these results should be accessible to a larger group of mathematicians.

1. Proof of main theorem, part (1)

Let L_X be the fundamental class of a hyperplane section of X. Recall that by cup product, there is an isomorphism

$$L_X\colon H^1(X,\mathbf{Q}) \xrightarrow{\sim} H^3(X,\mathbf{Q}) \quad \text{(hard Lefschetz)}$$

with inverse denoted by F_X. Note that F_X determines a corresponding class $[F_X]$ in $[H^3(X,\mathbf{Q})]^* \otimes H^1(X,\mathbf{Q}) = H^1(X,\mathbf{Q}) \otimes H^1(X,\mathbf{Q})$ (Poincaré duality), and in particular since F_X respects Hodge type, it follows from the Künneth formula that $[F_X]$ lies in $H^2(X \otimes X,\mathbf{Q}) \cap H^{1,1}(X \otimes X)$. By the Lefschetz (1, 1) theorem, $[F_X]$ is algebraic (over $\mathbf{Q}$). Choose an integer $N \neq 0$ for which $N[F_X]$ is integral algebraic, and in particular lets view (by abuse of notation) NF_X as a divisor on $X \times X$. If $Pr_j\colon X \times X \twoheadrightarrow X$ are the projections onto the first and second factors ($j = 1, 2$), then NF_X determines a homomorphism

$$\{NF_X\}_*: CH_0(X) \to CH^1(X) = Pic(X)$$

by the formula

$$\{NF_X\}_*(y) = Pr_{2*}(\{Pr_1^*(y) \cap NF_X\}_X).$$

Since taking intersections is an algebraic operation (explicitly $\{y_1 \cap y_2\}_X = \Delta^*(y_1 \times y_2)$ where Δ: $X \hookrightarrow X \times X$ is the diagonal), it follows from Poincaré duality that there is a cycle E in $CH^2(X \times X \times X)$ and commutative diagrams below:

(1.1)
$$\begin{tikzcd} CH_0(X) \otimes_{\mathbf{Z}} CH_0(X) \arrow[r, "\{NF_X\}_*^{\otimes 2}"] \arrow[d, twoheadrightarrow] & Pic(X) \otimes_{\mathbf{Z}} Pic(X) \arrow[d, "\cap"] \\ CH_0(X \times X) \arrow[r, "E_*"] & CH_0(X) \end{tikzcd}$$

(1.2)
$$\begin{tikzcd} H^{1,2}(X) \otimes_{\mathbf{C}} H^{1,2}(X) \arrow[r, "\sim"] \arrow[d, hook] & H^{0,1}(X) \otimes_{\mathbf{C}} H^{0,1}(X) \arrow[r, "\wedge"] & H^{0,2}(X) \arrow[r, twoheadrightarrow] \arrow[d, hook] & V^{0,2} \\ H^6(X \times X) \arrow[rr, "{[E]_*}"] & & H^2(X), & \end{tikzcd}$$

where the corresponding $[E]_*$ can be defined via the Künneth formula on $H^4(X \times X \times X)$ together with Poincaré duality (e.g., see [5: (2.3)]). We remark that since

$$[NF_X]_*: H^{1,2}(X) \xrightarrow{\sim} H^{0,1}(X)$$

is an isomorphism with NF_X integral, it follows that there is an induced isogeny $[NF_X]_*$: Alb$(X) \xrightarrow{\approx} Pic^0(X)$, and therefore

$$\{NF_X\}_*(A_0(X)) = Pic^0(X)$$

by *universality* of $\{\mathring{a}$: $A_0(X) \to Alb(X)\}$. From (1.1) we deduce:

(1.3) COROLLARY. *The image of* $\cap \circ \{Pic^0(X)^{\otimes_{\mathbf{Z}} 2}\}$ *is contained in* $E_*\{CH_0(X \times X)\}$.

We now apply Serre duality to (1.2) to arrive at the exact sequence

$$0 \longrightarrow V^{2,0} \longrightarrow H^{2,0}(X) \xrightarrow{[E]^*} H^{1,0}(X) \otimes_{\mathbf{C}} H^{1,0}(X) \subset H^{2,0}(X \times X),$$

where $V^{2,0} = \ker[E]^*$ is dual to $V^{0,2}$. We introduce the following objects. If V is a smooth variety, we denote by $\mathscr{S}^m(V)$ the m^{th}—symmetric product of V. The singular set of $\mathscr{S}^m(V)$ will be concentrated on $\{p_1 + \cdots + p_m | \text{not all the } p_i\text{'s are distinct}\}$. Likewise we define

$$\mathscr{S}^{n,m}(V) = \mathscr{S}^n(V) \times \mathscr{S}^m(V)$$

and corresponding maps

$$T_m \colon \mathscr{S}^m(V) \to CH_0(V) \text{ and } T_{n,m} \colon \mathscr{S}^{n,m}(V) \to CH_0(V)$$

where $T_{n,m}(y_1, y_2) = \{y_1 - y_2\} \in CH_0(V)$. It is clear that

$$CH_0(V) = \bigcup \{T_{n,m}(\mathscr{S}^{n,m}(V)) | n, m \geq 1\}.$$

Let w be a holomorphic k-form on V. There are canonically defined "k-forms" ω_m and $\omega_{n,m}$ which are regular outside the respective singular sets of $\mathscr{S}^m(V)$, $\mathscr{S}^{n,m}(V)$, defined as follows (see [7(2): §3]). For any cartesian product, let Pr_j be the jth projection. Then $w_m = \Sigma_j Pr_j^*(w)$ defines a k-form on V^m invariant under the action of the symmetric group on m letters, hence there is an induced ω_m on $\mathscr{S}^m(V)$. Likewise on $\mathscr{S}^{n,m}(V)$, we define $\omega_{n,m} = Pr_1^*(\omega_n) - Pr_2^*(\omega_m)$, and for convenience of notation in the discussion below, we will suppress the Pr_j^*'s and for example write $\omega_{n,m} = \omega_n - \omega_m$.

We also make use of the terminology "c-closed" from [7(2)] which means "*countable* union of closed subvarieties".

(1.4) PROPOSITION. *$T_{n,m}^{-1}(E_*(CH_0(X \times X))$ is c-closed in $\mathscr{S}^{n,m}(X)$.*

Proof. Immediate from [7(2): Lemma 5].

We now choose w to be a non-zero (two)-form in $V^{2,0}$ (i.e., $[E]^*(w) = 0$). Let $\underline{\Sigma}_{n,m}$ be an irreducible component of $T_{n,m}^{-1}(E_*(CH_0(X \times X))$ with desingularization $\Sigma_{n,m}$ and corresponding morphism

$$j_{n,m} \colon \Sigma_{n,m} \to \mathscr{S}^{n,m}(X).$$

The main technical assertion is:

(1.5) LEMMA. $j_{n,m}^*(\omega_{n,m}) = 0$.

Remark. Assuming the lemma, it follows from Mumford's theory (e.g., see [7(2): §3]) that $CH_0(X)/\mathrm{Im}(E_*)$ is "infinite dimensional", under the assumption $V^{0,2} \neq 0$.

Proof of Lemma. Using countability arguments, it follows that for given positive integers n and m, there exist positive integers q and r such that

$$T_{n,m}(j_{n,m}(\Sigma_{n,m})) \subset E_*\big(T_{q,r}(\mathscr{S}^{q,r}(X \times X))\big).$$

Set

$$\mathscr{W} = \big\{(y_1, y_2) \in \mathscr{S}^{q,r}(X \times X) \times \Sigma_{n,m} | E_*\big(T_{q,r}(y_1)\big) = T_{n,m}(j_{n,m}(y_2))\big\},$$

a c-closed subset of $\mathscr{S}^{q,r}(X \times X) \times \Sigma_{n,m}$ [7(2): Lemma 3]. It follows that there exists an irreducible component $\mathbf{W} \subset \mathscr{W}$ for which $Pr_2(\mathbf{W}) = \Sigma_{n,m}$. Let W be the desingularization of $\mathbf{W}$. By taking generic hyperplane sections of W and applying Bertini's theorem, there is no loss of generality in assuming $\dim W = \dim \Sigma_{n,m}$. It follows that there exists morphisms f and g and the commutative diagram below:

$$\begin{tikzcd} W \arrow[r, two heads, "f"] \arrow[d, "g"'] & \Sigma_{n,m} \arrow[r, "j_{n,m}"] \arrow[d, "T_{n,m} \circ j_{n,m}"] & \mathscr{S}^{n,m}(X) \arrow[dl, "T_{n,m}"] \\ \mathscr{S}^{q,r}(X \times X) \arrow[r, "E_* \circ T_{q,r}"] & CH_0(X) & \end{tikzcd} \tag{1.6}$$

To prove the lemma, it suffices to show that $(j_{n,m} \circ f)^*(\omega_{n,m}) = 0$. Now viewing E as a codimension 2-cycle in $X \times X \times X$, we may assume the irreducible components of E are in "sufficiently general" position (via rational equivalence & Chow's moving lemma). Specifically, on $\mathscr{S}^{q,r}(X \times X)$, E defines a corresponding *rational* map

$$\{E\}: \mathscr{S}^{q,r}(X \times X) \to \mathscr{S}^{u,v}(X)$$

for some positive integers u and v, moreover we may assume the restriction of $\{E\}$ to $g(W)$ is also rational. Let

$$H: \mathscr{S}^{u,v}(X) \times \mathscr{S}^{n,m}(X) \to \mathscr{S}^{u+m, v+n}(X)$$

be the map given by

$$H(y_1, y_2, y_3, y_4) = (y_1 + y_4, y_2 + y_3)$$

and

$$\mathbf{J}: W \to \mathscr{S}^{u+m, v+n}(X)$$

the rational map given by the formula

$$\mathbf{J} = H(\{E\} \circ g, j_{n,m} \circ f).$$

By construction of $\mathbf{J}$ and the commutivity of (1.6) it follows that $T_{u+m, v+n} \circ \mathbf{J}$ maps a non-empty Zariski open subset of W to a point in $CH_0(X)$. Therefore the pullback of $\omega_{u+m, v+n}$ to $\mathbf{J}(W)$ is zero [7(2): §3], a fortiori $\mathbf{J}^*(\omega_{u+m, v+n}) = 0$ in W. However

$$H^*(\omega_{u+m, v+n}) = \omega_{u,v} - \omega_{n,m}$$

and therefore

$$0 = \mathbf{J}^*(\omega_{u+m, v+n}) = (\{E\} \circ g)^*(\omega_{u,v}) - (j_{n,m} \circ f)^*(\omega_{n,m});$$

moreover

$$(\{E\} \circ g)^*(\omega_{u,v}) = g^*(\{E\}^*(\omega_{u,v})).$$

But recall $[E]^*(w) = 0$, and therefore $\{E\}^*(\omega_{u,v}) = 0$, a fortiori

$$(j_{n,m} \circ f)^*(\omega_{n,m}) = 0,$$

which concludes the proof of the lemma.

(1.7) Corollary. *Let m be a positive integer. Let Σ be an irreducible subvariety passing through a generic point of $\mathscr{S}^{m,m}(X)$ and contained in a fiber of the map*

$$\pi \circ T_{m,m}: \mathscr{S}^{m,m}(X) \xrightarrow{T_{m,m}} A_0(X) \xrightarrow{\pi} A_0(X)/\{\cap \circ Pic^0(X)^{\otimes_{\mathbf{Z}} 2}\}.$$

Then $\dim \Sigma \leq 2m$.

Proof. Immediate from (1.3), (1.5) and [7(2): §3].

We now pretend (0.3)(1) is *false*, i.e., Coker in (0.2) is finite dimensional, and arrive at a contradiction. By a standard argument, there exists a (possibly reducible) smooth curve E and $z \in CH^2(E \times X)$ such that

$$z_*: J(E) \twoheadrightarrow A_0(X)/\{\cap \circ Pic^0(X)^{\otimes_{\mathbf{Z}} 2}\}$$

is surjective. Let

$$\mathbf{H} = \{(a,b) \in \mathscr{S}^{m,m}(X) \times J(E) | \pi \circ T_{m,m}(a) = z_*(b)\},$$

which is c-closed using the general results in [7(2)]. As in the proof of the above lemma, we can find an irreducible $H \subset \mathbf{H}$ for which

$$Pr_1(H) = \mathscr{S}^{m,m}(X) \text{ and } \dim H = \dim \mathscr{S}^{m,m}(X) \ (= 4m).$$

Let N be the degree of $Pr_1|_H$: $H \twoheadrightarrow \mathscr{S}^{m,m}(X)$ and if we denote by [ ] the Pontryagin sum on $J(E)$, then define a rational map f: $\mathscr{S}^{m,m}(X) \to J(E)$ by the formula

$$f(p) = \left[Pr_2 \circ (Pr_1|_H)^{-1}(p)\right].$$

Note that by construction, $z_* \circ f(p) = N \cdot \pi \circ T_{m,m}(p)$ for generic $p \in \mathscr{S}^{m,m}(X)$. The results of this paper apply equally if we replace $T_{n,m}$ by $N \cdot T_{n,m}$, specifically if we replace $T_{m,m}$ by $N \cdot T_{m,m}$ in Corollary (1.7) (hence $\pi \circ T_{m,m}$ by $N \cdot \pi \circ T_{m,m}$), the conclusion of (1.7) remains the same.

Now let $p \in \mathscr{S}^{m,m}(X)$ be a generic point and Σ a subvariety of maximal dimension passing through p for which Σ is contained in a fiber of $N \cdot T_{m,m}$. By (1.7), $\dim \Sigma \le 2m$, on the other hand Σ contains an irreducible component of a fiber of f, hence

$$\dim \Sigma \ge \dim \mathscr{S}^{m,m}(X) - \dim J(E) = 4m - \dim J(E).$$

The desired contradiction is obtained by choosing $m > (\dim J(E))/2$.

2. Proof of main theorem, part (2)

We begin with an analysis of the simplest prototypical case of X mentioned in (0.4)(i), namely $X = E \times F$ where E, F are smooth curves, and prove:

(2.1) Proposition. *The following maps are onto*:

$$H^{0,1}(X) \otimes_{\mathbf{C}} H^{0,1}(X) \xrightarrow{\wedge} H^{0,2}(X) \tag{1}$$

$$Pic^0(X) \otimes_{\mathbf{Z}} Pic^0(X) \xrightarrow{\cap} T(X) \tag{2}$$

Proof. Part (1). By the Künneth formula

$$\begin{aligned} H^{0,2}(X) &= H^{0,1}(E) \otimes H^{0,1}(F) \\ &= \left(H^{0,1}(E) \otimes H^{0}(F)\right) \wedge \left(H^{0}(E) \otimes H^{0,1}(F)\right) \\ &= H^{0,1}(X) \wedge H^{0,1}(X). \end{aligned}$$

Part (2). Let $\mu = k_1(e_1, f_1) + \cdots + k_N(e_N, f_N) \in T(X)$. This means in particular that $\Sigma_j k_j = 0$ and that $\Sigma_j k_j e_j = 0$ in $J(E)$ and $\Sigma_j k_j f_j = 0$ in $J(F)$, using $\mathrm{Alb}(X) = J(E) \oplus J(F)$. By Abel's theorem, under rational equivalence

$$k_1 e_1 = -(k_2 e_2 + \cdots + k_N e_n), \quad k_1 f_1 = -(k_2 f_2 + \cdots + k_N f_N).$$

Using both of these equalities, we end up with

$$\mu = \sum_{2 \le j \le N} k_j\{(e_j, f_j) - (e_j, f_1)\} = \sum_{2 \le j \le N} k_j\{(e_j, f_j) - (e_1, f_j)\}.$$

and therefore

$$2\mu = \sum_{2 \le j \le N} k_j\{2(e_j, f_j) - (e_j, f_1) - (e_1, f_j)\}.$$

Now set

$$\begin{aligned} \beta_j &= \{(e_j, f_j) - (e_j, f_1) - (e_1, f_j) + (e_1, f_1)\} \\ &= \{(E \times f_j) - (E \times f_1)\} \cap \{(e_j \times F) - (e_1 \times F)\} \\ &\in Pic^0(X) \bigcap Pic^0(X), \end{aligned}$$

and

$$\beta = k_1\beta_1 + \cdots + k_N\beta_N \in \cap \circ Pic^0(X)^{\otimes_{\mathbf{Z}} 2}.$$

Then

$$\begin{aligned} 2\mu - \beta &= \sum_{2 \le j \le N} k_j\{(e_j, f_j) - (e_1, f_1)\} \\ &= \sum_{2 \le j \le N} k_j(e_j, f_j) - \Big(\sum_{2 \le j \le N} k_j\Big)(e_1, f_1); \end{aligned}$$

moreover $k_1 = -(\Sigma_{2 \le j \le N} k_j)$, a fortiori $2\mu - \beta = \mu$, i.e. $\mu = \beta$. Q.E.D.

We now attend to the proof of (0.3)(2), namely, "if $\mathscr{K}(X) \leq 1$, then $\mathbf{V}^{0,2} = 0$ implies Coker $= 0$". If $\mathscr{K}((X) \leq 1$ and $Pg(X) = 0$ then $T(X) = 0$ [1(2)] hence nothing to prove here. Likewise if $q(X) \leq 1$, then $V^{0,2} = 0$ implies $Pg(X) = 0$. Therefore:

(2.2) *We will assume the following for* X: $Pg(X) > 0$, $\mathscr{K}(X) \leq 1$, $q(X) \geq 2$, and $V^{0,2} = 0$; moreover we may assume X is a minimal surface, as $Pg(X)$, $\mathscr{K}(X)$, $q(X)$ and $T(X)$ (hence $V^{0,2}$) are birational invariants.

It therefore suffices to show Coker $= 0$. A tour of the classification of surfaces will reveal that X is either an abelian or an elliptic surface. The case X an abelian surface was established in [1(2): (A.2)]. We are now reduced to the case of an elliptic surface X with fibering $F\colon X \twoheadrightarrow E$, i.e. where F is a morphism, E a smooth curve, and the generic fiber $X_t = F^{-1}(t)$ is a smooth elliptic curve. Let $\Sigma \subset E$ be the finite set of singular points of the fibering F; i.e., $t \in \Sigma$ iff X_t is singular. Set $U = E - \Sigma$ the so-called smooth set; $j\colon U \hookrightarrow E$ the inclusion, $X^{\#} = F^{-1}(U)$ and $f = F|_{X^{\#}}\colon X^{\#} \twoheadrightarrow U$. There is the local invariant cycle theorem: $R^iF_*\mathbf{Q} \to j_*R^if_*\mathbf{Q}$ surjective $(i \geq 0)$ (see [9:(15.12)]), with kernel subsheaf $\mathscr{L}_\Sigma$ supported on Σ. For the remainder of this section, we will *assume that* X *has no multiple fibers*. Note that X *minimal* implies no exceptional curves of the first kind as well. We will handle the general case of elliptic X in Section 3 below. Then $R^iF_*\mathbf{Q} \tilde{\to} j_*R^if_*\mathbf{Q}$ is in fact an isomorphism, due to the following. Let Δ be a small disk in E centered at $s \in \Sigma$. Then $H^1(X_s, \mathbf{Q})$ and $H^0(\Delta, j_*R^1f_*\mathbf{Q})$ are both either 0 or $\mathbf{Q}$ (cf. Table 1 in [4: p. 604]), moreover

$$H^0(\Delta, R^1F_*\mathbf{Q}) \cong H^1(X_s, \mathbf{Q}) \cong H^1(X_t, \mathbf{Q})^T \cong H^0(\Delta, j_*R^1f_*\mathbf{Q})$$

where $t \in \Delta - \{s\}$ and T is the local monodromy transformation. Therefore

$$H^i(E, R^1F_*\mathbf{Q}) \cong H^i(E, j_*R^1f_*\mathbf{Q}) \text{ for } i \geq 0.$$

The Leray spectral sequence

$$E_2^{p,q} = H^p(E, R^qF_*\mathbf{Q}) \Rightarrow H^{p+q}(X, \mathbf{Q})$$

degenerates at E_2 [9: (15.15)]; moreover from this spectral sequence is an induced s.e.s.

$$0 \to E_2^{1,0} \to H^1(X, \mathbf{Q}) \to E_2^{0,1} \to 0.$$

Using our assumption on X in (2.2) we prove:

(2.3) Proposition. $E_2^{0,1} \neq 0$.

Proof. Suppose to the contrary that $E_2^{0,1} = 0$. Then

$$H^1(X, \mathbf{Q}) \cong H^1(E, j_* R^0 f_* \mathbf{Q});$$

moreover $j_* R^0 f_* \mathbf{Q}$ is the constant sheaf over E. If we work on the cycle level viz Poincaré duality: $H^1(X, \mathbf{Q}) \cong H_3(X, \mathbf{Q})$, it follows that every cycle μ in $H_3(X, \mathbf{Q})$ is of the form $\mu = f^{-1}(\beta)$ for some 1-cycle β in U. Let $\beta_1, \beta_2 \in H_1(U, \mathbf{Q})$ with corresponding $\mu_j = f^{-1}(\beta_j)$, $j = 1, 2$. Then $\beta_1 \cap \beta_2 \sim r\{t\}$ for some $r \in \mathbf{Q}$ and $t \in U$; therefore $\mu_1 \cap \mu_2 \sim r\{X_t\}$ which is algebraic. It follows that

$$H^1(X, \mathbf{Q}) \wedge H^1(X, \mathbf{Q}) \subset H^2(X, \mathbf{Q})_{alg} \subset H^{1,1}(X);$$

hence $V^{0,2} \neq 0$, a contradiction.

(2.4) Corollary. (i) *For $t \in E$, the natural restriction map*

$$E_2^{0,1} \to H^1(X_t, \mathbf{Q})$$

is an isomorphism (of Hodge structures); hence $R^1 F_* \mathbf{Q}$ is the constant sheaf $\mathbf{Q}^2$ over E, and the functional invariant is trivial.

In particular:

(ii) *F: $X \to E$ is an analytic fibration of smooth elliptic curves.*

Proof. Since $j_* R^0 f_* \mathbf{Q}$ is the constant sheaf, it follows that $E_2^{1,0} \cong H^1(E, \mathbf{Q})$ and hence $E_2^{0,1} \cong H^1(X, \mathbf{Q})/E_2^{1,0}$ has a Hodge structure of weight 1 (this is also true for general reasons [9]), a fortiori $E_2^{0,1}$ even dimensional. Therefore by (2.3), there exists global independent sections s_1, s_2 of $j_* R^1 f_* \mathbf{Q}$ over E. Since any section of $j_* R^1 f_* \mathbf{Q}$ is the same thing as a horizontal displacement of a cycle $\beta_t \in H^1(X_t, \mathbf{Q})$ over $t \in E$, it follows that $j_* R^1 f_* \mathbf{Q} \cong R^1 F_* \mathbf{Q}$ is a *constant* sheaf (trivial monodromy group); moreover, by [4: p. 604, Table 1]), $\Sigma = \varnothing$. Parts (i) and (ii) of (2.4) easily follow from this.

(2.5) Now recall that F: $X \twoheadrightarrow E$ is called Jacobian-elliptic if there is a holomorphic section μ: $E \to X$, i.e., $F \circ \mu = \mathrm{Id}_E$.

We conclude:

(2.6) Corollary. *If X is Jacobian-elliptic, then $X \cong E \times C$, where C is any fiber of F: $X \twoheadrightarrow E$.*

First proof. F: $X \twoheadrightarrow E$ and Pr_1: $E \times C \twoheadrightarrow E$ have the same homological and functional invariants; moreover $E \times C$ is also Jacobian-elliptic. By uniqueness of Jacobian-elliptic for the given homological and functional invariants [4], it follows that $X \cong E \times C$.

Second proof. Let $p \in X$ and $t = F(p)$, and let μ be given as in (2.5). There is the Abel-Jacobi (isomorphism) map

$$\int_t : X \to J(X_t),\ p \mapsto \{p - \mu(t)\} \in J(X_t),$$

defined by a process of integration, where $J(X_t) = H^1(X_t, \mathbf{R})/H^1(X_t, \mathbf{Z})$ (with $H^1(X_t, \mathbf{R})$ endowed with a suitable complex structure). However working with integral cohomology modulo torsion, it follows from (2.4)(i) that

$$J(X_t) \cong C =_{\text{def}} H^0(E, R^1F_*\mathbf{R})/H^0(E, R^1F_*\mathbf{Z}),$$

again where $H^0(E, R^1F_*\mathbf{R})$ is endowed with a suitable complex structure. One then arrives at a global holomorphic (isomorphism) map

$$\int : X \xrightarrow{\sim} E \times C$$

defined by the formula

$$\int(p) = \left(F(p), \int_{F(p)}(p)\right) \in \{t\} \times J(X_t) \cong \{t\} \times C$$

where $t = F(p)$.

(2.7) COROLLARY. *Let X be Jacobian-elliptic. Then Theorem* (0.3)(2) *holds for X.*

Proof. Use (2.6) and (2.1).

Now recall for every elliptic X with no multiple fibers and exceptional curves, there is a unique associated Jacobian elliptic Y having the same homological and functional invariants as X. According to [1(2): p. 138], one can construct a rational dominating map $g: X \to Y$ and a correspondence $z \in CH^2(Y \times X)$ such that $g_* \circ z_* = n^2$ and $z_* \circ g_* = n^2$ on $A_0(X)$, for some fixed integer n. Next, we may assume w.l.o.g. that g is a *morphism*, via a suitable blow-up of X, using the fact that $Pic^0(-)$ and $A_0(-)$ are birational invariants. Now using (2.7), g and z above, $T(X)$ torsionless [7], $T(X)$ divisible (for all X [1(2): p. 136]), together with the projection formula, there is a commutative diagram

$$\text{(2.8)}\quad \begin{array}{ccccc} Pic^0(Y)\otimes_{\mathbf{Z}} Pic^0(X) & \xrightarrow{g^*\otimes 1} & Pic^0(X)\otimes_{\mathbf{Z}} Pic^0(X) & \xrightarrow{\cap} & T(X) \\ \downarrow{\scriptstyle 1\otimes g_*} & & & & \downarrow{\scriptstyle g_*}\wr \\ Pic^0(Y)\otimes_{\mathbf{Z}} Pic^0(Y) & & \xrightarrow{\quad\cap\quad} & & T(Y) \end{array}$$

The conclusion of the proof of (0.3)(2) for all but the general case of elliptic X is obvious via an immediate inspection of (2.8).

3. Conclusion of the proof of (0.3)(2)

We assume X given as in (2.2), elliptic, but with the possiblity of multiple fibers. We prove:

(3.0) LEMMA. *$j_*R^1f_*\mathbf{Q}$ is the constant sheaf $\mathbf{Q}^2$ over E.*

Proof. Assume to the contrary. Then using the fact that $H^0(E, j_*R^1f_*\mathbf{Q})$ admits a Hodge structure of weight 1 [9] together with the same argument as in the proof of (2.4), it follows that

$$H^0(E, j_*R^1f_*\mathbf{Q}) = 0.$$

Therefore via the s.e.s.,

$$0 \to \mathscr{L}_\Sigma \to R^1F_*\mathbf{Q} \to j_*R^1f_*\mathbf{Q} \to 0,$$

we arrive at

$$H^0(\Sigma, \mathscr{L}_\Sigma) = E_2^{0,1}.$$

Now choose any Cech cocycle $s = \{s_{ij}\} \in E_2^{1,0}$ corresponding to an open cover $\{U_j\}_{j\in J}$ of E, where we may assume via a suitable refinement that $U_i \cap U_j \cap \Sigma = \varnothing$ for all $i \neq j$ in J. It follows that $s \smile \mathrm{E}_2^{0,1} = 0$ in $E_2^{1,1}$, a fortiori $E_2^{1,0} \smile E_2^{0,1} = 0$. The cup product

$$E_2^{p,q} \smile E_2^{r,s} \subset E_2^{p+r,q+s}$$

is compatible with the cup product on $H^*(X)$ vis-à-vis a "Leray" filtration on

$H^*(X)$; moreover it is easy to check that $E_2^{1,0} \cup E_2^{0,1} = 0$ implies

$$H^1(X,\mathbf{Q}) \cup H^1(X,\mathbf{Q}) \subset H^2(X,\mathbf{Q})_{\text{alg}} \subset H^{1,1}(X),$$

contradicting our assumptions on X in (2.2). Q.E.D.

According to [4: Thm. 6.3, p. 572]) there is induced, over a suitable finite ramified covering $k\colon S \twoheadrightarrow E$, an elliptic surface $G\colon Y \twoheadrightarrow S$ free from multiple fibers and exceptional curves, and commutative diagram

$$\text{(3.1)}\qquad \begin{array}{ccc} Y & \xrightarrow{h} & X \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F} \\ S & \xrightarrow{k} & E \end{array}$$

where (i) Y, S are smooth and (ii) $h\colon Y \twoheadrightarrow X$ is a finite abelian covering.
Moreover, we may assume (by slightly enlarging Σ if necessary) that

$$\Sigma_0 = \{s \in S \mid Y_s = G^{-1}(s) \text{ singular}\} = k^{-1}(\Sigma).$$

Also let $j_0\colon \Sigma_0 \hookrightarrow S$ be the inclusion and $g = G|_{G^{-1}(S-\Sigma_0)}$. It follows from (3.0) that $j_{0,*}R^1g_*\mathbf{Q}$ (and hence $R^1G_*\mathbf{Q}$) is the constant sheaf $\mathbf{Q}^2$ over S, and moreover the *same* arguments in Section 2 imply that

$$Pic^0(Y) \otimes Pic^0(Y) \xrightarrow{\cap} T(Y)$$

is surjective. There is a commutative diagram, with β defined by commutivity below:

$$\text{(3.2)}\qquad \begin{array}{ccccccccc} 0 \to & E_2^{1,0}(F) & \longrightarrow & H^1(X,\mathbf{Q}) & \longrightarrow & E_2^{0,1}(F) & \to 0 \\ & \| \wr & & & & & \\ & H^1(E,\mathbf{Q}) & & & & & \\ & \downarrow {\scriptstyle k^*} & & \downarrow {\scriptstyle h^*} & & \downarrow {\scriptstyle \beta} & \\ & H^1(S,\mathbf{Q}) & & & & & \\ & \| \wr & & & & & \\ 0 \to & E_2^{1,0}(G) & \longrightarrow & H^1(Y,\mathbf{Q}) & \longrightarrow & E_2^{0,1}(G) & \to 0. \end{array}$$

Moreover β is surjective by virtue of the commutative diagram

(3.3)
$$\begin{array}{ccc} E_2^{0,1}(F) & \xrightarrow{\ \beta\ } & E_2^{0,1}(G) \\ \downarrow & & \| \wr \\ \mathbf{Q}^2 = H^0(E, j_* R^1 f_* \mathbf{Q}) & \xrightarrow{\ \sim\ } & H^0(S, j_{0,*} R^1 g_* \mathbf{Q}). \end{array}$$

Next, there is a commutative diagram of exact sequences with the top row interpreted as the respective tangent spaces (with suitable complex structure) to the bottom row:

(3.4)
$$\begin{array}{ccccccccc} & & H^1(S,\mathbf{R}) & & & & & & \\ & & \| \wr & & & & & & \\ 0 & \to & E_2^{1,0}(G) \otimes \mathbf{R} & \longrightarrow & H^1(Y,\mathbf{R}) & \longrightarrow & E_2^{0,1}(G) \otimes \mathbf{R} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & Pic^0(S) & \xrightarrow{\ G^*\ } & Pic^0(Y) & \longrightarrow & \{Pic^0(Y)/G^*(Pic^0(S))\} & \to & 0. \end{array}$$

We therefore conclude from (3.4) and the surjectivity of β that:

(3.5) LEMMA. *The composite*

$$Pic^0(X) \xrightarrow{h^*} Pic^0(Y) \longrightarrow \{Pic^0(Y)/G^*(Pic^0(S))\}$$

is surjective.

The projection formula implies the commutative diagram

(3.6)
$$\begin{array}{ccccc} Pic^0(X) \otimes Pic^0(Y) & \xrightarrow{h^* \otimes 1} & Pic^0(Y) \otimes Pic^0(Y) & \xrightarrow{\ \cap\ } & T(Y) \\ & \searrow^{1 \otimes h_*} & & & \downarrow h_* \\ & & Pic^0(X) \otimes Pic^0(X) & \xrightarrow{\ \cap\ } & T(X) \end{array}$$

Now let $\beta_1, \beta_2 \in Pic^0(S)$. By Chow's moving lemma, we may assume $\beta_1 \cap \beta_2 = \varnothing$, and therefore $G^*(\beta_1) \cap G^*(\beta_2) = 0$ in $T(Y)$. Now let

$$\{Z\} \in \{Pic^0(Y)/G^*(Pic^0(S))\},$$

where Z can be chosen so that $Z(t) = G^{-1}(t) \cap Z$ is a 0-cycle (of degree 0) on $G^{-1}(t)$. Let $\beta \in \mathrm{Pic}^0(S)$. By (3.5), there exists $W \in Pic^0(X)$ such that

$$\{h^*(W)\} = \{Z\} \text{ in } \{Pic^0(Y)/G^*(Pic^0(S))\},$$

and therefore

$$G^*(\beta) \cap Z = G^*(\beta) \cap h^*(W) \text{ in } T(Y).$$

Finally,

$$h_*\{(G^*(\beta) \cap Z)_Y\} = h_*\{(G^*(\beta) \cap h^*(W))_Y\} = ((h_* \circ G^*(\beta)) \cap W)_X$$

by the projection formula, i.e., (3.6). It therefore follows from (3.6) that

$$Pic^0(X) \otimes Pic^0(X) \xrightarrow{\cap} T(X)$$

is surjective.

The techniques of Sections 2 and 3 imply the following:

(3.7) COROLLARY. *Let X be any surface of $\mathscr{K}(X) \leq 1$ with non-trivial pairing*

$$H^{0,1}(X) \otimes H^{0,1}(X) \xrightarrow{\wedge} H^{0,2}(X).$$

Then:

(i) *This pairing is surjective, and moreover,*

(ii) $Pic^0(X) \otimes Pic^0(X) \xrightarrow{\cap} T(X)$ *is likewise surjective.*

Proof. It follows that $Pg(X) \neq 0$ and $q(X) \geq 2$. By classification, X is birational to either an abelian surface or an elliptic surface. The rest of the proof, which follows along the line of reasoning in Sections 2 and 3, implies that the mapping in (3.7)(ii) is surjective. Now apply (0.3)(1) to conclude that the pairing in (i) is surjective.

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University of Saskatchewan
Saskatoon, Saskatchewan, Canada