ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 2, Summer 1991

FINITE 2-GROUPS OF CLASS 2 IN WHICH EVERY PRODUCT OF FOUR ELEMENTS CAN BE REORDERED[1]

BY

P. LONGOBARDI AND S.E. STONEHEWER

1. Introduction

If n is an integer greater than 1, then a group G belongs to the class P_n if every ordered product of n elements can be reordered in at least one way; in other words, to each n-tuple $(x_1, x_2, \ldots, x_n)$ of elements of G there corresponds a non-trivial element σ of the symmetric group Σ_n such that

$$x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

The union of the classes P_n, $n \geq 2$, is denoted by P. It was shown in [4] that P consists precisely of the finite-by-abelian-by-finite groups.

Clearly P_2 is the class of abelian groups, while $G \in P_3$ if and only if $|G'| \leq 2$ [3]. Graham Higman [6] characterised finite groups of odd order in P_4 and also proved that a group G with $G' \cong V_4$ (the 4-group) always belongs to P_4. Then in [8], improving a result in [1], it was shown that all P_4-groups are metabelian. Finally in [9] the non-nilpotent P_4-groups were classified and the nilpotent P_4-groups were shown to have class at most 4. We recall the details of these results in §2.

The present work is a further contribution to the classification of P_4-groups. We determine precisely which finite 2-groups of class 2 belong to P_4. Combining this work with the results of [9] it has been possible to classify *all* P_4-groups and a complete description by M. Maj and the present authors will appear elsewhere. The finite 2-groups of class 2, however, are most conveniently treated independently. If G is such a group in P_4, we shall see that G' has exponent at most 4. Our main results are:

THEOREM A. *Let G be a finite 2-group of class 2 with G' of exponent 4. Then $G \in P_4$ if and only if $G' \cong C_4$ and G has a subgroup B of index 2 with $|B'| = 2$.*

Received January 3, 1989
1980 Mathematics Subject Classification (1985 Revision). Primary 20E34; Secondary 20D60, 20E25, 20D15.
[1]The authors are grateful to the British Council and C.N.R. for financial support during the preparation of this paper.

THEOREM B. *Let G be a finite 2-group of class 2 with G' of exponent 2. Then $G \in P_4$ if and only if*
(i) *G has an abelian subgroup of index 2, or*
(ii) *$|G'| \leq 4$, or*
(iii) *$|G'| = 8$ and $G/Z(G)$ can be generated by 3 elements, or*
(iv) *$|G'| = 8$, $G/Z(G)$ can be generated by 4 elements and G is not the product of two abelian subgroups.*

Notation is as follows.

C_n	a cyclic group of order n,
V_4	the 4-group,
Σ_n	the symmetric group of degree n,
G'	derived subgroup of G,
$Z(G)$	centre of G,
C_G	centraliser in G,
$\Phi(G)$	Frattini subgroup of G,
$\lvert g\rvert$	order of element g,
g^x	$x^{-1}gx$,
$[x, y]$	$x^{-1}y^{-1}xy$,
$\exp G$	exponent of G.

2. Known results

First we state Higman's two contributions [6].[2]

2.1. *Let G be a group with $G' \cong V_4$. Then $G \in P_4$.*

2.2. *Let G be a finite group of odd order. Then $G \in P_4$ if and only if one of the following holds:*
(i) *G is abelian;*
(ii) *$|G'| = 3$;*
(iii) *$|G'| = 5$ and $|G/Z(G)| = 25$.*

The result which first suggested that a complete description of the class P_4 might be possible is:

2.3. [8]. *If a group G belongs to P_4, then G is metabelian.*

A useful result from [9] (namely 2.1.3) is:

[2]Since the proofs of 2.1 and 2.2 are not yet published, Professor Higman has kindly allowed M. Maj and the authors to include them in their complete description of the class P_4 which exists in the form of a set of typed notes and is available from the authors. We wish to record our gratitude to Professor Higman.

2.4. *Let G be a finite 2-group belonging to P_4 and let A be an abelian subgroup of G containing G'. If $G = A\langle x\rangle$, then one of the following holds*:
(1) $[A, x^2] = 1$;
(2) $G' \cong V_4$;
(3) $G' \cong C_4$ *and* $G' \leq Z(G)$.

Finally the main theorem of [9], which characterizes the non-nilpotent groups in P_4, is:

2.5. *A group G belongs to P_4 if and only if one of the following holds*:
(i) *G has an abelian subgroup of index* 2;
(ii) *G is nilpotent of class* ≤ 4 *and* $G \in P_4$;
(iii) $G' \cong V_4$;
(iv) $G = B\langle a, x\rangle$, *where* $B \leq Z(G)$, $|a| = 5$ *and* $a^x = a^2$.

3. Proofs of Theorems A and B

Throughout

G denotes a finite 2-group of class ≤ 2.

Our objective is to find necessary and sufficient conditions for G to belong to P_4. If $G \in P_4$, then since each element of G belongs to an abelian subgroup containing G', it follows from 2.4 that $\exp G' \leq 4$. In 3.1 we study the case where $\exp G' = 4$. It turns out then that $G' \cong C_4$ (3.1.2) in which case necessary and sufficient conditions for $G \in P_4$ are founded in Theorem A. The case when $\exp G' = 2$ is considered in 3.2. If $G \in P_4$, then either G has an abelian subgroup of index 2 or $|G'| \leq 8$. The complete description of this case is given in Theorem B.

3.1. *G' of exponent* 4.

Following Philip Hall we call a group *diabelian* if it is the product of two abelian subgroups. Then we have:

3.1.1. *Let G be diabelian with* $\exp G' = 4$. *If $G \in P_4$, then* $G' \cong C_4$.

Proof. We have $G = AX$ with A and X abelian and $Z(G) \leq A \cap X$. Let $a \in A$, $x \in X$ such that $|[a, x]| = 4$. Then $[a, x^2] \neq 1$ and so, by 2.4, $[A, x] \leq \langle [a, x]\rangle$. Similarly $[a, X] \leq \langle [a, x]\rangle$. Now for each $x_1 \in X$, either $|[a, x_1]| = 4$ or $|[a, xx_1]| = 4$. Thus either $[A, x_1] \leq \langle [a, x_1]\rangle$ or $[A, xx_1] \leq \langle [a, xx_1]\rangle$. Therefore $G' = [A, X] = \langle [a, x]\rangle$. □

Now we can dispense with the hypothesis that G is diabelian.

3.1.2. *Let* $G \in P_4$ *and* $\exp G' = 4$. *Then* $G' \cong C_4$.

Proof. Let $a, x \in G$ such that $|[a, x]| = 4$. Then

$$\langle a, x, y\rangle' = \langle [a, x]\rangle \qquad \textit{for all } y \in G. \tag{1}$$

For, write $X = \langle a, x, y\rangle$ and $b = [x, y]$. *Suppose that* $|b| \leq 2$. By 3.1.1, it suffices to show that

$$X \text{ is diabelian.} \tag{2}$$

Thus we may assume that $b \neq 1$. If $[a, x^2] \in \langle b\rangle$, then $[x, a^2 y] = 1$ and (2) follows. Assume therefore that $[a, x^2] \notin \langle b\rangle$. Since $X/\langle b\rangle$ is diabelian, 3.1.1 gives $(X/\langle b\rangle)' = \langle [a, x]\langle b\rangle\rangle$ and so

$$[a, y] \in \langle [a, x], b\rangle \cong C_4 \times C_2.$$

If $[a, y] = [a, x]^i$ for some integer i, then $[a, y^{-1}x^i] = 1$ and

$$X = \langle a, y^{-1}, x^i\rangle\langle x\rangle Z(X)$$

is diabelian, If $[a, y] = [a, x]^i b$, then $[xa^{-1}, a^{-i}y] = 1$ and again

$$X = \langle x\rangle\langle xa^{-1}, a^{-i}y\rangle Z(X)$$

is diabelian.

Now *suppose that* $|b| = 4$. Then $|[x, y^2]| = 2$ and by the previous case

$$[x, y^2] \in \langle a, x, y^2\rangle' = \langle [a, x]\rangle.$$

Therefore $[x, y^2] = [a^2, x]$ and $[x, ay]^2 = 1$. Thus again by the previous case (with y replaced by ay)

$$X' = \langle a, x, ay\rangle' = \langle [a, x]\rangle.$$

Now we have established (1).

Let $g, z \in G$. It suffices to show that $[g, z] \in \langle [a, x]\rangle$. By (1),

$$[a, g] \quad \textit{and} \quad [x, z] \qquad \textit{belong to } \langle [a, x]\rangle.$$

If $|[a, g]| = 4$, then again by (1)

$$\langle a, g, z\rangle' = \langle [a, g]\rangle = \langle [a, x]\rangle$$

and so $[g, z] \in \langle [a, x] \rangle$. If $|[a, g]| \leq 2$, then $|[a, xg]| = 4$ and (1) gives

$$\langle a, xg, z \rangle' = \langle [a, xg] \rangle = \langle [a, x] \rangle$$

and hence $[g, z] = [x, z]^{-1}[xg, z] \in \langle [a, x] \rangle$. □

If $G' \cong C_4$, then G does not necessarily belong to P_4. The following result (which we need for our classification purposes anyway) will enable us to construct an example of this fact.

3.1.3. *Suppose that $G' \cong C_4$. Then the following are equivalent:*
(i) $G \notin P_4$;
(ii) *there are elements $x_1, x_2, x_3, x_4 \in G$ such that $[x_1, x_2] = [x_2, x_3] = [x_3, x_4]$ of order 4 and $[x_1, x_3] = [x_1, x_4] = [x_2, x_4] = 1$.*

Proof. Suppose that $G \notin P_4$ and that the product $x_1x_2x_3x_4$ cannot be reordered. Then

$$x_1x_2x_3x_4 = x_4x_1x_2x_3a = x_4x_3x_1x_2b = x_4x_3x_2x_1c$$

with $G' = \{1, a, b, c\}$. Let $x_1x_2x_3x_4 = x_4x_1x_3x_2d$. Clearly $d \neq a$; and if $d = c$, then

$$x_1x_3x_2 = x_3x_2x_1$$

giving

$$1 = [x_1, x_3x_2] = [x_1, x_2x_3],$$

a contradiction. Therefore $d = b$ and so

$$[x_1, x_3] = 1.$$

In the same way we obtain $x_1x_2x_3x_4 = x_4x_2x_1x_3b$ and so $x_2x_1x_3 = x_3x_1x_2 = x_1x_3x_2$. Therefore $[x_2, x_1x_3] = 1$ and

$$[x_1, x_2] = [x_2, x_3].$$

Since $x_4^{-1}x_3^{-1}x_2^{-1}x_1^{-1}$ also cannot be reordered, we have, by the same argument,

$$[x_2, x_4] = 1 \quad \text{and} \quad [x_2, x_3] = [x_3, x_4].$$

Now consider $x_1x_2x_3x_4 = x_1x_4x_2x_3e$. If $e = b$, then $x_4x_3x_1x_2 = x_1x_4x_2x_3 = x_1x_2x_4x_3$ and $1 = [x_1x_2, x_4x_3] = [x_1x_2, x_3x_4]$, a contradiction. If $e = c$, then $x_4x_3x_2x_1 = x_1x_4x_2x_3 = x_1x_2x_4x_3$ and $x_1x_2x_3x_4 = x_3x_4x_2x_1$, again a contradiction. Therefore $e = a$ and

$$[x_1, x_4] = 1.$$

Finally, $a = [x_3, x_4] = [x_2, x_3] = [x_1, x_2]$ cannot have order 2, otherwise $x_1x_2x_3x_4 = x_2x_1x_4x_3$. Thus (i) implies (ii).

Conversely, if (ii) is true, a routine check shows that $x_1x_2x_3x_4$ cannot be reordered and so (i) follows. □

We can now construct an example of a finite 2-group G of class 2 with $G' \cong C_4$ and $G \notin P_4$. Thus let

$$G = (\langle x_2\rangle \times \langle x_4\rangle \times \langle a\rangle) \rtimes (\langle x_1\rangle \times \langle x_3\rangle)$$

where x_1, x_2, x_3, x_4 and a all have order 4,

$$[x_1, x_2] = [x_2, x_3] = [x_3, x_4] = a$$

and

$$[x_1, x_4] = [x_1, a] = [x_3, a] = 1.$$

Then $G' = \langle a\rangle \leq Z(G)$ and $G' \cong C_4$. Moreover the elements x_1, x_2, x_3, x_4 satisfy (ii) of 3.1.3 and so $G \notin P_4$.

The structure of the groups under consideration which belong to P_4 can now be described.

3.1.4. *Suppose that $G' \cong C_4$. Then $G \in P_4$ if and only if G has a subgroup B of index 2 with $|B'| = 2$.*

Proof. Suppose that $G \in P_4$ and let B be a subgroup of G, maximal subject to $|B'| = 2$. Then $Z(G) \leq B \lhd G$. We show that $|G/B| = 2$.

Since for all $g \in G$, $\langle B, g^2\rangle' = B'$, we have

G/B is elementary abelian.

Suppose, to the contrary, that G has 2 independent elements modulo B, say w, y. By choice of B, there is an element $x \in B$ such that $|[x,w]| = 4$. Put $a = [x,w]$. Thus $G' = \langle a\rangle$ and $B' = \langle a^2\rangle$. For some integer i, we have $[w, y] = [x^i, w]$ and so $[w, x^iy] = 1$. Therefore taking x^iy for y, we may assume that $[w, y] = 1$.

Now $B = C_B(w)\langle x\rangle$. If $[C_B(w), y] \leq B'$, then $|[x, y]| = 4$, since $\langle B, y\rangle' = \langle a\rangle$. Thus $[x, w^2y^2] = 1$ and so $\langle B, wy\rangle' = B'$, again contradicting our choice of B. Therefore there is an element $z \in C_B(w)$ such that

$$|[z, y]| = 4.$$

If $[x, z] \neq 1$, then $[x, z] = a^2$ and hence $[xy^2, z] = 1$; and $xy^2 \in B$, $[xy^2, w] = [x, w]$. Thus taking xy^2 for x, we may assume that

$$[x, z] = 1$$

and (replacing z by z^{-1} if necessary) that $[y, z] = a$. Also replacing x by xz if necessary, we may assume that $|[x, y]| = 4$. Then replacing x by xz^2 if necessary, we may assume that $[x, y] = a$.

Taking w, x, y^{-1}, z for x_1, x_2, x_3, x_4 respectively, we see that (ii) of 3.1.3 holds, contradicting $G \in P_4$. It follows that $|G/B| = 2$.

Conversely, suppose that there is a subgroup $B \lhd G$ with $|G/B| = |B'| = 2$. Assume, to the contrary, that $G \notin P_4$. By 3.1.3 there are elements $x_1, x_2, x_3, x_4 \in G$ such that $[x_1, x_3] = [x_1, x_4] = 1$ and $[x_1, x_2] = [x_3, x_4]$ of order 4. Thus $C_G(x_1)' = G'$. If $x_1 \notin B$, then $G = \langle B, x_1 \rangle$ and $C_G(x_1) = \langle x_1 \rangle C_B(x_1)$, giving $C_G(x_1)' = C_B(x_1)' \leq B'$, a contradiction. Therefore $x_1 \in B$. Hence $x_2 \notin B$ and so $C_G(x_1) \leq B$, again a contradiction. Then $G \in P_4$ as required. □

From 3.1.2 and 3.1.4 we obtain Theorem A.

3.2. *G' of exponent* 2.

Throughout this section (except for 3.2.6)

G denotes a finite 2-group of class ≤ 2 and with G' elementary.

First we show that if G can be generated by 3 elements, then $G \in P_4$. After this we find necessary and sufficient conditions for $G \in P_4$ when G is generated by 4 elements. The general case (Theorem B) is handled by studying the situation in which G/A is generated by 3 elements, for some maximal normal abelian subgroup A of G.

In the proof of the following result, and occasionally thereafter, we make use of the Burnside Basis Theorem (see, for example, [13]).

3.2.1. *Let G be generated by* 3 *elements. Then $G \in P_4$.*

Proof. Let B be a maximal subgroup of G. Then $B/\Phi(G)$ can be generated by 2 elements and so $|B'| \leq 2$, since $\Phi(G) \leq Z(G)$. Therefore $B \in P_3$.

Now let $x_1, x_2, x_3, x_4 \in G$ and suppose, for a contradiction, that $x_1x_2x_3x_4$ cannot be reordered. Then $\langle x_1, x_2, x_3 \rangle \notin P_3$ and so $G = \langle x_1, x_2, x_3 \rangle$. Thus

$$x_4 = x_1^i x_2^j x_3^k z$$

where $z \in Z(G)$ and $0 \le i, j, k \le 1$. If $i = 0$, then

$$x_1 x_2^{1-j} x_4 x_3^{1-k} x_2^j x_3^k = x_1 x_2^{1-j} \cdot x_2^j x_3^k z \cdot x_3^{1-k} x_2^j x_3^k = x_1 x_2 x_3 x_4,$$

a contradiction for all choices of j and k. Therefore $i = 1$ and so

$$\begin{aligned} x_4 x_2^{1-j} x_3^{1-k} x_1 x_2^j x_3^k &= x_4 x_2^{1-j} x_3^{1-k} x_4 z^{-1} \\ &= x_4 z^{-1} x_2^{1-j} x_3^{1-k} x_4 \\ &= x_1 x_2^j x_3^k x_2^{1-j} x_3^{1-k} x_4 \\ &= x_1 x_2 x_3 x_4 \qquad \text{if } j = k = 0 \quad \text{or} \quad j = 1, \end{aligned}$$

again a contradiction. It follows that $x_4 = x_1 x_3 z$ and so

$$\begin{aligned} x_1 x_2 x_3 x_4 &= x_1 x_2 x_3 (x_1 x_3 z) \\ &= x_3 x_1 x_2 (x_1 x_3 z)[x_1 x_2, x_3] \\ &= x_3 x_1 (x_3 x_2 x_1) z [x_2 x_1, x_3][x_1 x_2, x_3] \\ &= x_3 (x_1 x_3 z) x_2 x_1 \\ &= x_3 x_4 x_2 x_1, \end{aligned}$$

a final contradiction. Therefore $G \in P_4$. □

Now we proceed to the case when G can be generated by 4 elements and record first some routine observations.

3.2.2. *Let* $w, x, y, z \in G$ *and put* $a = [w, x]$, $b = [w, y]$, $c = [w, z]$, $d = [x, y]$, $e = [x, z]$, $f = [y, z]$. *Then the product* $wxyz$ *can be reordered if and only if at least one of the following elements is equal to* 1:

$$\begin{aligned} &a, d, f, \\ &ab, af, bd, de, ef, \\ &abc, abd, acf, bde, cef, def, \\ &abcd, abcf, acef, bcde, cdef, \\ &abcde, abcef, bcdef, \\ &abcdef. \end{aligned} \tag{*}$$

Proof. This follows by setting the product $wxyz$ equal to each of its 23 reorderings in turn. □

As a straightforward corollary, we have:

3.2.3. *With the same notation as 3.2.2, let r be the rank of $\langle w, x, y, z\rangle'$. Then the product wxyz cannot be reordered in each of the following cases*:

(i) $r = 5$, $b = 1$:
(ii) $r = 4$, $b = c = 1$;
(iii) $r = 4$, $b = e = 1$;
(iv) $r = 3$, $b = c = e = 1$;
(v) $r = 3$, $b = cd = ade = 1$;
(vi) $r = 3$, $b = cdf = adef = 1$.

Using these results we can establish:

3.2.4. *Suppose that $G = \langle w, x, y, z\rangle$ with $[w, y] = [x, z] = 1$ and $|G'| \geq 8$. Then $G \notin P_4$.*

Proof. We adopt the notation of 3.2.2 with $r = \operatorname{rank} G'$. By hypothesis $b = e = 1$. If $r = 4$, then $G \notin P_4$ by 3.2.3(iii). Therefore *suppose that* $r = 3$. Then there are $i, j, k, l \in \{0, 1\}$, not all 0, such that $a^i c^j d^k f^l = 1$. If $i = 1$, then replacing the 4-tuple (w, x, y, z) by (w, z, y, x), c is interchanged with a and so we may assume that $j = 1$. Similarly if $k = 1$, we can argue with (x, w, z, y) and if $l = 1$ with (y, x, w, z) so that we may assume $j = 1$ in all cases.

Thus $\{a, d, f\}$ is a basis for G' and

$$c = a^i d^k f^l.$$

If $c = 1$, then $G \notin P_4$ by 3.2.3(iv). For the other values of c, there are 4 elements (indicated in column 2 below), which we may substitute for w, x, y, z, whose product cannot be reordered, again by 3.2.3 (the relevant part being indicated in column 3).

c	4 elements	3.2.3
a	w, x, y, xz	(iv)
d	w, x, y, wyz	(v)
f	wy, x, y, z	(iv)
df	w, xy, y, wyz	(vi)
af	wy, x, y, wxz	(v)
ad	w, wx, wy, wz	(v)
adf	wy, x, y, xz	(iv)

Thus $G \notin P_4$. □

Now we can exclude from P_4 those 4-generator groups G with $|G'| > 8$.

3.2.5. *Let G be generated by 4 elements and $|G'| > 8$. Then $G \notin P_4$.*

Proof. Since G has a quotient with derived subgroup of order 16, we may assume that $|G'| = 16$. Suppose that $G = \langle w, x, y, z\rangle$ with $[w, y] = 1$. By 3.2.4 we may assume that $[x, z] \neq 1$. Let $N = \langle x, z\rangle'$ and write $\overline{G} = G/N$, $\bar{g} = Ng$ for all $g \in G$. Then $\overline{G} = \langle \overline{w}, \bar{x}, \bar{y}, \bar{z}\rangle$, $|\overline{G}'| = 8$ and $[\overline{w}, \bar{y}] = [\bar{x}, \bar{z}] = 1$ and so $\overline{G} \notin P_4$, by 3.2.4. Therefore $G \notin P_4$.

Thus we can assume that

$$\textit{among any 4 elements which generate } G\textit{, no two commute.} \qquad (3)$$

Let $G = \langle w, x, y, z\rangle$ and $X = \langle [w, x], [w, y], [w, z]\rangle$. By (3), $|X| = 8$ and so $|G'/X| = 2$. Therefore at least one of the commutators $[x, y]$, $[x, z]$, $[x, yz]$ belongs to X and clearly we may assume that $[x, y] \in X$. By (3) we have

$$[x, y] \notin \langle [w, x], [w, y]\rangle;$$

for, if, for example, $[x, y] = [w, x][w, y]$, then $[wx, xy] = 1$, contradicting (3). Hence $[x, y] = [w, x^i y^j z]$ for some integers i, j. Then $G/\langle x, y\rangle' \notin P_4$ by 3.2.4. Therefore $G \notin P_4$. □

If $|G'| = 2$, then $G \in P_3$ [3] and if $G' \cong V_4$, then $G \in P_4$ (2.1). Thus among the 4-generator groups G, we have to consider only those with $|G'| = 8$. In this case G has an abelian subgroup of index 4. For, if V is a 4-dimensional vector space over a finite field, then for any 3 antisymmetric bilinear forms on V, there is a subspace of dimension 2 on which all 3 forms are trivial [5]. Take $G/\Phi(G)$ for V and let N_i $(i = 1, 2, 3)$ be subgroups of order 4 in G' with $N_1 \cap N_2 \cap N_3 = 1$. Writing $\bar{g} = \Phi(G)g$ for all $g \in G$, and observing that $\Phi(G) \leq Z(G)$,

$$(\bar{g}_1, \bar{g}_2) = N_i[g_1, g_2]$$

defines an antisymmetric bilinear form in V for each i, and so there is a subgroup $A/\Phi(G)$ of order 4 such that, for all $a_1, a_2 \in A$, $[a_1, a_2] \in N_i$ $(i = 1, 2, 3)$, i.e. *A is abelian*.

An alternative argument suggested by Caranti may have independent interest.

3.2.6. *Let G be a 4-generator finite p-group of class 2 with G' elementary of rank 3. Then G has an abelian subgroup of index p^2.*

Proof. Since $\Phi(G) \leq Z(G)$, we may assume that $G/\Phi(G)$ has rank 4 and it suffices to show that G has 2 commuting elements which are independent modulo $\Phi(G)$. Consider $V = G/\Phi(G)$ and G' as vector spaces over $GF(p)$.

Then there is a natural linear map from the wedge product $\Lambda^2 V$ to G', namely

$$\bar{g}_1 \wedge \bar{g}_2 \to [g_1, g_2],$$

where $\bar{g} = \Phi(G)g$ for all $g \in G$. Let K be the kernel of this map. We must show that K contains a decomposable tensor $\bar{g}_1 \wedge \bar{g}_2 \neq 0$, i.e., that K intersects non-trivially the (affine) Grassman manifold $\mathscr{G}$ of decomposable elements of $\Lambda^2 V$. Now $\mathscr{G}$ is defined by a single quadratic equation in the 6-dimensional space $\Lambda^2 V$. In fact $\mathscr{G}$ consists of all $(\lambda_1, \dots, \lambda_6)$ such that $\lambda_1\lambda_6 - \lambda_2\lambda_5 + \lambda_3\lambda_4 = 0$. (See [11], page 234.) Since K has dimension 3, it is defined by 3 linear equations. The sum of the degrees of these 4 equations is $5 < 6$, and so by the theorem of Chevalley-Warning (see [12]), the 4 equations have a common non-trivial solution. □

Remark. This result will be used later in 3.2.12 and the proof of Theorem B, and a chain of results terminating with those proofs now follows.

Reverting to our convention that G is a finite 2-group of class ≤ 2 with G' elementary, we have:

3.2.7. *Let G be generated by* 4 *elements. Then $G \in P_4$ if and only if*
(i) *G has an abelian subgroup of index* 2, *or*
(ii) $|G'| \leq 4$, *or*
(iii) $|G'| = 8$ *and G is not diabelian.*

Proof. Let $G \in P_4$. Then, by 3.2.5, $|G'| \leq 8$. If $|G'| = 8$ and G does not have an abelian subgroup of index 2, then G is not diabelian by 3.2.4.

Conversely, if (i) or (ii) holds, then $G \in P_4$ by 2.5. Thus suppose that

$$|G'| = 8 \quad \textit{and} \quad G \textit{ is not diabelian}.$$

For a contradiction, assume that there are elements w, x, y, z in G such that

wxyz cannot be reordered.

Let $H = \langle w, x, y, z \rangle$. Then

$$H = G. \tag{4}$$

For, if $H < G$, then $H\Phi(G)/\Phi(G) \cong H/H \cap \Phi(G)$ can be generated by 3 elements and hence $H/Z(H)$ can be generated by 3 elements. But then $H \in P_4$, by 3.2.1, a contradiction. Therefore (4) is true.

Adopt the notation for commutators used in 3.2.2. Then the elements (*) are all different from 1. It follows that

$$G' = \langle a \rangle \times \langle bcde \rangle \times \langle f \rangle.$$

Consider the element $abcef$. By 3.2.2 this element is not equal to 1 or $abcdef$ and so it must be equal to one of a, $abcde$, af, $bcde$, $bcdef$ or f. Therefore

$$\text{(i)}\ \ bcef = 1, \ \text{or}\ \ \text{(ii)}\ \ df = 1, \ \text{or}\ \ \text{(iii)}\ \ bce = 1, \ \text{or}$$
$$\text{(iv)}\ \ adf = 1, \ \text{or}\ \ \text{(v)}\ \ ad = 1, \ \text{or}\ \ \text{(vi)}\ \ abce = 1.$$

Since $z^{-1}y^{-1}x^{-1}w^{-1}$ also cannot be reordered, the situation is symmetric in a and f, b and e and hence it suffices to consider only the cases (i)–(iv).

Case (i). $bcef = 1$. Then $G' = \langle a \rangle \times \langle d \rangle \times \langle f \rangle$ and from 3.2.2 it follows that either $e = 1$ or $e = ad$. But if $e = 1$, then $[x, z] = 1$ and hence $bcef = bcf = [wy, wz] \neq 1$, since G is not diabelian, a contradiction. Also if $e = ad$, then $[x, wyz] = 1$ and again $bcef = abcdf = [wy, wxz] \neq 1$ for the same reason.

Case (ii). $df = [y, xz] = 1$. Now $G' = \langle a \rangle \times \langle bce \rangle \times \langle f \rangle$ and 3.2.2 gives $e = a$, af, bce or $bcef$. Each possibility implies respectively that $[x, wz]$, $[x, wyz]$, $[w, yz]$ or $[wy, wz]$ is 1, contradicting the fact that G is *not* diabelian.

Case (iii). $bce = 1$. We have $G' = \langle a \rangle \times \langle d \rangle \times \langle f \rangle$. If $b = e = adf$, then $[w, z] = c = 1$ and $[wy, wxz] = abcdf = c = 1$, again contradicting G not diabelian. Thus by the symmetry referred to above, we may assume that $b \neq adf$. Then the only possibility consistent with 3.2.2 is $b = [w, y] = 1$ and hence $[wx, z] = ce = 1$, giving G diabelian.

Case (iv). $adf = 1$. Now $G' = \langle a \rangle \times \langle bce \rangle \times \langle f \rangle$. Since $b = [w, y]$ and $e = [x, z]$, b and e cannot both be 1. Thus we may assume that $b \neq 1$ and then 3.2.2 implies that $b = bce$, i.e. $[wx, z] = ce = 1$. Also $[wxy, xz] = acdef = 1$, contradicting G not diabelian. □

Now we move towards the general situation which involves considering G modulo a maximal abelian subgroup under different conditions. These results build up to a proof of Theorem B.

We need the following result from [10]:

If G is a group with proper subgroups H_1, H_2, H_3, then

$$G = H_1 \cup H_2 \cup H_3$$

if and only if

$$H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3 \quad \textit{and} \quad G/H_1 \cap H_2 \cong V_4.$$

3.2.8. *Let* $G = \langle A, x, y\rangle \in P_4$ *where* A *is a maximal abelian subgroup of* G *with* G/A *not cyclic and suppose that* $[x, y] = 1$. *Then* $|G'| \leq 4$.

Proof. Assume first that $A \leq C_G(x) \cup C_G(y) \cup C_G(xy)$. Then A is covered by the 3 proper subgroups $C_A(x), C_A(y), C_A(xy)$. Thus, by [10],

$$A/Z(G) = A/C_A(x) \cap C_A(y) \cong V_4$$

and so $A/Z(G)$ is generated by 2 elements. Therefore $G = Z(G)\langle a, b, x, y\rangle$ for some $a, b \in A$ and $|G'| \leq 4$ by 3.2.4.

Now suppose that $A \nleq C_G(x) \cup C_G(y) \cup C_G(xy)$. Then there is an element $a \in A$ such that $|\langle a, x, y\rangle'| = 4$. If $b \in A$, again by 3.2.4 we have $|\langle a, b, x, y\rangle'| = 4$. Therefore $G' = [A, \langle x, y\rangle]$ has order 4. □

If $[x, y] \neq 1$, we have:

3.2.9. *Let* $G = \langle A, x, y\rangle \in P_4$ *where* A *is a maximal abelian subgroup of* G *with* G/A *not cyclic and* $[x, y] \neq 1$. *Then either* $|G'| \leq 4$ *or* $|G'| = 8$ *and* $G/Z(G)$ *can be generated by* 4 *elements.*

Proof. Arguing as in the first part of 3.2.8 and using 3.2.5, we may assume that

$$A \nleq C_G(x) \cup C_G(y) \cup C_G(xy)$$

and so for some $a \in A$,

$$|\langle [a, x], [a, y]\rangle| = 4. \tag{5}$$

If $[x, y] \in [A, x][A, y]$, then $[x, y] = [a, x][b, y]$ for suitable $a, b \in A$ and $G = \langle A, ay, bx\rangle$ with $[ay, bx] = 1$. Thus, by 3.2.8, $|G'| \leq 4$. Therefore we may assume that

$$[x, y] \notin [A, x][A, y]. \tag{6}$$

Then $A/\langle [x, y]\rangle$ is a maximal abelian subgroup of $G/\langle [x, y]\rangle$ and so, by 3.2.8,

$$|(G/\langle [x, y]\rangle)'| \leq 4 \quad \text{and} \quad |G'| \leq 8,$$

as required. Also $|[A, x]| \leq 4$ and $|[A, y]| \leq 4$. Therefore

$$|A : C_A(x)| \leq \quad \text{and} \quad |A : C_A(y)| \leq 4.$$

Suppose that $C_A(x) \nleq C_A(y)$ and $C_A(y) \nleq C_A(x)$. Then there are elements $b, c \in A$ such that

$$[b, x] = [c, y] = 1, \qquad [b, y] \neq 1 \neq [c, x].$$

Let $X = \langle b, c, x, y \rangle$. Thus $|X'| \leq 4$, by 3.2.4, and hence $[b, y] = [c, x]$, by (6). Therefore, by (5) and (6), $[b, y] = [c, x] = [a, x^\gamma y^\delta]$, for some $\gamma, \delta \in \{0, 1\}$, not both 0. Let

$$Y = \langle ab^{\delta(1-\gamma)}c^\gamma, x^\gamma y^\delta, b^{\delta(1-\gamma)}c^\gamma, x^{\delta(1-\gamma)}y^\gamma \rangle.$$

Using (5) and (6) it is straightforward to check that $|Y'| = 8$. But

$$\left[ab^{\delta(1-\gamma)}c^\gamma, x^\gamma y^\delta\right] = \left[b^{\delta(1-\gamma)}c^\gamma, x^{\delta(1-\gamma)}y^\gamma\right] = 1,$$

contradicting 3.2.4.

It follows that either $C_A(x) \leq C_A(y)$ or $C_A(y) \leq C_A(x)$ and hence $|A : Z(G)| \leq 4$. Thus $G/Z(G)$ can be generated by 4 elements. □

The next 3 results deal with the case when G/A can be generated by 3 (and not 2) elements.

3.2.10. *Let $G = \langle A, x_1, x_2, x_3 \rangle \in P_4$ where A is a maximal abelian subgroup of G, $\langle x_1, x_2, x_3 \rangle$ is abelian and G/A cannot be generated by 2 elements. Then either G has an abelian subgroup of index 2 or $|G'| \leq 4$.*

Proof. Suppose that G does not have an abelian subgroup of index 2. By 3.2.8, we have for any i, j, $1 \leq i \neq j \leq 3$,

$$|\langle A, x_i, x_j \rangle'| \leq 4; \tag{7}$$

and for any $a, b \in A$, independent modulo $Z(G)$,

$$|\langle a, b, x_1, x_2, x_3 \rangle'| \leq 4, \tag{8}$$

since $H = \langle a, b, x_1, x_2, x_3 \rangle$ will not be cyclic modulo a maximal abelian subgroup B containing $\langle x_1, x_2, x_3 \rangle$. In fact, if $aB = bB$, then

$$ab^{-1} \in B \cap \langle a, b \rangle \leq Z(G) \quad \text{and} \quad a, b \in Z(G) \cap A \leq Z(H) \leq B.$$

Assume, to the contrary, that $|G'| \geq 8$. Since $[A, x_1] \neq 1$, there is an element $a_1 \in A$ such that $[a_1, x_1] \neq 1$. It is easy to see from 3.2.7 that

$A/Z(G)$ has rank ≥ 2, otherwise

$$G = Z(G)\langle a, x_1, x_2, x_3\rangle$$

for suitable $a \in A$ with $|\langle a, x_1, x_2, x_3\rangle'| \geq 8$ and $\langle a, x_1, x_2, x_3\rangle$ diabelian. If all elements of A, which are independent of a_1 modulo $Z(G)$, commute with x_2 and x_3 modulo $\langle[a_1, x_1]\rangle$, then $\langle A, x_2, x_3\rangle' \leq \langle A, x_1\rangle'$ and $G' = \langle A, x_1\rangle'$ has order ≤ 4, by (7), a contradiction. Thus there is an element $a_2 \in A$, independent of a_1 modulo $Z(G)$, such that (interchanging x_2 and x_3 if necessary)

$$|\langle[a_1, x_1], [a_2, x_2]\rangle| = 4.$$

Now rank $A/Z(G) \geq 3$, by (8), otherwise $G' = \langle a_1, a_2, x_1, x_2, x_3\rangle'$ of order at most 4. In the same way we find an element $a_3 \in A$ with a_1, a_2, a_3 independent modulo $Z(G)$ and

$$X = \langle[a_1, x_1], [a_2, x_2], [a_3, x_3]\rangle \text{ has order } 8.$$

Hence $\langle A, x_i, x_j\rangle' = \langle[a_i, x_i]\rangle \times \langle[a_j, x_j]\rangle$, for any $i \neq j$, by (7), and so $G' = X$.

Let $\{i, j, k\} = \{1, 2, 3\}$. Then $[a_i, x_j] = [a_i, x_i]^l[a_j, x_j]^m$ for some l, m, $0 \leq l, m \leq 1$. If $l = 1$, then $G' \leq \langle A, x_j, x_k\rangle'$, contradicting (7). Thus $l = 0$. If $m = 1$, then

$$G' \leq \langle a_i, a_k, x_1, x_2, x_3\rangle',$$

contradicting (8). Therefore $[a_i, x_j] = 1$ and so $G' = \langle a_1a_2a_3, x_1, x_2, x_3\rangle'$, again contradicting (8). Hence $|G'| \leq 4$. □

Next we assume that $|\langle x_1, x_2, x_3\rangle'| = 2$.

3.2.11. *Let* $G = \langle A, x_1, x_2, x_3\rangle \in P_4$ *where* A *is a maximal abelian subgroup of* G, $|\langle x_1, x_2, x_3\rangle'| = 2$ *and* G/A *cannot be generated by* 2 *elements. Then either* $|G'| \leq 4$ *or* $|G'| = 8$ *and* $G/Z(G)$ *can be generated by* 4 *elements.*

Proof. suppose that $|G'| > 4$ and without loss of generality

$$[x_1, x_2] \neq 1, \qquad [x_1, x_3] = [x_2, x_3] = 1. \tag{*}$$

Let $X = \langle A, x_1, x_2\rangle'$, $Y = \langle A, x_1, x_2x_3\rangle'$, $T = \langle A, x_2, x_1x_3\rangle'$. Suppose that $|X|$, $|Y|$, $|T| \leq 4$. Then $|X \cap T| \leq 2$, otherwise $X = T$ and $G' = X$, a contradiction. Thus $[A, x_2] \leq X \cap T = \langle[x_1, x_2]\rangle$ and so $G' = Y$, again a contradiction. Hence at least one of X, Y, T has order 8 (using 3.2.9) and we may assume without loss of generality that

$$|\langle A, x_1, x_2\rangle'| = 8 \tag{9}$$

Let $H = \langle A, x_1, x_2\rangle$. If $[x_1, x_2] = [a_1, x_1]^i[a_2, x_2]^j$ for some $a_1, a_2 \in A$, $0 \le i,\ j \le 1$, then $[a_2^j x_1, a_1^i x_2] = 1$. Therefore $H = \langle A, a_2^j x_1, a_1^i x_2\rangle$ and $|H'| \le 4$, by 3.2.8, contradicting (9). Thus

$$[x_1, x_2] \notin [A, \langle x_1, x_2\rangle]. \tag{10}$$

We claim that there is an element $a \in A$ such that

$$|\langle a, x_1, x_2, x_3\rangle'| = 8. \tag{11}$$

For, certainly $|\langle a, x_1, x_2, x_3\rangle'| \le 8$ for all $a \in A$, by 3.2.5. Thus suppose to the contrary that $|\langle a, x_1, x_2, x_3\rangle'| \le 4$ for all $a \in A$. Then, by (10), $|\langle [a, x_1], [a, x_2]\rangle| \le 2$ and so

$$A \le C_A(x_1) \cup C_A(x_2) \cup C_A(x_1x_2).$$

Hence $A/Z(H) \cong V_4$ (by [10]) and $H = \langle Z(H), h_1, h_2, x_1, x_2\rangle$ for some $h_1, h_2 \in A$. Moreover we can choose h_1, h_2 such that $[h_1, x_1] = [h_2, x_2] = 1$. But then $H' = \langle h_1, h_2, x_1, x_2\rangle'$ has order 8, by (9), contradicting 3.2.4. Thus (11) holds for some $a \in A$.

From 3.2.4, (10) and $(*)$ we obtain

$$\langle a, x_1, x_2, x_3\rangle' = \langle [x_1, x_2], [a, x_1], [a, x_2]\rangle = K \text{ say.} \tag{12}$$

Now using 3.2.4 and $(*)$ again, it follows that $[a, x_3] \in \langle [x_1, x_2]\rangle$. Thus *suppose first that* $[a, x_3] = 1$. Since A is a maximal abelian subgroup, it follows from $(*)$ that there is an element $b \in A$ such that $[b, x_3] \ne 1$. Also, by 3.2.5, (11) and (12),

$$\langle a, b, x_1, x_2\rangle' = \langle a, b, x_1, x_2x_3\rangle' = K \text{ of order } 8.$$

Therefore, by (10), $[b, x_2] \in \langle [a, x_1], [a, x_2]\rangle = L$ say. Since (9) holds with x_2 replaced by x_2x_3, so does (10), i.e. $[x_1, x_2] \notin [A, \langle x_1, x_2x_3\rangle]$. Thus $[b, x_2x_3] \in L$ and therefore $[b, x_3] \in L$. Write $[b, x_3] = [a, x_1]^i[a, x_2]^j$, $0 \le i,\ j \le 1$, i, j not both 0. Then one checks (using (10)) that

$$\langle ax_3, bx_1^i x_2^j, x_1^j x_2^{i-j}, x_3\rangle$$

has derived subgroup of order 8 and so does not belong to P_4, by 3.2.4, a contradiction. Therefore

$$[a, x_3] = [x_1, x_2]. \tag{13}$$

Now by 3.2.8, $|\langle A, x_i, x_3\rangle'| = 4$, for $i = 1, 2$, and by (11) and (13)

$$\langle A, x_1, x_3\rangle' \neq \langle A, x_2, x_3\rangle'.$$

Thus

$$\begin{aligned}[A, x_3] &= \langle [a, x_3]\rangle, \quad \text{and} \\ [A, x_1] &= \langle [a, x_1]\rangle, [A, x_2] = \langle [a, x_2]\rangle \text{ by (10)}.\end{aligned} \tag{14}$$

Since (9) holds with x_1 replaced by $x_1x_2x_3$, in the same way we obtain $|[A, x_1x_2x_3]| = 2$ and hence

$$[A, x_1x_2x_3] = \langle [a, x_1x_2x_3]\rangle.$$

Therefore $A = \langle a, C_A(x_1x_2x_3)\rangle$. Finally, from (11)–(14), it follows that

$$C_A(x_1x_2x_3) = C_A(x_1) \cap C_A(x_2) \cap C_A(x_3) = Z(G)$$

and thus $G = \langle Z(G), a, x_1, x_2, x_3\rangle$ and $|G'| = 8$. □

Having considered (in 3.2.10 and 3.2.11) the situation when $\langle x_1, x_2, x_3\rangle'$ has order at most 2, there remains the case $|\langle x_1, x_2, x_3\rangle'| \geq 4$.

3.2.12. *Let $G = \langle A, x_1, x_2, x_3\rangle \in P_4$, where A is a maximal abelian subgroup of G, $|\langle x_1, x_2, x_3\rangle'| \geq 4$ and G/A cannot be generated by 2 elements. Then either $|G'| \leq 4$ or $|G'| = 8$ and $G/Z(G)$ can be generated by 4 elements.*

Proof. Let $X = \langle x_1, x_2, x_3\rangle$. We distinguish two cases.

Case (i). *Suppose that* $|X'| = 4$. Without loss of generality we may assume that

$$[x_1, x_2] = 1.$$

If $|\langle a, X\rangle'| = 4$ for all $a \in A$, then $|G'| = 4$. Therefore suppose that

$$|\langle a, X\rangle'| = 8 \tag{15}$$

for some $a \in A$ (using 3.2.5). Now

$$[a, x_3] \notin \langle [x_1, x_3] \rangle \times \langle [x_2, x_3] \rangle = X', \tag{16}$$

otherwise $[ax_1^i x_2^j, x_3] = 1 = [x_1, x_2]$, for some $0 \leq i, j \leq 1$, and $|\langle x_1, x_2, ax_1^i x_2^j, x_3 \rangle'| = 8$, by (15), contradicting 3.2.4. Hence

$$\langle a, X \rangle' = \langle [a, x_3] \rangle \times \langle [x_1, x_3] \rangle \times \langle [x_2, x_3] \rangle.$$

Likewise, by 3.2.4,

$$[a, x_1] \neq [a, x_3] \quad \text{and} \quad [a, x_2] \neq [a, x_3]. \tag{17}$$

Assume to the contrary that $G/Z(G)$ cannot be generated by 4 elements. Then there exists $b \in A$ such that a and b are independent modulo $Z(G)$ and

$$\langle a, b, x_1, x_2 \rangle' \neq 1, \tag{18}$$

otherwise $A\langle x_1, x_2 \rangle$ $(> A)$ would be abelian.

Let $H = \langle a, b, X \rangle$. So $|H'| \geq 8$, by (16). We claim that

$$H/Z(H) \textit{ cannot be generated by 4 elements}. \tag{19}$$

For, since $Z(H) \cap A \leq Z(G)$, the elements a and b, which are independent modulo $Z(G)$, are also independent modulo $Z(H)$; $|A \cap H/Z(H) \cap A| \geq 2^2$, and $|H/Z(H) \cap A| \geq 2^5$ since

$$H/A \cap H \cong AH/A = G/A.$$

If $H/Z(H)$ can be generated by 4 elements, then $|H/Z(H)| \leq 2^4$ and there is an element $g \in Z(H) \setminus A$. Moreover $g \not\equiv x_3 \bmod A$, by (16), and so

$$G = AX = \langle A, g, x_i, x_3 \rangle,$$

for $i = 1$ or 2. But $|\langle g, x_i, x_3 \rangle'| = 2$ and thus, by 3.2.11, $G/Z(G)$ can be generated by 4 elements, contradicting our assumption. Therefore (19) must be true.

Also we claim that

$$H \textit{ does not have an abelian subgroup of index 2}. \tag{20}$$

For, suppose that B is such a subgroup. Suppose also that $a, b \in B$. Then, by (16), $x_3 \notin B$ and hence, by (17), $x_1, x_2 \in B$, contradicting (18). Therefore

$H = B\langle a, b\rangle$ and $B \cap \langle a, b\rangle \le Z(H)$, contradicting the independence of a and b modulo $Z(H)$. Thus (20) is established.

Now $H = \langle x_1, x_2\rangle^H \langle a, b, x_3\rangle$ and $\langle x_1, x_2\rangle^H$ is abelian. Let A_1 be a maximal abelian subgroup of H containing $\langle x_1, x_2\rangle$. Thus H/A_1 is not cyclic (by (20)) and cannot be generated by 2 elements (by 3.2.8 and 3.2.9, using (19)). Therefore, by 3.2.11,

$$|\langle a, b, x_3\rangle'| = 4.$$

By 3.2.9, $|\langle A, x_i, x_3\rangle'| \le 8$, for $i = 1, 2$. If $|[A, \langle x_i, x_3\rangle]| = 8$, then $[x_i, x_3] = [a_1, x_i][a_2, x_3]$, for some $a_1, a_2 \in A$, and $[a_2 x_i, a_1 x_3] = 1$, contradicting 3.2.8. It follows that

$$|[A, \langle x_i, x_3\rangle]| \le 4, \qquad i = 1, 2.$$

Thus

$$[A, \langle x_1, x_3\rangle] = [A, \langle x_2, x_3\rangle] = \langle a, b, x_3\rangle' = \langle [a, x_3]\rangle \times \langle [b, x_3]\rangle. \tag{21}$$

If $[a, x_1] = [a, x_2] = 1$, then from 3.2.9 applied to $H = \langle x_1, x_2, a\rangle^H \langle b, x_3\rangle$ with $\langle x_1, x_2, a\rangle^H$ abelian, we see that $H/Z(H)$ can be generated by 4 elements, contradicting (19). Therefore we may assume that

$$[a, x_1] \ne 1.$$

Thus, since $[a, x_1] \ne [a, x_3]$ (by (17)), (21) gives

$$[A, \langle x_1, x_3\rangle] = \langle [a, x_1]\rangle \times \langle [a, x_3]\rangle. \tag{22}$$

It follows that

$$\langle a, x_1, x_2\rangle' = \langle [a, x_1]\rangle. \tag{23}$$

For, if not, then (17), (21) and (22) give $[a, x_2] = [a, x_1][a, x_3]$ and so $[a, x_1 x_2 x_3] = 1$; then, by 3.2.4, $|\langle a, x_1 x_2 x_3, x_1, x_2\rangle'| \le 4$, contradicting (15). Therefore (23) holds.

Now, by (21) and (22),

$$[b, x_3] \in \langle a, X\rangle' = \langle [a, x_3]\rangle \times \langle [x_1, x_3]\rangle \times \langle [x_2, x_3]\rangle.$$

Thus, for suitable i, j, k,

$$[b, x_3] = [a, x_3]^i[x_1, x_3]^j[x_2, x_3]^k$$

and so $[a^ibx_1^jx_2^k, x_3] = 1$. Therefore

$$H = \langle a^ibx_1^jx_2^k, x_3\rangle^H\langle a, x_1, x_2\rangle$$

with the first factor abelian and $|\langle a, x_1, x_2\rangle'| = 2$, by (23). As we observed before, H cannot be generated by 2 elements modulo a maximal abelian subgroup, and so 3.2.11 shows that $H/Z(H)$ can be generated by 4 elements, contradicting (19). Thus $G/Z(G)$ can be generated by 4 elements and then $|G'| = 8$, by 3.2.5.

Case (ii). *Suppose that* $|X'| = 8$. By 3.2.5, $\langle a, X\rangle' = X'$, for all $a \in A$, and so $G' = X'$. Assume to the contrary that $G/Z(G)$ cannot be generated by 4 elements. Then there are elements $a, b \in A$ which are independent modulo $Z(G)$. Let $H = \langle a, b, X\rangle$. As in case (i), $H/Z(H)$ *cannot be generated by* 4 *elements*. For otherwise there is an element $g \in Z(H) \setminus A$ and $G = \langle A, g, x_i, x_j\rangle$, for suitable $i \neq j$, $1 \leq i, j \leq 3$, contradicting 3.2.11.

Let $K = \langle a, X\rangle$. By 3.2.6, there are generators y_1, y_2, y_3, y_4 of K with $[y_1, y_2] = 1$. We claim that

$$a, y_1, y_2 \textit{ are dependent modulo } \Phi(K). \tag{24}$$

For, if not, then $K = \langle a, y_1, y_2, y_i\rangle$, $i = 3$ or 4. Thus

$$H = \langle a, b\rangle^H\langle y_1, y_2, y_i\rangle$$

and $|\langle y_1, y_2, y_i\rangle'| \leq 4$. Let A_1 be a maximal abelian subgroup of H containing $\langle a, b\rangle$. It is easy to see that H does not have an abelian subgroup of index 2. For, assume $H = D\langle y\rangle$, where D is abelian and $y^2 \in D$. If, for example, $x_1, x_2 \notin D$ and $x_3 \in D$, then from $x_1D = x_2D$ it follows that $x_1^{-1}x_2 \in D$ and $[x_1, x_3] = [x_2, x_3]$, a contradiction. If $x_1, x_2, x_3 \notin D$, then $[x_1^{-1}x_2, x_1^{-1}x_3] = 1$, again a contradiction. So

$$H/A_1 \textit{ cannot be generated by } 2 \textit{ elements}, \tag{25}$$

by 3.2.8 and 3.2.9. But this contradicts 3.2.10, 3.2.11 or case (i) above. Therefore (24) is true.

Since y_1, y_2 are independent modulo $\Phi(K)$ (otherwise K would be 3-generator and H 4-generator), we have (interchanging y_1 and y_2 if necessary)

$$K = \langle a, y_1, y_3, y_4 \rangle$$

and $[a, y_1] = 1$. Thus without loss of generality we may assume that $[a, x_1] = 1$. Then $[b, x_1] \neq 1$, by (25). Therefore arguing analogously with $\langle b, x_1, x_2, x_3 \rangle$, we may assume that $[b, x_2] = 1$. Thus

$$H = \langle a, x_1 \rangle^H \langle b, x_2, x_3 \rangle,$$

$|\langle b, x_2, x_3 \rangle'| \leq 4$ and we obtain a contradiction just as we did when establishing (25). □

The classification of the groups considered in this section which belong to P_4 can now be given.

Proof of Theorem B. Suppose that $G \in P_4$ and proceed by induction on $|G|$. Suppose that there is a maximal subgroup M of G with $|M'| \geq 8$. Then, by induction, M has a normal abelian subgroup B ($\geq G'$) with M/B elementary abelian of rank ≤ 2 (using 3.2.6). Let A be a maximal abelian subgroup of G containing B. Thus G/A can be generated by 3 elements. If G/A is cyclic, then (i) holds. Otherwise if G/A can be generated by 2 elements, then (ii), (iii) or (iv) holds, using 3.2.7, 3.2.8 and 3.2.9. (Observe that if G is diabelian, then so is every subgroup K of G such that $G = Z(G)K$). If G/A cannot be generated by 2 elements, then the result follows using 3.2.7, 3.2.10, 3.2.11 and 3.2.12. If G can be generated by 4 elements, then 3.2.7 suffices. Therefore we may assume that every 4-generator subgroup H of G has $|H'| \leq 4$. In this case it is known that $|G'| \leq 4$ (Theorem A of [2]).

Conversely if G satisfies (i) or (ii), then $G \in P_4$, by 2.5. If G satisfies (iii), then $G \in P_4$, by 3.2.1; and if G satisfies (iv), then $G \in P_4$, by 3.2.7. □

REFERENCES

1. M. BIANCHI, R. BRANDL and A. GILLIO BERTA MAURI, *On the 4-permutational property,* Arch. Math., vol. 48 (1987), pp. 281–285.
2. J. BRIDE, *Second nilpotent BFC-groups*, J. Australian Math. Soc., vol. 11 (1970), pp. 9–18.
3. M. CURZIO, P. LONGOBARDI and M. MAJ, *Su di un problema combinatorio di teoria dei gruppi*, Atti Acc. Lincei Rend. Sci. Mat. Fis. Nat., vol. 74 (1983), pp. 136–142.
4. M. CURZIO, P. LONGOBARDI, M. MAJ and D.J.S. ROBINSON, *On a permutational property of groups*, *Arch. Math.*, vol. 44 (1985), pp. 385–389.
5. H. HEINEKEN, *Vektorräume mit mehreren antisymmetrischen Bilinearformen*, Arch. Math., vol. 18 (1967), pp. 449–455.
6. G. HIGMAN, *Rewriting products of group elements*, Lectures given in Urbana in 1985 (unpublished).

7. A.G. Kurosh, *The theory of groups*, 2nd edition (2 vols.), Chelsea, New York, 1960.
8. P. Longobardi, M. Maj, *On groups in which every product of four elements can be reordered*, Arch. Math., vol. 49 (1987), pp. 273–276.
9. M. Maj and S.E. Stonehewer, *Non-nilpotent groups in which every product of four elements can be reordered*, to appear.
10. G. Scorza, *I gruppi finiti che possono pensarsi come somma di tre loro sottogruppi*, Boll. U.M.I., vol. 5 (1926), pp. 216–218.
11. J.G. Semple and L. Roth, *Introduction to algebraic geometry*, O.U.P., 1985.
12. J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer-Verlag, New York, 1973.
13. M. Suzuki, *Group theory I*, A Series of Comprehensive Studies in Mathematics, vol. 247, Springer-Verlag, New York, 1982.

Universitá di Napoli
Napoli, Italy
University of Warwick
Coventry, England