ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 3, Fall 1991

REARRANGEMENT TECHNIQUES IN MARTINGALE SETTING

BY

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The concept of rearrangement function was introduced by Hardy-Littlewood [5] about sixty years ago. It played a remarkable role in Lorentz space theory and its related interpolation theory. But for a long time, people preferred the distribution function technique to the rearrangement one. It was Herz [6], Bennett-Sharpley [2] and Bagby-Kurtz [1], etc., who showed that there was no reason for this preference. In this article, we will study some examples to show what are the superiority or inferiority of the rearrangement technique in obtaining several typical inequalities in martingale theory.

Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability space with $\{\mathscr{F}_n\}_{n\geq 0}$ a nondecreasing sequence of sub-σ-fields such that $\mathscr{F} = V_n\mathscr{F}_n$, and each $(\Omega, \mathscr{F}_n, \mu)$ is complete. $f = (f_n)_{n\geq 0}$ is said to be a martingale (with respect to $\{\mathscr{F}_n\}_{n\geq 0}$), if each $f_n \in L^1(\Omega, \mathscr{F}_n, \mu)$, and $E(f_{n+1}|\mathscr{F}_n) = f_n$, $\forall n$. The Doob maximal function and the square function of the martingale $f = (f_n)_{n\geq 0}$ are defined as

$$Mf = \sup_n |f_n|, \qquad M_nf = \sup_{k\leq n}|f_k|, \tag{1}$$

$$Sf = \left(\sum_0^{\infty}|\Delta_n f|^2\right)^{1/2}, \qquad S_nf = \left(\sum_{k=0}^{n}|\Delta_k f|^2\right)^{1/2}, \tag{2}$$

where $\Delta_k f = f_k - f_{k-1}$, $k \geq 1$, $\Delta_0 f = f_0$. In what follows, we make the convention that for any process $\lambda = (\lambda_n)_{n\geq 0}$, λ_{-1} is taken to be equal to 0, unless otherwise specified. Let f be a measurable function on $(\Omega, \mathscr{F}, \mu)$. Its distribution function, rearrangement function, and averaged rearrangement function are defined respectively as

$$\sigma_f(\lambda) = |\{\omega \in \Omega : |f(\omega)| > \lambda\}|_\mu = |\{|f| > \lambda\}|, \qquad \lambda > 0, \tag{3}$$

$$f^*(t) = \inf\{\lambda : \sigma_f(\lambda) \leq t\}, \qquad t > 0, \tag{4}$$

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t > 0. \tag{5}$$

Received December 19, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 60G42; Secondary 60E15.
[1]Supported by the National Science Foundation of China.

We will need a few results about convex functions and Orlicz spaces. Let $\Phi(u)$ be a non-decreasing, non-negative function on $\mathbf{R}^+$ such that $\Phi(0) = 0$, and $\lim_{u\to\infty} \Phi(u) = \infty$. If Φ is also of moderate growth (i.e., $\Phi(2u) \le c\Phi(u)$), we will call it "general". If Φ is of moderate growth and convex, we will call it "moderate convex". For any convex Φ we use two indices

$$p_\Phi = \sup_{u>0} \frac{u\varphi(u)}{\Phi(u)}, \quad q_\Phi = \inf_{u>0} \frac{u\varphi(u)}{\Phi(u)}, \qquad \varphi(u) = \Phi'(u). \tag{6}$$

And we consider the Orlicz space L^Φ defined by

$$L^\Phi = \left\{ f \text{ measurable} : \|f\|_\Phi = \inf\left\{ \lambda : \int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) d\mu \le 1 \right\} < \infty \right\}. \tag{7}$$

It is well known that $\|f\|_\Phi$ is equivalent to

$$N_\Phi(f) = \sup\left\{ \left| \int_\Omega fg\, d\mu \right| : \rho_\Psi(g) = \int_\Omega \Psi(|g|)\, d\mu \le 1 \right\}, \tag{7'}$$

where Ψ is the Young complementary function of Φ, and that

$$\int_\Omega \Phi\left(\frac{|f|}{\|f\|_\Phi}\right) d\mu \le 1. \tag{7''}$$

For all of these facts, see Zygmund [9]. The function $\Psi(u)$ can be defined as follows

$$\Psi(u) = \int_0^u \psi(v)\, dv, \qquad \psi(u) = \inf\{ v : \varphi(v) \ge u \}. \tag{8}$$

For the Φ-inequalities between pairs (F, G) of non-negative measurable functions on $(\Omega, \mathscr{F}, \mu)$, we need the following lemmas.

LEMMA 1 (Garsia-Neveu). *Let $\Phi(u)$ be a convex function and (F, G) be a pair as above and such that*

$$\int_{\{F>\lambda\}} (F - \lambda)\, d\mu \le \int_{\{F>\lambda\}} G\, d\mu, \qquad \forall \lambda > 0. \tag{9}$$

Then we have

$$\int_\Omega \Phi(F)\, d\mu \le \int_\Omega \varphi(F) G\, d\mu. \tag{10}$$

LEMMA 2. *Let Φ be a convex function, and (F,G) be as above. Assume that*

$$\int_\Omega F\varphi(F)\,d\mu < \infty, \qquad \int_\Omega F\varphi(F)\,d\mu \le \int_\Omega G\varphi(F)\,d\mu. \tag{11}$$

Then

$$\int_\Omega \Phi(F)\,d\mu \le \int_\Omega \Phi(G)\,d\mu. \tag{12}$$

Proof. See Dellacherie [4].

Combining Lemmas 1 and 2, we see that (9) implies

$$\int_\Omega \Phi(F)\,d\mu \le \int_\Omega \Phi(p_\Phi G)\,d\mu, \qquad \|F\|_\Phi \le p_\Phi \|G\|_\Phi. \tag{13}$$

For convex Φ-inequalities with $q_\Phi > 1$, we have the following lemma.

LEMMA 3. *Let Φ be convex and such that $q_\Phi > 1$, and let (F,G) be a pair satisfying*

$$\lambda|\{F>\lambda\}| \le \int_{\{F>\lambda\}} G\,d\mu, \qquad \forall\lambda > 0. \tag{14}$$

Then we have

$$\|F\|_\Phi \le q'_\Phi \|G\|_\Phi, \tag{15}$$

where q'_Φ denotes the conjugate index of q_Φ.

Proof. See Dellacherie [4].

Now we will use the rearrangement technique to obtain several inequalities. We will first obtain the Φ-inequality between $M_a f = M(|f|^a)^{1/a}$ and $f_a^\#$ defined by

$$f_a^\# = \sup_{n\ge 0} \rho_n = \sup_{n\ge 0} E\big(|f-f_{n-1}|^a|\mathscr{F}_n\big)^{1/a}, \qquad 1 \le a < \infty, \tag{16}$$

for any L^1-bounded martingale $f = (f_n)_{n\ge 0}$.

LEMMA 4. *For any martingale* $f = (f_n)_{n\geq 0}$, *we have*

$$(Mf)^*(t) \leq 4f_a^{\#*}\left(\frac{t}{2}\right) + (Mf)^*(2t), \qquad t > 0, \tag{17}$$

$$(M_a f)^*(t) \leq 6f_a^{\#*}\left(\frac{t}{4}\right) + (M_a f)^*\left(\frac{5t}{4}\right), \qquad t > 0. \tag{17'}$$

Proof. Let $t > 0$ and f be given. When $(Mf)^*(2t)$ or $f_a^{\#*}(t/2)$ is ∞, then there is nothing to prove. When one or both of them is 0, we replace it by ε in the following argument and then let $\varepsilon \to 0$. Now assume $0 < (Mf)^*(2t)$, $f_a^{\#*}(t/2) < \infty$. Following the idea in Long [8], we define three stopping times

$$S = \inf\{n : |f_n| > (Mf)^*(2t)\},$$
$$T = \inf\left\{n : |f_n| > 4f_a^{\#*}\left(\frac{t}{2}\right) + (Mf)^*(2t)\right\},$$
$$R = \inf\left\{n : \rho_n > f_a^{\#*}\left(\frac{t}{2}\right)\right\}.$$

Notice that $S \leq T$, and

$$\{T < \infty\} = \left\{Mf > 4f_a^{\#*}\left(\frac{t}{2}\right) + (Mf)^*(2t)\right\},$$
$$\{S < \infty\} = \{Mf > (Mf)^*(2t)\},$$
$$\{R < \infty\} = \left\{f_a^{\#} > f_a^{\#*}\left(\frac{t}{2}\right)\right\},$$
$$|\{S < \infty\}| \leq 2t, \quad |\{R < \infty\}| \leq \frac{t}{2}.$$

We have

$$\begin{aligned}\{T < \infty\} &= \{T < \infty, S < R\} \cup \{T < \infty, R \leq S\}\\ &\subset \{R < \infty\} \cup \{T < \infty, S < R\},\end{aligned}$$
$$\{T < \infty, S < R\} \subset \left\{S < R, |f_T - f_{S-1}| > 4f_a^{\#*}\left(\frac{t}{2}\right)\right\}.$$

And so

$$\begin{aligned}|\{T < \infty, S < R\}| &\leq \left(4f_a^{\#*}\left(\frac{t}{2}\right)\right)^{-1} \int_{\{S<R\}} |E(f - f_{S-1}|\mathscr{F}_T)|\, d\mu\\ &\leq \left(4f_a^{\#*}\left(\frac{t}{2}\right)\right)^{-1} \int_{\{S<R\}} E(E(|f - f_{S-1}|\mathscr{F}_T)|\mathscr{F}_S)\, d\mu\\ &\leq \left(4f_a^{\#*}\left(\frac{t}{2}\right)\right)^{-1} \int_{\{S<R\}} E(|f - f_{S-1}|^a|\mathscr{F}_S)^{1/a}\, d\mu\\ &\leq \frac{1}{4}|\{S < \infty\}| \leq \frac{t}{2}.\end{aligned}$$

Thus we get

$$\left|\left\{Mf > 4f_a^{\#*}\left(\frac{t}{2}\right) + (Mf)^*(2t)\right\}\right| \le |\{R < \infty\}| + \frac{t}{2} \le t,$$

and hence (17) follows. Noticing that

$$E\left(|f|^a|\mathscr{F}_n\right)^{1/a} \le E\left(|f - f_{n-1}|^a|\mathscr{F}_n\right)^{1/a} + |f_{n-1}|,$$
$$M_a f \le f_a^{\#} + Mf,$$

and that similarly,

$$Mf \le f_a^{\#} + M_a f,$$

we get

$$(M_a f)^*(t) \le f_a^{\#*}\left(\frac{t}{4}\right) + (Mf)^*\left(\frac{3t}{4}\right) \le f_a^{\#*}\left(\frac{t}{4}\right) + 4f_a^{\#*}\left(\frac{3t}{8}\right) + (Mf)^*\left(\frac{6t}{4}\right)$$
$$\le 6f_a^{\#*}\left(\frac{t}{4}\right) + (M_a f)^*\left(\frac{5t}{4}\right).$$

The proof is finished.

Remark. (1) The result in the classical case is due to Bennett-Sharpley [2].
(2) Let Φ be any convex function, and $f = (f_n)_{n\ge 0}$ be such that

$$\int_\Omega \Phi\left(f_a^{\#}\right) d\mu < \infty.$$

Then from (17) and its consequence

$$\int_\Omega \Phi(Mf)\, d\mu \le C\int_\Omega \Phi\left(f_a^{\#}\right) d\mu < \infty$$

which we will show later, we get that $f = (f_n)_{n\ge 0}$ is at least L^1-bounded. This means that it is reasonable to consider only L^1-bounded martingales.

In order to get the Φ-inequality between $M_a f$ and $f_a^{\#}$ we need a few lemmas.

LEMMA 5. *Let (F, G) be a pair of non-negative measurable functions on $(\Omega, \mathscr{F}, \mu)$ satisfying*

$$F^*(t) \le CG^*\left(\frac{2}{t}\right) + F^*(2t), \qquad t > 0. \tag{18}$$

Then with the same C, we have

$$F^*(t) \leq 2CG^*\left(\frac{t}{2}\right) + \frac{C}{\log 2}\int_t^\infty \frac{G^*(s)}{s}\,ds, \qquad t > 0. \tag{19}$$

Proof. See Bagby-Kurtz [1].

Remark. The same assertion holds when $F^*(2t)$ is replaced by $F^*(\alpha t)$, $\alpha > 1$, and $G^*(t/2)$ by $G^*(\beta t)$, $\beta < 1$, in (18), with a modified constant.

LEMMA 6. *Let Φ be a convex function, and $1 \leq \alpha < \infty$. Let*

$$\Phi_\alpha(u) = \int_0^u \varphi_\alpha(v)\,dv,$$

where

$$\varphi_\alpha(v) = 1 + \log^{+\alpha} v,$$

and let Ψ_α be Φ_α's Young complementary function. Then Hardy's average operators

$$T : f \to Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \qquad x > 0 \tag{20}$$

$$T^* : g \to T^*g(x) = \int_x^\infty \frac{g(t)}{t}\,dt, \qquad x > 0, \tag{20'}$$

satisfy

$$\|Tf\|_\Phi \leq q'_\Phi \|f\|_\Phi, \qquad \forall f \in L^1_{\mathrm{loc}}(0,\infty), \tag{21}$$

$$\|Tf\|_{\Phi_{\alpha-1}} \leq C_\alpha \|f\|_{\Phi_\alpha}, \qquad \forall f \in L^1_{\mathrm{loc}}(0,1), \tag{22}$$

$$\|T^*g\|_\Phi \leq p_\Phi \|g\|_\Phi, \qquad \forall g \in L^1\left(\varepsilon,\infty,\frac{dt}{t}\right), \quad \forall \varepsilon > 0, \tag{23}$$

$$\|T^*g\|_{\Psi_\alpha} \leq C_\alpha \|g\|_{\Psi_{\alpha-1}}, \qquad \forall g \in L^1\left(\varepsilon,1,\frac{dt}{t}\right), \quad \forall \varepsilon > 0, \tag{24}$$

where the norms in (22) *and* (24) *are defined on* $(0,1)$.

Proof. Let us first study T. We can assume that the function f is non-negative and nonincreasing, since $|Tf| \leq Tf^*(x)$, and $\|f\|_\Phi = \|f^*\|_\Phi$. For $\lambda > 0$ given, let x_0 be the solution of $Tf(x) = \lambda$. Then

$$\{Tf(x) > \lambda\} = (0, x_0), \qquad x_0 = \frac{1}{\lambda}\int_0^{x_0} f(t)\,dt,$$

which just says that

$$|\{Tf > \lambda\}| = \frac{1}{\lambda}\int_{\{Tf>\lambda\}} f(t)\,dt.$$

Using Lemma 3, we get (21).

The proof of (23) is similar. Without loss of generality, we can assume that g is nonnegative. Let λ be given, and x_0 be such that $T^*g(x_0) = \lambda$. Then

$$\{T^*g > \lambda\} = (0, x_0)$$

and

$$\begin{aligned}\int_{\{T^*g>\lambda\}} (T^*g - \lambda)\,dx &= \int_0^{x_0}\left(\int_x^{x_0}\frac{g(t)}{t}\,dt + \int_{x_0}^{\infty}\frac{g(t)}{t}\,dt\right)dx - \lambda x_0 \\ &= \int_0^{x_0} g(t)\,dt = \int_{\{T^*g>\lambda\}} g(x)\,dx.\end{aligned}$$

Using Lemmas 1 and 2 we get (23).

If we notice that

$$\frac{u}{2}\left(1 + \log^{+\alpha}\frac{u}{2}\right) \le \Phi_\alpha(u) \le u(1 + \log^{+\alpha} u),$$

we see that, to prove (22) it is enough to show that

$$\int_0^1 |Tf|(1 + \log^{+(\alpha-1)}|Tf|)\,dx \le C_\alpha \int_0^1 |f|(1 + \log^{+\alpha}|f|)\,dx,$$
$$\forall f, \operatorname{supp} f \subset (0,1). \tag{25}$$

To get (22) from (25) we apply (25) to $\|f\|_{\Phi_\alpha}^{-1} f$, and get

$$\int_0^1 \Phi_{\alpha-1}\left(\frac{|Tf|}{\|f\|_{\Phi_\alpha}}\right)dx \le C_\alpha.$$

If we assume that $C_\alpha \ge 1$, we get, using the convexity of $\Phi_{\alpha-1}$,

$$\int_0^1 \Phi_{\alpha-1}\left(\frac{|Tf|}{C_\alpha\|f\|_{\Phi_\alpha}}\right)dx \le 1, \qquad \|Tf\|_{\Phi_{\alpha-1}} \le C_\alpha\|f\|_{\Phi_\alpha}.$$

Now let us prove (25). We can assume without loss of generality, that

$\|f\|_1 \geq 1$. We have

$$\begin{aligned}
\int_0^1 (Tf)^*(t)\log^{+(\alpha-1)}(Tf)^*(t)\,dt &\leq \int_0^1 \frac{1}{t}\int_0^t f^*(\tau)\,d\tau \log^{(\alpha-1)}\frac{\|f\|_1}{t}\,dt \\
&= \int_0^1 \int_\tau^1 \frac{1}{t}\log^{\alpha-1}\frac{\|f\|_1}{t}\,dt f^*(\tau)\,d\tau \\
&\leq C_\alpha \int_0^1 f^*(t)\log^{\alpha}\frac{\|f\|_1}{t}\,dt \\
&= C_\alpha\left(\int_{\{f^*(t)\leq(\|f\|_1/t)^{1/2}\}} + \int_{\{f^*(t)>(\|f\|_1/t)^{1/2}\}}\right) \\
&\leq C_\alpha\|f\|_1 + C_\alpha \int_0^1 f^*(t)\log^{+\alpha} f^*(t)\,dt \\
&= C_\alpha \int_0^1 f^*(1+\log^{+\alpha} f^*)\,dt.
\end{aligned}$$

Thus (22) is proved.

Now we prove (24) by duality. Since Φ_α is of moderate growth, the set $S = \{\text{all simple functions on } (0,1)\}$ is dense in $L^{\Phi_\alpha}(0,1)$. If we notice the set identity

$$\left\{f \in L^{\Phi_\alpha}(0,1) : \|f\|_{\Phi_\alpha} \leq 1\right\} = \left\{f \in L^{\Phi_\alpha}(0,1) : \int_0^1 \Phi_\alpha(|f|)\,dx \leq 1\right\},$$

we get

$$\begin{aligned}
\|T^*g\|_{\Psi_\alpha} &\leq C\sup\left\{\left|\int_0^1 T^*gf\,dx\right| : f \in S, \int_0^1 \Phi_\alpha(|f|)\,dx \leq 1\right\} \\
&= C\sup\left\{\left|\int_0^1 gTf\,dx\right| : f \in S, \|f\|_{\Phi_\alpha} \leq 1\right\} \\
&\leq C\|g\|_{\Psi_{\alpha-1}} \sup_f \|Tf\|_{\Phi_{\alpha-1}} \leq C\|g\|_{\Psi_{\alpha-1}}.
\end{aligned}$$

The proof of the lemma is finished.

Remark 1. The restriction of T^* to the set $\{f$: non-negative and nonincreasing$\}$ seems to be Φ-bounded for any general Φ, since it can be shown that it is Φ-bounded when Φ satisfies $\Phi(u+v) \leq \Phi(u) + \Phi(v)$ (for example, any nondecreasing (having an infinite limit at infinity) and concave Φ

satisfies the condition). In fact for f nonnegative and nonincreasing, we have

$$\int_0^\infty \Phi\left(\int_x^\infty \frac{f(t)}{t}\,dt\right)dx \le \int_0^\infty \sum_{k=0}^\infty \Phi\left(\int_{2^k x}^{2^{k+1}x} \frac{f(t)}{t}\,dt\right)dx$$
$$\le \sum_{k=0}^\infty \int_0^\infty \Phi(f(2^k x))\,dx = 2\int_0^\infty \Phi(f)\,dx.$$

We do not know whether it holds without added conditions imposed on Φ.

2. Ψ_α is essentially like the function $e^{(u-1)^{1/\alpha}}\chi_{\{u\ge 1\}}$, for $0<\alpha<\infty$. It is well known that $h\in L^\Psi(X)$ if and only if $\exists\theta_h>0$ such that $\int_X \Psi(\theta_h|h|)\,dx<\infty$. So (24) is of exponential type.

Now we have the following Φ-inequality between $M_a f$ and $f_a^\sharp$.

THEOREM 7. *Let $1\le a<\infty$, and Φ be a moderate convex function. We denote by Ψ_α the Young complementary function of Φ_α. Then*

$$\|M_a f\|_\Phi \le Cp_\Phi \|f_a^\sharp\|_\Phi, \qquad \forall f=(f_n)_{n\ge 0}, \tag{26}$$

$$\|M_a f\|_{\Psi_\alpha} \le C_\alpha \|f_a^\sharp\|_{\Psi_{\alpha-1}}, \qquad \forall f=(f_n)_{n\ge 0}. \tag{27}$$

Proof. From (17′) and Lemmas 5 and 6, we get

$$(M_a f)^*(t) \le Cf_a^{\sharp *}\left(\frac{t}{2}\right) + C\int_t^\infty \frac{f_a^{\sharp *}(s)}{s}\,ds,$$

$$\|M_a f\|_\Phi \le Cp_\Phi \|f_a^\sharp\|_\Phi, \qquad \|M_a f\|_{\Psi_\alpha} \le C_\alpha \|f\|_{\Psi_{\alpha-1}}.$$

The proof is finished.

Remark. In Long [8], an inequality for "general" function Φ has been obtained. Here we get the inequality valid only for moderate convex Φ, and for functions Φ satisfying

$$\Phi(u+v) \le \Phi(u) + \Psi(v)$$

(as shown in the Remark 1, after Lemma 6). But as a compensation, we get better constants, and a new inequality of exponential type. When $\alpha=1$ the exponential type inequality has been known before, since in this case $L^{\Psi_{\alpha-1}} = L^\infty$, and $\|f_a^\sharp\|_{\Psi_{\alpha-1}} = \|f\|_{BMO_a}$.

Now we want to prove the inequalities between Mf and Sf. We want to work on comparatively general objects. Let (A, B) be a pair of non-negative nondecreasing processes. Assume that A is adapted, B is predictable and $B_0 = 0$, and that there exist some constants a, q such that

$$E\Big(\big(A_T - A_{T\wedge(\tau-1)}\big)^q\Big) \leq a^q E\big(B_T^q \chi_{\{\tau<\infty\}}\big), \quad \forall \text{ stopping times } T, \tau. \tag{28}$$

Lemma 8. *Let (A, B) be as above. Then*

$$A_\infty^*(t) \leq 4^{1/q} a B_\infty^*\left(\frac{t}{2}\right) + A_\infty^*(2t). \tag{29}$$

Proof. Let $t > 0$ be given. It is enough to consider the case

$$0 < A_\infty^*(2t), \qquad B_\infty^*\left(\frac{t}{2}\right) < \infty.$$

Define the stopping times

$$T = \inf\left\{n : B_{n+1} > B_\infty^*\left(\frac{t}{2}\right)\right\}, \qquad \tau = \inf\{n : A_n > A_\infty^*(2t)\}.$$

Then with $c = 4^{1/q}a$, we have

$$\left\{A_\infty > cB_\infty^*\left(\frac{t}{2}\right) + A_\infty^*(2t)\right\} \subset \{T < \infty\} \cup \left\{A_T > cB_\infty^*\left(\frac{t}{2}\right) + A_\infty^*(2t)\right\}.$$

From our conventions, we have $A_{T\wedge(\tau-1)} = 0$ on $\{\tau = 0\}$, and so we have set inclusion

$$\left\{A_T > cB_\infty^*\left(\frac{t}{2}\right) + A_\infty^*(2t)\right\} \subset \left\{A_T - A_{T\wedge(\tau-1)} > cB_\infty^*\left(\frac{t}{2}\right)\right\}.$$

Thus we get

$$\begin{aligned}\left|\left\{A_T > cB_\infty^*\left(\frac{t}{2}\right) + A_\infty^*(2t)\right\}\right| &\leq |\{T < \infty\}| + \left(cB_\infty^*\left(\frac{t}{2}\right)\right)^{-q} \\ &\quad \times E\Big(\big(A_T - A_{T\wedge(\tau-1)}\big)^q\Big) \\ &\leq \frac{t}{2} + a^q\left(cB_\infty^*\left(\frac{t}{2}\right)\right)^{-q} E\big(B_T^q\chi_{\{\tau<\infty\}}\big) \\ &\leq t.\end{aligned}$$

This proves $A_\infty^* \leq cB_\infty^*(t/2) + A_\infty^*(2t)$, $\forall t > 0$. The proof is finished.

We now apply this lemma to several pairs $(A(f), B(f))$ associated with some martingale $f = (f_n)_{n\geq 0}$ having a predictable control $D = (D_n)_n \geq 0$ in the sense: D is non-negative, adapted, nondecreasing and such that $|\Delta_n f| \leq D_{n-1}$, $\forall n$.

THEOREM 9. *Let $f = (f_n)_{n\geq 0}$ be a martingale having a predictable control $D = (D_n)_{n\geq 0}$. Then with both sets of definition for $(A(f), B(f))$, namely*

$$\begin{aligned} A(f) &= (A_n)_{n\geq 0} = \big(M_n(f)\big)_{n\geq 0}, B(f) = (B_n)_{n\geq 0} \\ &= \big(S_{n-1}(f) + D_{n-1}\big)_{n\geq 0}, \end{aligned}$$

and

$$\begin{aligned} A(f) &= (A_n)_{n\geq 0} = \big(S_n(f)\big)_{n\geq 0}, B(f) = (B_n)_{n\geq 0} \\ &= \big(M_{n-1}(f) + D_{n-1}\big)_{n\geq 0}, \end{aligned}$$

we have

$$A_\infty^*(t) \leq cB_\infty^*\left(\frac{t}{2}\right) + A_\infty^*(2t), \qquad t > 0. \tag{30}$$

Proof. We have only to verify that for any stopping times T and τ, we have (28). In fact, say $A = Mf$, $B = Sf + D_\infty$,

$$\begin{aligned} E\Big(\big(A_T - A_{T\wedge(\tau-1)}\big)^2\Big) &= E\Big(\big(M_T f - M_{T\wedge(\tau-1)}(f)\big)^2\Big) \\ &\leq E\Big(M\big(f^{(T)} - f^{(T)}_{\tau-1}\big)^2\Big) \\ &\leq a^2 E\Big(S\big(f^{(T)} - f^{(T)}_{\tau-1}\big)^2\Big) \\ &= a^2 E\Big(S_T(f)^2 - S_{T\wedge(\tau-1)}(f)^2\Big) \\ &\leq a^2 E\Big(B_T^2 \chi_{\{\tau<\infty\}}\Big). \end{aligned}$$

This completes the proof.

COROLLARY 10. *The same assertions as those in Theorem 7 hold for the pairs*

$$(Mf, Sf + D_\infty) \text{ and } (Sf, Mf + D_\infty).$$

Let $f = (f_n)_{n \geq 0}$ still be a martingale having a predictable control $D = (D_n)_{n \geq 0}$. We use the notations

$$\begin{aligned} &A = (A_n)_{n\geq 0}, \qquad A_\infty = Mf \vee Sf, \qquad A_n = M_n f \vee S_n f, \\ &B = (B_n)_{n\geq 0}, \qquad B_\infty = Mf \wedge Sf + D_\infty, \\ &B_n = M_{n-1} f \wedge S_{n-1} f + D_{n-1}. \end{aligned}$$

It is known (see Lenglart-Lepingle-Pratelli [7]) that

$$E\Big(\big(A_T - A_{T \wedge (\tau - 1)}\big)^2\Big) \leq a^2 E\big(B_T^2 \chi_{\{\tau < \infty\}}\big), \qquad \forall \text{ stopping times } T, \tau.$$

So we also have:

THEOREM 11. *For any martingale $f = (f_n)_{n\geq 0}$ having predictable control $D = (D_n)_{n\geq 0}$, we have*

$$(Mf \vee Sf)^*(t) \leq C(Mf \wedge Sf + D_\infty)^*\left(\frac{t}{2}\right) + (Mf \vee Sf)^*(2t), \qquad t > 0. \quad (31)$$

COROLLARY 12. *The same assertions as those in Theorem 7 hold for the pair*

$$(Mf \vee Sf, Mf \wedge Sf + D_\infty).$$

The inequalities for moderate convex function Φ in Corollaries 10 and 12 can be extended to any martingale without any "predictability" as follows.

THEOREM 13. *Let Φ be a moderate convex function. Then*

$$\|Mf\|_\Phi \sim \|Sf\|_\Phi, \qquad \forall f = (f_n)_{n\geq 0}, \quad (32)$$

with the constant of equivalence $\leq Cp_\Phi^2$, and

$$\|Mf \vee Sf\|_\Phi \leq Cp_\Phi^2 \|Mf \wedge Sf\|_\Phi, \qquad \forall f = (f_n)_{n\geq 0}. \quad (33)$$

Proof. We only prove (33). Using Davis' decomposition, (see [3], Chapter 3, 14), we get $f = g + h$, with

$$|\Delta_n g| \leq 4 d_{n-1}^*, \qquad \forall n \quad (\text{where } d_n = \Delta_n f),$$

and

$$\sum_{n=0}^{\infty} |\Delta_n h| \leq \sum_{n=0}^{\infty} \left\{2(d_n^* - d_{n-1}^*) + 2E(d_n^* - d_{n-1}^* | \mathscr{F}_{n-1})\right\}. \tag{34}$$

We have

$$d^* \leq (2Mf) \wedge Sf, \qquad Mh \vee Sh \leq \sum_0^{\infty} |\Delta_n h|,$$

$$Mf \vee Sf \leq Mg \vee Sg + Mh \vee Sh,$$

$$Mg \wedge Sg \leq \min(Mf + Mh, Sf + Sh) \leq Mf \wedge Sf + \sum_0^{\infty} |\Delta_n h|.$$

Since g has a predictable control d^*, we have, using Corollary 12,

$$\begin{aligned} \|Mf \vee Sf\|_\Phi &\leq \|Mg \vee Sg\|_\Phi + \left\| \sum_0^{\infty} |\Delta_n h| \right\|_\Phi \\ &\leq Cp_\Phi(\|Mg \wedge Sg\|_\Phi + \|d^*\|_\Phi) + \left\| \sum_0^{\infty} |\Delta_n h| \right\|_\Phi \\ &\leq Cp_\Phi\left(\|Mf \wedge Sf\|_\Phi + \left\| \sum_0^{\infty} |\Delta_n h| \right\|_\Phi \right). \end{aligned}$$

But it is easy to show that

$$\left\| \sum_0^{\infty} |\Delta_n h| \right\|_\Phi \leq Cp_\Phi \|d^*\|_\Phi \leq Cp_\Phi \|Mf \wedge Sf\|_\Phi,$$

by applying Lemmas 1 and 2 to

$$F = (F_n)_{n \geq 0}, \qquad F_n = \sum_{k=0}^{n} E(\varepsilon_k | \mathscr{F}_{k-1}),$$

$$G = (G_n)_{n \geq 0}, \qquad G_n = \sum_{k=0}^{n} \varepsilon_k,$$

where $\varepsilon_k = d_k^* - d_{k-1}^*$. Thus we get (33). The proof is finished.

Remark. Both (32) and (33) without the constant estimates are known before. For (33), see [7].

Finally, we want to establish the rearrangement inequality for the pair (Sf, Mf) related to a non-negative martingale $f = (f_n)_{n \geq 0}$.

THEOREM 14. *Let $f = (f_n)_{n \geq 0}$ be a non-negative martingale. Then*

$$(Sf)^*(t) \leq 3(Mf)^*\left(\frac{t}{2}\right) + (Sf)^*(2t), \qquad t > 0. \tag{35}$$

Proof. Let $t > 0$ be given. Define the stopping times

$$\tau = \inf\left\{n : |f_n| > (Mf)^*\left(\frac{t}{2}\right)\right\},$$
$$T = \inf\{n : S_n f > (Sf)^*(2t)\}.$$

It is enough to prove

$$(Sf)^{*2}(t) \leq 9(Mf)^{*2}\left(\frac{t}{2}\right) + (Sf)^{*2}(2t), \quad t > 0. \tag{35'}$$

We have

$$\left\{S(f)^2 > 9(Mf)^{*2}\left(\frac{t}{2}\right) + (Sf)^{*2}(2t)\right\}$$
$$\subset \{\tau < \infty\} \cup \left\{S_{\tau-1}(f)^2 > 9(Mf)^{*2}\left(\frac{t}{2}\right) + (Sf)^{*2}(2t)\right\}.$$

Let us estimate

$$\left|\left\{S_{\tau-1}(f)^2 > 9(Mf)^{*2}\left(\frac{t}{2}\right) + (Sf)^{*2}(2t)\right\}\right|.$$

Without loss of generality we can assume that $\tau < \infty$, a.e., otherwise we consider $\tau_m = \tau \Lambda (T + m)$ instead, and then let $m \to \infty$. We have

$$\left\{S_{\tau-1}(f)^2 > 9(Mf)^{*2}\left(\frac{t}{2}\right) + (Sf)^{*2}(2t)\right\}$$
$$\subset \left\{T < \tau, S_{\tau-1}(t)^2 - S_T(f)^2 > 8(Mf)^{*2}\left(\frac{t}{2}\right)\right\};$$

here we have used the fact that on the set $\{T < \tau\}$, we have

$$S_T(f)^2 \leq S_{T-1}(f)^2 + (f_T - f_{T-1})^2 \leq (Mf)^{*2}\left(\frac{t}{2}\right) + (Sf)^{*2}(2t).$$

If we notice the identity

$$\sum_{k=T+1}^{\tau-1} (\Delta_k f)^2 = -2 \sum_{k=T+1}^{\tau} f_{k-1}\Delta_k f - f_T^2 - f_{\tau-1}^2 + 2f_{\tau-1}f_\tau, \tag{36}$$

and

$$E\left(\sum_{k=T+1}^{\tau} f_{k-1}\Delta_k f | \mathscr{F}_T \right) = E\left(\sum_1^\infty E\left(\chi_{\{T+1 \le k \le \tau\}} f_{k-1}\Delta_k f | \mathscr{F}_{k-1}\right) | \mathscr{F}_T \right) = 0,$$

we get

$$\begin{aligned} &\left|\left\{T < \tau, S_{\tau-1}(f)^2 - S_T(f)^2 > 8(Mf)^{*2}(2t)\right\}\right| \\ &\quad \le \left(8(Mf)^{*2}\left(\frac{t}{2}\right)\right)^{-1} E\left(\left(S_{\tau-1}(f)^2 - S_T(t)^2\right)\chi_{\{T<\tau\}}\right) \\ &\quad \le \left(8(Mf)^{*2}\left(\frac{t}{2}\right)\right)^{-1} E\left(E(2f_{\tau-1}f_\tau|\mathscr{F}_T)\chi_{\{T<\tau\}}\right) \le \frac{1}{4}|\{T<\infty\}| \le \frac{t}{2}. \end{aligned}$$

This completes the proof of the theorem.

Remark. The proof is essentially due to Burkholder [3].

COROLLARY 15. *For Φ and Ψ_α as in Theorem 7, and any non-negative martingale $f = (f_n)_{n \ge 0}$, we have*

$$\|Sf\|_\Phi \le Cp_\Phi \|Mf\|_\Phi, \tag{37}$$

$$\|Sf\|_{\Psi_\alpha} \le C_\alpha \|Mf\|_{\Psi_{\alpha-1}}. \tag{38}$$

The author would like to express his deep thanks to Professor D.L. Burkholder for his kind and valuable help.

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