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ON THE BRAUER GROUP AND QUOTIENT SINGULARITIES

BY

TIMOTHY J. FORD[1]

Let k be an algebraically closed field with characteristic 0. Let $\mathrm{B}(\cdot)$ denote the Brauer group functor as defined in [AG]. Let A be a regular local ring which is a k-algebra essentially of finite type. Let G be a finite group of k-automorphisms of A. Suppose no height 1 prime of A ramifies over A^G. Let P be a prime ideal of height ≥ 2 in A^G and let R be the local ring $(A^G)_P$. Set $S = A \otimes_{A^G} R$. Then G acts on S and $S^G = R$. So S is a finite R-module and no height 1 prime of S ramifies over R. If $K = K(A^G)$ denotes the quotient field, we have the following inclusion relations:

$$\begin{array}{ccccc} A & \subseteq & S & \subseteq & K(A) \\ \uparrow & & \uparrow & & \uparrow \\ A^G & \subseteq & R & \subseteq & K \end{array}$$

Therefore S is a localization of A in the field of fractions $K(A)$ hence is a regular domain. Since S is finite over R and R is local, S is a semilocal ring. We say that *the ring R has quotient singularities* if S is a local ring. The maximal ideals of S correspond to the prime ideals of A lying over P, so we see that R has quotient singularities if and only if there is a unique prime ideal Q of A lying over P.

In this short note, we investigate the kernel $\mathrm{B}(K/R)$ of the natural map $\tau\colon \mathrm{B}(R) \to \mathrm{B}(K)$. If R is regular, it is known that $\mathrm{B}(K/R) = (0)$ [AG, Theorem 7.2, p. 388]. For this reason we are primarily interested in the situation where R actually has singularities. This study was motivated by similar questions about the Brauer group and rational singularities on surfaces that were answered in Section 1 of [FS]. Theorem 1 below can also be considered an attempt to correct Theorem 12 of [DF] which is false; a counterexample is given in [DFM]. The example is a normal algebraic surface X with isolated rational singular point P such that $\ker \tau$ is finite and non-trivial.

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THEOREM 1. *If R is a local ring with quotient singularities and quotient field K, the natural map* $\mathrm{B}(R) \to \mathrm{B}(K)$ *is injective.*

Proof. Let $(\tilde{R}, \tilde{m})$ be the strict henselization of (R, m), where m is the maximal ideal of R [R1]. Then $k(\tilde{m}) = \tilde{R}/\tilde{m}$ is the algebraic closure of $k(m) = R/m$. Let (R^h, m^h) be the henselization of R. By our hypotheses on R, S is a local ring. Therefore, by [F, Corollary 9.3, p. 40], $\mathrm{Cl}(S) = \mathrm{Pic}(S) = 0$. Set $\tilde{S} = S \otimes_R \tilde{R}$ and $S^h = S \otimes_R R^h$. Since S is finite over R, S^h is finite over R^h and $\tilde{S}$ is finite over $\tilde{R}$. Thus S^h is the henselization of S and $\tilde{S}$ is the strict henselization of S. Since S is regular, S^h and $\tilde{S}$ are regular and by [F, Corollary 9.3, p. 40], $\mathrm{Cl}(S^h) = 0$ and $\mathrm{Cl}(\tilde{S}) = 0$. Let $M, M^h, \tilde{M}$ be the maximal ideals of $S, S^h, \tilde{S}$ respectively. Then $S/M \cong S^h/M^h$ is a finite extension field of $k(m)$ and $\tilde{S}/\tilde{M} \cong k(\tilde{m})$. Also G acts on S^h and $\tilde{S}$ such that $(S^h)^G = R^h$ and $\tilde{S}^G = \tilde{R}$. Since $R \subseteq R^h \subseteq \tilde{R}$ are faithfully flat extensions of local rings, there are embeddings of divisor class groups $\mathrm{Cl}(R) \subseteq \mathrm{Cl}(R^h) \subseteq \mathrm{Cl}(\tilde{R})$ [F, Corollary 6.11, p. 35]. From Corollary 1.8 of [CGO] (or [G] if R has a unique singular point) we find that the kernel $\mathrm{B}(K/R)$ of the map $\mathrm{B}(R) \to \mathrm{B}(K)$ embeds in the quotient $\mathrm{Cl}(\tilde{R})/\mathrm{Cl}(R)$. Thus it suffices to prove:

LEMMA 2. *In the above context,* $\mathrm{Cl}(R) = \mathrm{Cl}(\tilde{R})$.

Proof. By Theorem 16.1 of [F, p. 82] there are natural isomorphisms

$$\mathrm{Cl}(R) \cong \mathrm{H}^1(G, S^*), \quad \mathrm{Cl}(R^h) \cong \mathrm{H}^1(G, S^{h*}), \quad \mathrm{Cl}(\tilde{R}) \cong \mathrm{H}^1(G, \tilde{S}^*).$$

Let $\mu(\tilde{S})$ be the group of roots of unity in $\tilde{S}$. Then $\mu(\tilde{S}) \cong \mathbf{Q}/\mathbf{Z}$ since k is algebraically closed of characteristic 0. Consider the sequence

$$1 \to \mu(\tilde{S}) \to \tilde{S}^* \to V \to 1 \tag{1}$$

where V is defined to be the quotient $\tilde{S}^*/\mu(\tilde{S})$. Then (1) is split exact since $\mu(\tilde{S})$ is divisible. If $\alpha \in \tilde{S}^*$, then for every integer $l \geq 1$, $x^l - \alpha$ splits over $\tilde{S}/\tilde{M}$ hence splits over $\tilde{S}$ by the henselian property. Thus $\tilde{S}^*$ is a divisible abelian group. Consequently V is a uniquely divisible abelian group. So multiplication by l is an automorphism on $\mathrm{H}^1(G, V)$. But G is finite and $\mathrm{H}^1(G, V)$ is annihilated by $|G|$. So $\mathrm{H}^1(G, V) = 1$ and (1) gives $\mathrm{H}^1(G, \mu(\tilde{S})) \cong \mathrm{H}^1(G, \tilde{S}^*)$. Since $\tilde{S}$ is a domain, $\mu(S) = \mu(S^h) = \mu(\tilde{S})$. So $\mathrm{H}^1(G, \mu(\tilde{S})) = \mathrm{H}^1(G, \mu(S))$. The sequence $1 \to \mu(S) \to S^*$ is split exact since $\mu(S)$ is divisible. Thus $\mathrm{H}^1(G, \mu(S)) \to \mathrm{H}^1(G, S^*)$ is injective. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{Cl}(\tilde{R}) & \xrightarrow{\cong} & \mathrm{H}^1(G, \tilde{S}^*) \xleftarrow{\cong} \mathrm{H}^1(G, \mu(\tilde{S})) & = & \mathrm{H}^1(G, \mu(S)) \\
\uparrow{\scriptstyle 1-1} & & & & {\scriptstyle 1-1}\downarrow \\
\mathrm{Cl}(R) & & \xrightarrow{\qquad\cong\qquad} & & \mathrm{H}^1(G, S^*)
\end{array} \tag{2}$$

Since G acts trivially on $\mu(S)$, $H^1(G, \mu(S)) = \text{Hom}(G, \mu(S))$ [R2, p. 280] which is finite since G is finite. Hence all groups in (2) are finite and $\text{Cl}(R) = \text{Cl}(\tilde{R})$. ■

Example 3. Let k be an algebraically closed field of characteristic 0. Let R be a regular local ring essentially of finite type over k having residue field k. Let f, g be local equations for a divisor on Spec R with normal crossings. Let

$$S = R\left((fg)^{1/n}\right) \quad \text{and} \quad T = R\left(f^{1/n}, g^{1/n}\right).$$

Let $X = \text{Spec}\, T$. Then $S = T^G$ where $G = \langle \sigma \rangle$ and σ is defined by

$$f^{1/n} \mapsto \zeta f^{1/n}, \quad g^{1/n} \mapsto \zeta^{-1} g^{1/n}$$

for a primitive nth root of 1ζ. The map $X \to \overline{X} = \text{Spec}\, S$ is unramified at all height 1 primes. Thus S has quotient singularities. If L is the quotient field of S, we have $B(L/S) = 0$ by Theorem 1.

References

[AG] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., vol. 97 (1960), pp. 367–409.

[CGO] L. Childs, G. Garfinkel and M. Orzech, *On the Brauer group and factoriality of normal domains*, J. Pure Appl. Algebra, vol. 6 (1975), pp. 111–123.

[DF] F. DeMeyer and T. Ford, *On the Brauer group of surfaces*, J. Algebra, vol. 86 (1984), pp. 259–271.

[DFM] F. DeMeyer, T. Ford and R. Miranda, *Rational singularities and the Brauer group*, J. Algebra, to appear.

[FS] T. Ford and D. Saltman, *Division algebras over henselian surfaces*, Israel Math. Conf. Proc., vol. 1 (1989), pp. 320–336.

[F] R. Fossum, *The divisor class group of a Krull domain*, Springer-Verlag, New York, 1973.

[G] A. Grothendieck, "Le groupe de Brauer II" in: *Dix Exposés sur la Cohomologie des schémas*, North Holland, Amsterdam, 1968.

[M] J. Milne, *Etale cohomology*, Princeton University Press, Princeton 1980.

[R1] M. Raynaud, *Anneaux Locaux Henseliens*, Lecture Notes in Math., Vol. 169, Springer-Verlag, Berlin 1970.

[R2] J. Rotman, *An introduction to homological algebra*, Academic Press, Orlando 1979.

Florida Atlantic University
Boca Raton, Florida