ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 3, Fall, 1991

BOUNDEDNESS OF CERTAIN MULTIPLIER OPERATORS IN FOURIER ANALYSIS ON WEIGHTED LEBESGUE SPACES

BY

DONALD KRUG[1]

1. Introduction

The purpose of this paper is to extend known results on the relationship between Hardy-Littlewood type maximal functions and certain multi-directional generalizations of the Hilbert transform to the case of weighted Lebesgue spaces. It is well known that the boundedness of the Hardy-Littlewood maximal function on the spaces $L^p(\mathbf{R})$, $1 < p < \infty$, is closely related to the boundedness of the Hilbert transform on these same spaces. In their paper, *On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier Analysis* [3], A. Cordoba and R. Fefferman study the relationship between two operators, one related to the Hardy-Littlewood maximal function and one to the Hilbert transform, whose boundedness properties are not so well known.

Specifically, let $\theta_1 > \theta_2 > \theta_3 > \dots$ be a decreasing sequence of angles, $0 < \theta_i < \pi/2$. Define the maximal function M_θ on $L^p(\mathbf{R}^2)$ by

$$M_\theta(f)(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where each rectangle $R \subset \mathbf{R}^2$ is oriented in one of the directions θ_i. Let P_θ be the subset of the plane shown in Fig. 1. Consider the multiplier T_θ (defined initially on $L^2(\mathbf{R}^2)$) given by $\widehat{T_\theta(f)}(t) = X_{p_\theta}(t)\hat{f}(t)$, where X_{p_θ} is the characteristic function of P_θ, and $\hat{g}$ denotes the Fourier transform of g.

Cordoba and Fefferman have proven the following two results giving the relationship between the boundedness of M_θ and T_θ:

THEOREM A. *If for some $p > 2$, M_θ is a bounded operator on $L^{(p/2)'}(\mathbf{R}^2)$, then T_θ is also bounded, but on the space $L^p(\mathbf{R}^2)$.*

Received September 11, 1989.

1980 Mathematics Subject Classification (1985 Revision) Primary 42B15; Secondary 42B25.

[1]This work is part of the author's dissertation completed at Indiana University under the supervision of Alberto Torchinsky.

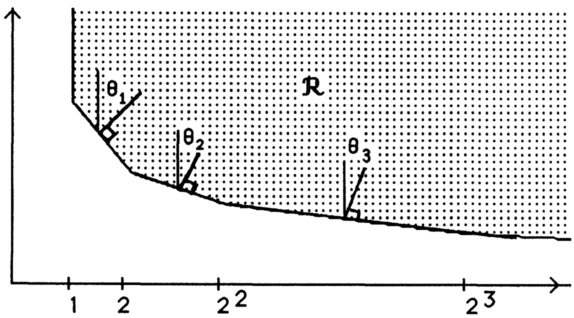

FIG. 1

THEOREM B. *If for some $p > 2$, T_θ is bounded on $L^p(\mathbf{R}^2)$, then under the additional assumption that $|\{M_\theta(X_E) > 1/2\}| \leq C|E|$ for all measurable $E \subset \mathbf{R}^2$, it follows that M_θ is of weak type $[(p/2)', (p/2)']$.*

In this paper we extend Theorem A to the case of weighted spaces. Let $w(x, y)$ be a locally integrable nonnegative function of two variables. Let $d\mu = w d\lambda$, where $d\lambda$ denotes Lebesgue measure on $\mathbf{R}^2$. (In this paper $d\mu$ always denotes $w d\lambda$, for other weights we will use $d\nu$ or $d\sigma$.) We obtain two versions of Theorem A; for the precise definitions see the material that follows.

THEOREM 4B. *If $p > 2$, $w \in A_2(\theta)$ and $w \in A_2(\mathbf{R}^2)$, and if M_θ is bounded on $L_\nu^{(p/2)'}(\mathbf{R}^2)$ where $\nu = w^{1-(p/2)'}$, then T_θ is bounded on $L_\mu^p(\mathbf{R}^2)$, with norm depending only on the $A_2(\theta)$ and $A_2(\mathbf{R}^2)$ constants of w and the norm of M_θ.*

THEOREM 6. *If $p_0 > 2$ and M_θ is bounded on $L_\nu^{p'}(\mathbf{R}^2)$, where $\nu = w^{1-p'}$, and $p \in [p_0, p_0 - \varepsilon)$ for some $\varepsilon > 0$, then the multiplier operator T_θ is bounded on $L_\mu^{p_0}(\mathbf{R}^2)$, with norm depending only on the norm of M_θ and the $A_p(\theta)$ constant of w.*

Theorem 4B gives, in particular, the result of Cordoba and Fefferman in the case $w \equiv 1$. However, Theorem 6 implies that $w \in A_{p_0}(\theta)$ (rather than A_2) which is what one would hope for.

In proving Theorem 4B we obtain a result relating a weighted integral inequality to vector-valued inequalities, and in proving Theorem 6 we use results related to extrapolation. In Section 6 we consider a result on extrapolation in the directions θ_i. We finish the paper with several applications of the results mentioned above, including a weighted version of the angular Littlewood-Paley inequality.

I would like to express my appreciation to Professor Alberto Torchinsky for his constant support and encouragement. I especially want to thank him

for reading an earlier version of this paper and offering numerous helpful suggestions.

2. A condition which implies boundedness

In this section we use a condition, shown by J.L. Rubio de Francia [14], [15] to be related to vector-valued inequalities and interpolation, which will imply the boundedness of the operator T_θ on L^p_μ. In later sections we will relate this condition to the boundedness of M_θ. The boundedness condition is:

$BC(\mu, p)$. Let $p > 2$ be given. For each g in $L^{(p/2)'}_\mu(\mathbf{R}^2)$, $g \geq 0$, there is a $G \geq g$, G in $L^{(p/2)'}_\mu(\mathbf{R}^2)$ such that

$$\|G\|_{L^{(p/2)'}_\mu} \leq C_p \|g\|_{L^{(p/2)'}_\mu}$$

and $G \cdot w \in A_2(\theta)$. Here c depends only on p and μ, and is independent of g.

$A_p(\theta)$ denotes the class of all those functions w such that

$$\sup_R \left(\frac{1}{|R|} \int_R w(x, y)\, dx\, dy \right) \left(\frac{1}{|R|} \int_R w(x, y)^{-1/p-1}\, dx\, dy \right)^{p-1} = C < \infty$$

where the supremum is taken over all rectangles in one of the directions θ_i, $i = 1, 2, 3, \ldots$. The number C is called the $A_p(\theta)$-constant for w. We also let $A_p(\mathbf{R}^2)$ be the A_p condition with the supremum taken over rectangles oriented in the direction of the coordinate axes.

THEOREM 1. *Given $p > 2$ and $w \in A_p(\mathbf{R}^2)$, assume that the boundedness condition $BC(\mu, p)$ is true. Then T_θ is bounded on $L^p_\mu(\mathbf{R}^2)$ with norm depending only on the $A_p(\theta)$ and $A_p(\mathbf{R}^2)$-constants of w, and the A_2-constant of $G \cdot w$.*

Proof. Consider the infinite strip

$$E_k = \{(x, y) \in \mathbf{R}^2 \colon 2^k \leq x < 2^{k+1}\}.$$

Define the multiplier operator S_k by $\widehat{S_k}(f)(x) = X_{E_k}(x) \cdot \hat{f}(x)$. Kurtz [10] shows that

$$\|f\|_{L^p_\mu} \approx \left\| \left(\sum |S_k f|^2 \right)^{1/2} \right\|_{L^p_\mu} \quad \text{for} \quad w \in A_p(\mathbf{R}^2).$$

($A \approx B$ means there are constants c and c' such that $cA \leq B \leq c'A$.) Let F_k be the half-plane shown in Fig. 2. Define H_k, initially on $L^2(\mathbf{R}^2)$, by

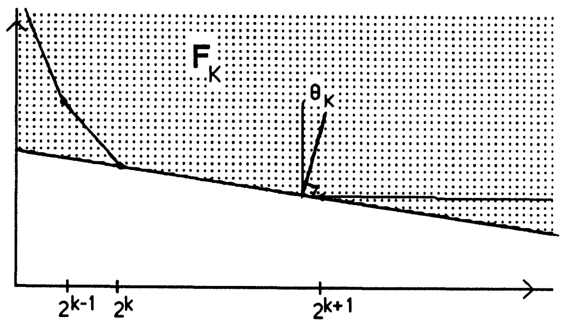

FIG. 2

$(H_k f)\hat{}(x) = X_{F_k}(x) \cdot \hat{f}(x)$. Then H_k is essentially the Hilbert transform oriented in the direction θ_k and consequently is bounded on $L^p_\mu(\mathbf{R}^2)$ for $\mu \in A_p(\theta_k)$ (i.e., A_p where the supremum is taken over all rectangles oriented in the direction θ_k). Note, also, that $S_k T_\theta(f) = H_k S_k(f)$: Indeed,

$$(S_k T_\theta)\hat{}(f) = X_{E_k} \cdot X_{P_0} \cdot \hat{f} = X_{F_k} \cdot X_{E_k} \cdot \hat{f} = (H_k S_k)\hat{}(f).$$

Thus

$$\begin{aligned} \|T_\theta(f)\|^p_{L^p_\mu} &\le C\left\| \left(\sum_k |S_k T_\theta(f)|^2 \right)^{1/2} \right\|^p_{L^p_\mu} \\ &= C\left\| \left(\sum_k |H_k S_k(f)|^2 \right)^{1/2} \right\|^p_{L^p_\mu} \\ &= C\left\| \sum_k |H_k S_k(f)|^2 \right\|^{p/2}_{L^{p/2}_\mu}. \end{aligned}$$

We estimate this last norm using duality. Choose $g \ge 0$, $g \in L^{(p/2)'}_\mu(\mathbf{R}^2)$ with $\|g\|_{L^{(p/2)'}_\mu} \le 1$, and note that

$$\begin{aligned} &\int_{\mathbf{R}^2} \sum_k |H_k S_k(f)(x)|^2 g(x) w(x)\, dx \\ &\quad = \sum_k \int_{\mathbf{R}^2} |H_k S_k(f)(x)|^2 g(x) w(x)\, dx \\ &\quad \le \sum_k \int_{\mathbf{R}^2} |H_k S_k(f)(x)|^2 G(x) w(x)\, dx \\ &\quad \le C \sum_k \int_{\mathbf{R}^2} |S_k(f)(x)|^2 G(x) w(x)\, dx, \end{aligned}$$

where G is the function from the boundedness condition, and the last inequality follows since H_k is bounded for $G \cdot w \in A_2(\theta_k)$. Hence,

$$\begin{aligned}
&\int_{\mathbf{R}^2} \sum_k |H_k S_k(f)(x)|^2 g(x) w(x)\, dx \\
&\quad \leq c \int_{\mathbf{R}^2} \sum_k |S_k(f)(x)|^2 G(x) w(x)\, dx \\
&\quad \leq c \left[\int_{\mathbf{R}^2} \left(\sum_k |S_k(f)(x)|^2 \right)^{p/2} w(x)\, dx \right]^{2/p} \\
&\qquad \times \left[\int_{\mathbf{R}^2} (G(x))^{(p/2)'} w(x)\, dx \right]^{1-2/p} \\
&\quad \leq c \|f\|^2_{L^p_\mu} \cdot \|g\|_{L^{(p/2)'}_\mu} \leq c \|f\|^2_{L^p_\mu}.
\end{aligned}$$

If we take the supremum of all such functions g it follows that

$$\|T_\theta(f)\|_{L^p_\mu} \leq c \|f\|_{L^p_\mu},$$

where c depends only on the $A_2(\theta)$-constant of $G \cdot w$ and the $A_p(\mathbf{R}^2)$-constant of w.

In the next three sections we discuss conditions on M_θ that imply the boundedness condition, and hence boundedness of T_θ.

3. Weighted vector-valued inequalities

Rubio de Francia [14] has shown that the boundedness condition $BC(\mu, p)$, is equivalent to certain weighted vector-valued inequalities. We will show that the same type of result is true for the weights in $Ap(\theta)$.

THEOREM 2. *Given a weight w and $p > 2$, the following conditions are equivalent*:

(a) $$\left\| \left(\sum_j |M_j f_j|^2 \right)^{1/2} \right\|_{L^p_\mu} \leq C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p_\mu}$$

where M_j is the maximal function with supremum taken over all rectangles oriented in the direction θ_j, $f_j \in L^p_\mu(\mathbf{R}^2)$, $j = 1, 2, \dots$.

(b)
$$\left\| \left(\sum_j |H_j f_j|^2 \right)^{1/2} \right\|_{L^p_\mu} \le C'_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p_\mu}$$

and

$$\left\| \left(\sum_j |H_j^{\perp} f_j|^2 \right)^{1/2} \right\|_{L^p_\mu} \le C''_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p_\mu}$$

where H_j is the Hilbert transform in the direction θ_j and $H_j^{\perp}$ the Hilbert transform in the direction $\theta_j + \pi/2$, and $f_j \in L^p_\mu(\mathbf{R}^2)$, $j = 1, 2, \dots$.

(c) *The boundedness condition $BC(\mu, p)$: For each $g \ge 0$, $g \in L^{(p/2)'}_\mu$, there is a $G \ge g$ with*

$$\|G\|_{L^{(p/2)'}_\mu} \le K \|g\|_{L^{(p/2)'}_\mu}$$

and $G \cdot w \in A_2(\theta)$.

Here c_p, c'_p, and c''_p depend only on K and p and the $A_2(\theta)$-constant of $G \cdot w$.

To prove this we will need to assume the following result of Rubio de Francia [15]:

THEOREM. *Let $F = \{T_j\}$ be a family of sublinear operators T_j, $T_j\colon L^q_\mu \to L^q_\mu$, then*

$$\left\| \left(\sum_j |T_j f_j|^2 \right)^{1/2} \right\|_{L^q_\mu} \le c_q \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^q_\mu}$$

if and only if for $r = (q/2)'$ and $g \ge 0$, $g \in L^r_\mu$, there is a $G \in L^r_\mu$ such that $\|G\|_{L^r_\mu} \le c\|g\|_{L^r_\mu}$ with $G \ge g$ and

$$\int |T_j f|^2 G(x)\, d\mu \le c_p \int |f|^2 G(x)\, d\mu,$$

for $j = 1, 2, \dots$, with c_p independent of j.

We will assume this result and move on to the proof of Theorem 2.

(*c*) *implies* (*a*).

$$\left\| \left[\sum_j |M_j f_j|^2 \right]^{1/2} \right\|_{L^p_\mu}^2 = \sup \sum_j \int |M_j f_j(x)|^2 g(x) w(x)\, dx$$

where the sup is taken over $g \geq 0$, $\|g\|_{L_\mu^{(p/2)'}} \leq 1$

$$\leq \sup_g \sum_j \int |M_j f_j(x)|^2 G(x) w(x)\, dx$$

$$\leq c_p \left(\sup_g \sum_j \int |f_j(x)|^2 G(x) w(x)\, dx \right)$$

$$\leq c_p \left(\sup_g \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p_\mu}^2 \|G\|_{L_\mu^{(p/2)'}} \right)$$

$$\leq c_p \left\| \left[\sum_j |f_j|^2 \right]^{1/2} \right\|_{L^p_\mu}^2 .$$

The second inequality follows since M_j is bounded on $L^2_\mu(\mathbf{R}^2)$ for $\mu = w(x)\, dx$ in $A_2(\theta_j)$.

(*c*) *implies* (*b*). Since both H_j and $H_j{}^\perp$ are bounded on $L^2_\mu(\mathbf{R}^2)$ for w in $A_2(\theta_j)$, for $j = 1, 2, \ldots,$ the proof is exactly as (c) implies (a) above, with M_j replaced by H_j (or $H_j{}^\perp$).

(*a*) *implies* (*c*). Since the vector valued inequality holds then, by the theorem of Rubio de Francia stated above, for any nonnegative g in $L_\mu^{(p/2)'}$, there is a G with $g \leq G$,

$$\|G\|_{L_\mu^{(p/2)'}} \leq c' \|g\|_{L_\mu^{(p/2)'}}$$

and

$$\int |M_j f|^2 G(x) w(x)\, dx \leq c \int |f|^2 G(x) w(x)\, dx$$

for c independent of $f \in L^p_\mu$ and $j = 1, 2, 3, \ldots$. This implies that $G \cdot w \in$

$A_2(\theta_j)$ with $A_2(\theta_j)$ constant independent of j. Hence $G \cdot w \in A_2(\theta)$ and thus G satisfies the condition for boundedness $BC(\mu, p)$.

(b) *implies* (c). This follows as above, since we get both

$$\int |H_j f(x)|^2 G(x) w(x)\, dx \leq c_1 \int |f(x)|^2 G(x) w(x)\, dx$$

and

$$\int \left| H_j^{\perp} f(x) \right|^2 G(x) w(x)\, dx \leq c_2 \int |f(x)|^2 G(x) w(x)\, dx$$

for all j, which implies that $G \cdot w \in A_2(\theta_j)$ with $A_2(\theta_j)$-constant independent of j. Hence $G \cdot w \in A_p(\theta)$. *Note*: In the above proof we used the fact that if H_j and $H_j^{\perp}$ are bounded on $L^2_\mu(\mathbf{R}^2)$, then $w \in A_2(\theta_j)$. The idea behind this is as follows. For convenience of notation let's suppose the Hilbert transforms H_x and H_y are bounded on L^2_μ; here H_x is the Hilbert transform in the direction of the x-axis and similarly for H_y. We will also let M_x and M_y be the one-dimensional Hardy-Littlewood maximal functions in the direction of the x and y axes respectively. Then $w(\cdot, y) \in A_2(\mathbf{R}^1)$ with constant independent of y and $w(x, \cdot\,) \in A_2(\mathbf{R}^1)$ with constant independent of x.

Hence

$$\|M(f)\|_{L^2_\mu(\mathbf{R}^2)} \leq \|M_x(M_y f)\|_{L^2_\mu} \leq c\|f\|_{L^2_\mu}$$

and this implies $w \in A_2(\mathbf{R}^2)$.

4. A weighted integral inequality

The main result of this section is a weighted integral inequality for the strong maximal function. From this the first of the theorems in which boundedness of M_θ implies that of T_θ follows.

LEMMA 1. *If $w \in A_p(\mathbf{R}^2)$, $1 < p < \infty$, then for some $\varepsilon > 0$, $w \in A_{p-\varepsilon}(\mathbf{R}^2)$ with $A_{p-\varepsilon}(\mathbf{R}^2)$-constant and ε depending only on p and the $A_p(\mathbf{R}^2)$-constant of w.*

Proof. A.P. Calderon has proven that if $w \in A_p(\mathbf{R})$ then $w \in A_{p-\varepsilon}(\mathbf{R})$ where ε and the $A_{p-\varepsilon}(\mathbf{R})$-constant of w depend only on p and the $A_p(\mathbf{R})$ constant of w. (See [1], Theorems 1 and 2 as well as the proof of Theorem 2.)

In this case, for each x, $w(x, -) \in A_p(\mathbf{R})$ with constant independent of x. Hence $w(x, -) \in A_{p-\varepsilon}(\mathbf{R})$ with ε and the $A_{p-\varepsilon}(\mathbf{R})$-constant of $w(x, -)$

independent of x. Similarly $w(-, y) \in A_{p-\varepsilon}(\mathbf{R})$ with ε and constant independent of y. So $w \in A_{p-\varepsilon}(\mathbf{R}^2)$ with ε and $A_{p-\varepsilon}(\mathbf{R}^2)$-constant depending only on p and the $A_{p-\varepsilon}(\mathbf{R}^2)$-constant of w.

More notation will be needed before we state Lemma 2. For f in $L^p(\mathbf{R}^2)$ let $M_x(f)$ be the Hardy-Littlewood maximal function of f in the x-variable only (similarly for $M_y(f)$). For a weight w define

$$M_\mu(f)(x) = \sup_R \frac{1}{\mu(R)} \int_R f(x)\, d\mu(x)$$

where the supremum is taken over all rectangles R containing x with sides parallel to the axes and $d\mu(x) = w(x)\,dx$.

LEMMA 2. *If $w \in A_p(\mathbf{R}^2)$, $1 < p < \infty$, then*

$$\int_{\mathbf{R}^2} |M_x(M_y(f))(x)|^p g(x)\, d\mu(x) \le c_p \int_{\mathbf{R}^2} |f(x)|^p M_\mu(M_\mu(g))(x)\, d\mu(x)$$

with c_p independent of f and g and depending only on p and the $A_p(\mathbf{R}^2)$-constant of w.

Proof. Let $q = p - \varepsilon$. Then for any rectangle R,

$$\begin{aligned}
&\left(\frac{1}{|R|}\int_R g(w)w(x)\,dx\right)\left(\frac{1}{|R|}\int_R (M_w(g)(x)w(x))^{-1/q-1}\,dx\right)^{q-1} \\
&\quad\le \left(\frac{1}{|R|}\int_R g(x)w(x)\,dx\right) \\
&\qquad \times \left(\frac{1}{|R|}\int_R \left(\frac{w(R)}{\int_R g(y)(w(y)\,dy}\right)^{1/q-1} w(x)^{-1/q-1}\,dx\right)^{q-1} \\
&\quad\le \frac{1}{|R|} w(R) \left(\frac{1}{|R|}\int_R w(x)^{-1/q-1}\,dx\right)^{q-1} \\
&\quad\le A_q(\mathbf{R}^2)\text{-constant of } w.
\end{aligned}$$

So $(gw, (M_w g)w) \in A_{p-\varepsilon}(\mathbf{R}^2)$ with constant no more than the $A_{p-\varepsilon}(\mathbf{R}^2)$ constant of w.

As in Lemma 1, this implies that $(gw, (M_w(g))w) \in A_{p-\varepsilon}(\mathbf{R}^1)$ uniformly in each variable. Hence,

$$\int_{\{M_x(M_y(f))>x\}} g(x,y)w(x,y)\,dx \le \frac{c}{\alpha^{p-\varepsilon}}\int_{\mathbf{R}^2} M_y(f)^{p-\varepsilon} M_w(g)w(x)\,dx.$$

By interpolating with the trivial $L^\infty_\mu(\mathbf{R}^2)$ result, the corresponding strong-type inequality holds for p. Hence, integrating in the x_1-variable alone,

$$\int_{\mathbf{R}} M_x\big(M_y(f)\big)^p(x)g(x)w(x)\,dx_1 \le \int_{\mathbf{R}} M_y(f)(x)\big)^p M_\mu(g)(x)w(x)\,dx_1.$$

Likewise, $(M_\mu(g)w,\ M_\mu(M_\mu(g))w) \in A_{p-\varepsilon}(\mathbf{R}^2)$ and so proceeding as above and integrating in the x_2-variable we now have

$$\iint_{\mathbf{R}^2} M_x\big(M_y(f)\big)^p(x)g(x)w(x)\,dx_1\,dx_2$$
$$\le c\iint_{\mathbf{R}^2} |f(x)|^p M_\mu\big(M_\mu(g)\big)(x)w(x)\,dx_1\,dx_2.$$

This lemma will enable us to prove two theorems which show that the boundedness of M_θ implies that of T_θ. Define

$$M_{\mu,i}(f)(x) = \sup_{R\ni x}\frac{1}{\mu(R)}\int_R f(x)\,d\mu(x)$$

where the supremum is taken over all rectangles oriented in the direction θ_i. Let

$$M_{\mu,\theta}(f)(x) = \sup_i M_{\mu,i}(f)(x).$$

THEOREM 3. *If $p > 2$, $w \in A_2(\theta)$ and $w \in A_2(\mathbf{R}^2)$ and if*

$$\|M_{\mu,\theta}(f)\|_{L_\mu^{(p/2)'}} \le c\|f\|_{L_\mu^{(p/2)'}} \quad \textit{for all } f \in L_\mu^{(p/2)'}(\mathbf{R}^2)$$

then

(A)

$$\left\|\left(\sum_j |M_j f_j|^2\right)^{1/2}\right\|_{L^p_\mu} \le c'_p \left\|\left(\sum_j |f_j|^2\right)^{1/2}\right\|_{L^p_\mu} \quad \textit{for all } f_j \in L^p_\mu,$$

and

(B) *T_θ is bounded on $L^p_\mu(\mathbf{R}^2)$ with norm depending only on the constant c above, and on the $A_p(\mathbf{R}^2)$ and $A_2(\mathbf{R}^2)$ constants of w.*

Proof. By duality. Let

$$g \in L_\mu^{(p/2)'}(\mathbf{R}^2),\ \|g\|_{L_\mu^{(p/2)'}} \le 1.$$

Then

$$\begin{aligned}
&\int_{\mathbf{R}^2} \sum_j |M_j f_j|^2 g(x)\, d\mu(x) \\
&\le c \sum_j \int_{\mathbf{R}^2} |f_j|^2 M_{\mu,j}\big(M_{\mu,j}(g)\big)(x)\, d\mu(x) \\
&\le c \int_{\mathbf{R}^2} \Big(\sum_j |f_j|^2\Big) M_{\mu,\theta}\big(M_{\mu,\theta}(g)\big)(x)\, d\mu(x) \\
&\le c \left\| \Big(\sum_j |f_j|^2\Big)^{1/2} \right\|_{L_\mu^p}^2 \|M_{\mu,\theta}\big(M_{\mu,\theta}(g)\big)\|_{L_\mu^{(p/2)'}} \\
&\le c \left\| \Big(\sum_j |f_j|^2\Big)^{1/2} \right\|_{L_\mu^p}.
\end{aligned}$$

Part (B) follows immediately from part (A), Theorem 1 and Theorem 2.

THEOREM 4. *If $p > 2$, $w \in A_2(\theta)$ and $w \in A_2(\mathbf{R}^2)$, and if M_θ is bounded on $L_\nu^{(p/2)'}(\mathbf{R}^2)$, $\nu = w^{1-(p/2)'}$, then*

(A)

$$\left\| \Big(\sum_j |M_j f_j|^2\Big)^{1/2} \right\|_{L_\mu^p} \le c \left\| \Big(\sum_j |f_j|^2\Big)^{1/2} \right\|_{L_\mu^p}, \quad d\mu(x) = w(x)\, d\lambda,$$

and

(B) *T_θ is bounded on $L_\mu^p(\mathbf{R}^2)$ with norm depending only on the norm of M_θ and on the $A_p(\mathbf{R}^2)$ and $A_2(\mathbf{R}^2)$-constants of w.*

Proof. We use another version of duality. The dual of $L_\mu^{p/2}(\mathbf{R}^2)$ is $L_\nu^{(p/2)'}(\mathbf{R}^2)$, where $\langle f, g\rangle = \int fg\, dx$. Let $g \in L_\nu^{(p/2)'}(\mathbf{R}^2)$ with norm less than or equal to 1, then

$$\begin{aligned}\int_{\mathbf{R}^2}\Big(\sum_j |M_j f_j|^2\Big) g(x)\,dx &\le c\sum_j \int_{\mathbf{R}^2} |f_j|^2 M_j\big(M_j(g)\big)(x)\,dx\\ &\le c\int_{\mathbf{R}^2}\sum_j |f_j|^2 M_\theta\big(M_\theta(g)\big)\left(\frac{w^{2/p}}{w^{2/p}}\right) dx\\ &\le c\left\|\Big(\sum_j |f_j|^2\Big)^{1/2}\right\|^2_{L^p_\mu} \|M_\theta(M_\theta(g))\|_{L_\nu^{(p/2)'}}\\ &\le c\left\|\Big(\sum_j |f_j|^2\Big)^{1/2}\right\|^2_{L^p_\mu}.\end{aligned}$$

Taking the supremum over all such functions g gives us part (A).
Part (B) follows from Theorems 1, 2 and part A.

In the case of Lebesgue measure both theorems 3B and 4B give the result of Cordoba and Fefferman. However, both demand a stronger condition than $w \in Ap(\theta)$. In the next section we will prove a result which will only require $w \in Ap(\theta)$, but we will also assume M_θ to be bounded on $L^p_\nu(\mathbf{R}^2)$, $\nu = w^{1-p'}\,d\lambda$, a stronger condition than the above.

5. The main result

The results in this and the following section are related to an extrapolation theorem of Garcia-Cuerva. Before we proceed we will need some notation.

We say that a pair (w, v) of nonnegative locally integrable functions satisfies the $Ap(F)$-condition, $1 < p < \infty$, and write $(w, v) \in Ap(F)$ if for all rectangles R in the family F,

$$\left(\frac{1}{|R|}\int_R w(y)\,dy\right)\left(\frac{1}{|R|}\int_R v(y)^{-1/(p-1)}\,dy\right)^{p-1} \le c,$$

with c independent of R. The smallest such c is called the $Ap(F)$-constant of (w, v).

A well known result is that the weak-type inequality for the Hardy-Littlewood maximal function,

$$\int_{\{Mf>\lambda\}} w(y)\,dy \le \frac{c_p}{\lambda^p}\int_{\mathbf{R}} |f|^p v(y)\,dy,$$

is true if and only if $(w, v) \in Ap(\mathbf{R})$. Here c_p depends only on the Ap constant of (w, v).

We are now ready to prove the following.

THEOREM 5. *If $p_0 > 2$ and M_θ is bounded on $L_\nu^{p'}(\mathbf{R}^2)$, $\nu = w^{1-p'}$, for $p \in (p_0 - \varepsilon, p_0)$, $\varepsilon > 0$, then the boundedness condition $BC(\mu, p_0)$ is true: For each $g \geq 0$, $g \in L_\mu^{(p_0/2)'}(\mathbf{R}^2)$, there is a $G \geq g$, $G \in L_\mu^{(p_0/2)'}(\mathbf{R}^2)$ with*

$$\|G\|_{L_\mu^{(p_0/2)'}} \leq c\|g\|_{L_\mu^{(p_0/2)'}}$$

and $G \cdot w \in A_2(\theta)$, and the $A_2(\theta)$-constant of $G \cdot w$ depends only on the $Ap_0(\theta)$-constant of w.

To prove this theorem we need the following lemma:

LEMMA 4. *Assume that $p > 2$ and M_θ is bounded on $L_\nu^{p'}(\mathbf{R}^2)$, $\nu = w^{1-p'}$. For $0 < \eta < 1$ and $g \in L^{p'/\eta}(\mathbf{R}^2)$ let*

$$G(y) = \left[M_\theta(g^{1/\eta} \cdot w)(y)/w(y)\right]^\eta.$$

Then
(i) $G \geq g$,
(ii) $(gw, Gw) \in A_{\eta + p(1-\eta)}(\theta)$,
(iii) $\|G\|_{L_\mu}^{p'/\eta} \leq c\|g\|_{L_\mu}^{p'/\eta}$,
where both c and the $A_{\eta+p(1-\eta)}(\theta)$-constant of (gw, Gw) depend only on the $Ap(\theta)$ constant of w.

We will postpone the proof of the lemma until after that of the theorem.

Proof (*of Theorem* 5). Choose $\eta = (p_0 + \varepsilon' - 2)/(p_0 - 1)$ for some $\varepsilon', \varepsilon > \varepsilon' > 0$. Then from Lemma 4 we obtain a G such that
(i) $g \leq G$,
(ii) $(gw, Gw) \in A_{2-\varepsilon'}(\theta)$,
(iii) $\|G\|_{L_\mu^{(p_0/2)'}} \leq c\|g\|_{L_\mu^{(p_0/2)'}}$
where c and the $A_{2-\varepsilon'}(\theta)$ constant depend only on the norm of M_θ.

Part (i) is obvious from the definition of G.

Part (ii) is true since for $p = p_0 - \varepsilon$,

$$\eta + p(1 - \eta) = 1 + \frac{p-1}{p_0 - 1}(1 - \varepsilon) < 2 - \varepsilon$$

and so

$$A_{\eta+p(1-\eta)}(\theta) \subseteq A_{2-\varepsilon}(\theta).$$

Part (iii) follows from the lemma by interpolating part (iii) of the lemma with

$$\|G\|_{L^\infty_\mu} \leq C\|g\|_{L^\infty_\mu};$$

since

$$(p_0/2)' = p_0'/\eta' \quad \text{where} \quad \eta' = \frac{p_0-2}{p_0-1} < \eta$$

so

$$(p_0/2)' = p_0'/\eta' > p_0'/\eta.$$

The next step is to replace G by a function H such that $H \cdot w \in A_2(\theta_i)$, independent of i. We will show this in the case $\theta_i = 0$, that is, $H \cdot w \in A_2$ in the x_1 and x_2 directions. One rotates to obtain the result in each direction θ_i, but the notation gets out of hand.

By the Lebesgue differentiation theorem, $(gw, G \cdot w) \in A_{2-\varepsilon'}(\theta)$ implies that $(gw, Gw) \in A_{2-\varepsilon'}$ in each direction θ_i independently. Thus (assuming for the moment that $\theta_i = 0$)

$$M_1\colon L^{2-\varepsilon'}_\nu(\mathbf{R}^1) \to L^{2-\varepsilon'}_\sigma(\mathbf{R}^1) \quad \text{and}$$
$$M_2\colon L^{2-\varepsilon'}_\nu(\mathbf{R}^1) \to L^{2-\varepsilon'}_\sigma(\mathbf{R}^1)$$

are of weak type ($\nu = gw$ and $\sigma = Gw$). So by interpolation with the trivial L^∞ result,

$$M_1\colon L^2_\nu(x_1) \to L^2_\sigma(x_1) \quad \text{and} \quad M_2\colon L^2_\nu(x_2) \to L^2_\sigma(x_2)$$

are (strong-type) bounded. Since $M(f) \leq M_1(M_2(f))$ it follows that

$$M\colon L^2_\nu(x_1, x_2) \to L^2_\sigma(x_1, x_2)$$

is (strong-type) bounded, with norm depending only on the norm of M_θ.

Now let $g_0 = g$, $g_1 = G$ and $\nu_i = g_i \cdot w$. Then

$$\|g_1\|_{L^{(p_0/2)'}_\mu} \leq c\|g_0\|_{L^{(p_0/2)'}_\mu},$$

and

$$\|M(f)\|_{L^2_{\nu_1}} \leq K\|f\|_{L^2_{\nu_0}} \quad \text{for} \quad f \in L^{(p_0/2)}_{\mu},$$

with c and K depending only on the norm of M_θ.

Proceeding inductively, given g_j we can obtain $g_{j+1} \geq g_j$ and $\nu_{j+1} = g_{j+1} \cdot w$ so that

$$\|g_{j+1}\|_{L^{(p_0/2)'}_{\mu}} \leq c\|g_j\|_{L^{(p_0/2)'}_{w}} \leq c^{j+1}\|g_0\|_{L^{(p_0/2)'}_{\mu}}$$

and

$$\|M(f)\|_{L^2_{\nu_{j+1}}} \leq K\|f\|_{L^2_{\nu_j}}.$$

Now let

$$H(g) = \sum_{j=0}^{+\infty} \frac{g_j(y)}{(c+1)^j}.$$

Since

$$\frac{\|g_j(y)\|_{L^{(p_0/2)'}_{\mu}}}{(c+1)^j} \leq \left[\frac{c}{c+1}\right]^j \|g\|_{L^{(p_0/2)'}_{\mu}}$$

the series converges, and also $H \geq g$ and

$$\|H\|_{L^{(p_0/2)'}_{\mu}} \leq (c+1)\|g\|_{L^{(p_0/2)'}_{\mu}}.$$

Now if we let $\nu = H \cdot w$, since

$$\|M(f)\|_{L^2_{\nu_{j+1}}} \leq K\|f\|_{L^2_{\nu_j}},$$

we have

$$\|M(f)\|_{L^2_{\nu}} \leq \sum_{K=0}^{\infty} \frac{K}{(c+1)^j}\|f\|_{L^2_{\nu}} = c\|f\|_{L^2_{\nu}}.$$

This last inequality implies $H \cdot w \in A_2(\mathbf{R}^2)$ with $A_2(\mathbf{R}^2)$ norm depending only on the norm of M_θ. By replacing the strong maximal function M by the maximal function with rectangles oriented in the direction θ_i, one obtains similarly $H \cdot w \in A_2(\theta_i)$ with the $A_2(\theta_i)$ constant depending only on the norm of M_θ. Since this is independent of i, it follows that $H \cdot w \in A_2(\theta)$.

Proof of Lemma 4. Note that M_θ is bounded on $L_\nu^{p'}(\mathbf{R}^2)$, $\nu = w^{1-p'}$, $w \in Ap(\theta)$.

(i) It is obvious that $G \geq g$.

(ii) We must show that $(gw, Gw) \in A_{\eta + p(1-\eta)}(\theta)$, i.e.,

$$\left(\frac{1}{|R|}\int_R g(y)w(y)\,dy\right)\left(\frac{1}{|R|}\int_R\left(\frac{M_\theta(g^{1/\eta}w)(y)}{w(y)}\right)^{-\eta/(q-1)}dy\right)^{q-1} \leq c,$$

for any rectangle R oriented in any of the directions θ_i. Here $q = \eta + p(1-\eta)$ so $q - 1 = (p-1)(1-\eta) > 0$. Hence $q > 1$.

By Holder's inequality with indices $1/\eta$, $1/1-\eta$,

$$\frac{1}{|R|}\int_R g(y)w(y)\,dy \leq \left(\frac{1}{|R|}\int_R g(y)^{1/\eta}w(y)\,dy\right)^{\eta}\left(\frac{1}{|R|}\int_R w(y)\,dy\right)^{1-\eta}.$$

Also for $y \in R$,

$$M_\theta(g^{1/\eta}w)(y) \geq \frac{1}{|R|}\int_R g(x)^{1/\eta}w(x)\,dx.$$

Then

$$\left[\frac{1}{|R|}\int_R\left(\frac{M_\theta(g^{1/\eta}w)(y)}{w(y)}\right)^{-\eta/(q-1)} w(y)^{-1/(\eta-1)}\right]^{q-1}$$

$$\leq \left(\frac{1}{|R|}\int_R g(x)^{1/\eta}w(x)\right)^{-\eta}\cdot\left[\frac{1}{R}\int_R w(y)^{(\eta-1)/(q-1)}\,dy\right]^{q-1}$$

$$= \left(\frac{1}{|R|}\int_R g(x)^{1/\eta}w(x)\,dy\right)^{-\eta}\left(\frac{1}{|R|}\int_R w(y)^{-1/(p-1)}\,dy\right)^{(p-1)(1-\eta)}$$

So, the $Ap(\theta)$ condition is bounded by

$$\left[\frac{1}{|R|}\int_R w(y)\,dy\right]^{1-\eta}\left[\frac{1}{|R|}\int_R w(y)^{-1/(p-1)}\,dy\right]^{(p-1)(1-\eta)}$$

$$\leq [Ap(\theta)\text{ constant of } w]^{1-\eta}.$$

(iii)

$$\int\left[\frac{M_\theta(g^{1/\eta}w)(y)}{w(y)}\right]^{\eta\cdot p'/\eta} w(y)\,dy$$

$$= \int M_\theta(g^{1/\eta}w)^{p'} w(y)^{1-p'}\,dy \le c\int g^{p'/\eta}w(y)\,dy.$$

Finally we have the main result of this section:

THEOREM 6. *If $p_0 > 2$ and M_θ is bounded on $L_\nu^{p'}(\mathbf{R}^2)$, for $p \in (p_0 - \varepsilon, p_0)$, some $\varepsilon > 0$, then the multiplier operator T_θ is bounded on $L_\mu^{p_0}(\mathbf{R}^2)$ with norm depending only on the norm of M_θ and the $Ap(\theta)$-constant of w.*

Proof. This follows directly from Theorems 1 and 5.

6. An extrapolation result

In this section we generalize an extrapolation theorem of Garcia-Cuerva to weights in $Ap(\theta)$. The theorem of Garcia-Cuerva states that for any sublinear operator T, if T is bounded on $L_\mu^{p_0}(\mathbf{R}^2)$ where $d\mu(x) = w(x)\,dx$, for some p_0, $1 < p_0 < \infty$, and all $w \in Ap_0(\mathbf{R}^2)$, then T is bounded on $L_\mu^p(\mathbf{R}^2)$ for all p, $1 < p < \infty$, and all $w \in Ap(\mathbf{R}^2)$. For more on extrapolation see [7], [15], or [17].

THEOREM 7. *Assume that T is a sublinear operator satisfying the following conditions:*

There is a p_0, $1 < p_0 < \infty$, such that for every $w \in Ap_0(\theta)$,

$$\|Tf\|_{L_\mu^{p_0}} \le c\|f\|_{L_\mu^{p_0}},$$

$d\mu(x) = w(x)\,dx$, where c is independent of f and depends only on the $Ap_0(\theta)$ constant of w.

(i) *For $p_0 < p < +\infty$ assume that*

(a) $$\|M_\theta f\|_{L_\mu^r} \le c'\|f\|_{L_\mu^r} \text{ for all } w \in Ap_0(\theta),\ c'$$

independent in $Ap_0(\theta)$ and $r \in (p_0 - \varepsilon, p_0]$ for some $\varepsilon > 0$, and

(b) $$\|M_\theta f\|_{L_\mu^q} \le K\|f\|_{L_\mu^q} \text{ if and only if } \|M_\theta f\|_{L_\nu^{q'}} \le K'\|f\|_{L_\nu^{q'}},$$

for $1 < q < +\infty$, $1/q' + 1/q = 1$ and $d\nu(x) = w(x)^{1-q'}\,dx$.

(ii) *For* $1 < p < p_0$ *assume that* $\|M_\theta f\|_{L^p_\mu} \le c\|f\|_{L^p_\mu}$ *for* c *independent in* $Ap(\theta)$.

Then $\|Tf\|_{L^p_\mu} \le K(p)\|f\|_{L^p_\mu}$ *for all* p, $1 < p < \infty$, *and for all* $w \in Ap(\theta)$ *where* $K(p)$ *is independent in* $Ap(\theta)$.

Proof. Case (*i*). For $p_0 < p < +\infty$, let $w \in Ap(\theta)$ and $f \in L^p_\mu(\mathbf{R}^2)$. To begin we need the following lemma.

LEMMA 5. *If* $\|M_\theta f\|_{L^r_\beta} \le c\|f\|_{L^r_\beta}$, $d\beta(x) = w(x)^{1-r}\,dx$, $r \in (p' - \varepsilon, p']$ *for some* $\varepsilon > 0$, *and* $f \in L^r_\sigma(\mathbf{R}^2)$, *then for each non-negative* $g \in L^{(p/p_0)'}_\mu(\mathbf{R}^2)$ *there is a* $G \ge g$ *such that*

$$\|G\|_{L^{(p/p_0)'}_\mu} \le c\|g\|_{L^{(p/p_0)'}_\mu} \text{ and } G \cdot w \in Ap_0(\theta). \text{ Here } c \text{ is independent of } g.$$

The proof of Lemma 5 is the same as the proof of Theorem 5, with 2 replaced by p_0.

To complete the proof of Case (i), note that M_θ bounded on $L^{p_0}_\mu(\mathbf{R}^2)$ implies, by interpolation with the trivial L^∞ result, that M_θ is bounded on $L^{p_0}_\mu(\mathbf{R}^2)$, $p_0 \le p \le +\infty$. So by hypothesis M_θ is bounded on $L^{p'}_\nu(\mathbf{R}^2)$, $d\nu(x) = w^{1-p'}(x)\,d\lambda$ for $p_0 \le p \le +\infty$. We may apply Lemma 5 in this range and get

$$\begin{aligned}
\|Tf\|^{p_0}_{L^p_\mu} &= \|\,|Tf|^{p_0}\|^{p_0}_{L^{p/p_0}_\mu} \\
&= \sup \int_{\mathbf{R}^2} |Tf(y)|^{p_0} g(y) w(y)\,dy, \\
&\qquad \left(\text{where the sup is taken over } \|g\|_{L^{(p/p_0)'}_\mu} \le 1,\ g \ge 0\right) \\
&\le \sup \int_{\mathbf{R}^2} |Tf(y)|^{p_0} G(y) w(y)\,dy \\
&\le \sup c \int_{\mathbf{R}^2} |f(y)|^{p_0} G(y) w(y)\,dy \\
&\le \sup c \|\,|f|^{p_0}\|_{L^{p/p_0}_\mu} \|G\|_{L^{(p/p_0)'}_\mu} \\
&\le c\|f\|^{p_0}_{L^p_\mu},
\end{aligned}$$

with c independent of f and μ.

Case (*ii*). For $1 < p < p_0$, let $w \in Ap(\theta)$ and $f \in L^p_\mu(\mathbf{R}^2)$. We also need a lemma here.

LEMMA 6. *Assume that* $1 < p < p_0$, $w \in Ap(\theta)$ *and that*

$$\|M_\theta f\|_{L^p_\mu} \leq c\|f\|_{L^p_\mu}$$

c independent of f. Then for each non-negative $g \in L_\mu^{p/(p-p_0)}(\mathbf{R}^2)$, *we can find* $G \geq g$ *such that*

$$\|G\|_{L_\mu^{p/(p-p_0)}} \leq c'\|g\|_{L_\mu^{p/(p-p_0)}}$$

and $G^{-1}w \in Ap_0(\theta)$, *with both c′ and the* $Ap_0(\theta)$*-constant of* $G^{-1}w$ *dependent only on the* $Ap(\theta)$*-constant of w.*

Proof. This is the dual to Lemma 5 and is proved exactly as in [17], Chapter 9, Proposition 7.5.

To complete the proof of the theorem let

$$g(x) = \left(|f(x)| / \|f\|_{L^p_\mu}\right)^{p_0-p},$$

where $f(x) \neq 0$, $g(x) = 0$ elsewhere. Note that

$$\int_{\{f\neq 0\}} |f(x)|^{p_0} g(x)^{-1} w(x)\, dx = \|f\|^{p_0}_{L^p_\mu}$$

and

$$\|g\|_{L_\mu^{p/p_0-p}} = 1.$$

Apply Lemma 6 to obtain $G \geq g$ with the given properties. Then

$$\begin{aligned}
\|Tf\|^{p_0}_{L^p_\mu} &= \left[\int_{\mathbf{R}^2}\left[\frac{|T(f)(x)|^{p_0}}{G(x)}\right]^{p/p_0} G(x)^{p/p_0} w(x)\, dx\right]^{p_0} \\
&\leq \|G\|_{L_\mu^{p/p_0-p}} \int_{\mathbf{R}^2} |T(f)(x)|^{p_0} G(x)^{-1} w(x)\, dx \\
&\leq c\int_{\{f\neq 0\}} |f(x)|^{p_0} G^{-1}(x) w(x)\, dx \\
&\leq c\int_{\{f\neq 0\}} |f(x)|^{p_0} g^{-1}(x) w(x)\, dx = \|f\|^{p_0}_{L^p_\mu}.
\end{aligned}$$

7. Applications

In this section we use the results proven above to obtain two applications. The first concerns an infinite class θ where M_θ is known to be bounded. The second application is a weighted version of the angular Littlewood-Paley operator.

DEFINITION. A sequence $\{\theta_K\}$ is called *lacunary* provided there is a constant $r < 1$ such that $0 < \theta_{K+1} < r\theta_K$, $K = 1, 2, \ldots$.

THEOREM. *If $\theta = \{\theta_K\}$ is lacunary then $\|M_\theta\|_{L^p_\mu} \le c\|f\|_{L^p_\mu}$, $d\mu(x) = w(x)\,dx$, if and only if $w \in Ap(\theta)$, where c depends only on the $Ap(\theta)$-constant of w.*

For a proof of this result see reference [8].

THEOREM 8. *If $w \in Ap(\theta)$ and $Ap(\mathbf{R}^2)$, and if $\theta = \{\theta_K\}$ is lacunary, then T_θ is bounded on $L^p_\mu(\mathbf{R}^2)$ $1 < p < +\infty$, $d\mu(x) = w(x)\,d\lambda$.*

Proof. For $p > 2$ the theorem immediately above, combined with Theorem 6, gives the result. For $1 < p \le 2$ we apply Theorem 7, the result on extrapolation.

THEOREM 9. *Let $\theta = \{\theta_K\}$ be lacunary and let H_K be the Hilbert transform in the direction θ_K. If $1 < p < \infty$, $w \in Ap(\theta)$ and $Ap(\mathbf{R}^2)$ and if $f \in L^p_\mu(\mathbf{R}^2)$ then*

$$\left\| \left(\sum_k |H_k(f)|^2 \right)^{1/2} \right\|_{L^p_\mu} \le c\|f\|_{L^p_\mu}.$$

Here c depends on w and p and is independent of f.

Proof. We may assume $p > 2$ and apply Theorem 7 to finish the proof. Since M_θ is bounded on $L^p_\mu(\mathbf{R}^2)$, $1 < p < \infty$, by Theorem 5, the condition $BC(\mu, p)$ is true for $p > 2$. Then by theorem 2,

$$\left\| \left(\sum_k |H_k f_k|^2 \right)^{1/2} \right\|_{L^p_\mu} \le c \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p_\mu}.$$

If S_k is the dyadic Littlewood-Paley operator defined on $L^2(\mathbf{R}^2)$ by

$$\widehat{S_k}(f)(x) = X_{R_k}(x)\hat{f}(x)$$

where each R_k is a dyadic rectangle, then Kurtz [9] has shown:

$$\left\| \left(\sum_k |S_k f|^2 \right)^{1/2} \right\|_{L^p_\mu(\mathbf{R}^2)} \le c \|f\|_{L^p_\mu(\mathbf{R}^2)} \tag{1}$$

and

$$\left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p_\mu(\mathbf{R}^2)} \approx \left\| \left(\sum_k |S_k f_k|^2 \right)^{1/2} \right\|_{L^p_\mu(\mathbf{R}^2)} \tag{2}$$

where c is independent of f.

Using this result, it then follows that

$$\begin{aligned} \left\| \left(\sum_k |H_k f|^2 \right)^{1/2} \right\|_{L^p_\mu} &\le c \left\| \left(\sum_k |S_k H_k f|^2 \right)^{1/2} \right\|_{L^p_\mu} \\ &= c \left\| \left(\sum_k |H_k S_k f|^2 \right)^{1/2} \right\|_{L^p_\mu} \\ &\le c \left\| \left(\sum_k |S_k f|^2 \right)^{1/2} \right\|_{L^p_\mu} \\ &\le c \|f\|_{L^p_\mu}, \end{aligned}$$

where c is independent of f, and depends on w and p.

As an immediate corollary we have the following version of the Angular Littlewood-Paley inequality.

THEOREM 10. *Let $\theta = \{\theta_k\}$ be lacunary, $0 < \theta_k < \pi/2$, and define the sector σ_k by*

$$\sigma_k = \{x \in \mathbf{R}^2 : \theta_k < \text{argument}(x) \le \theta_{k+1}\},$$

for $k = 0, 1, 2, \dots$. Set $\hat{T}_k(f)(x) = X_{\sigma_k}(x)\hat{f}(x)$. Then if f is supported in $\bigcup_k \sigma_k$, $f \in L^p_\mu(\mathbf{R}^2)$ and $f \in L^2_\mu(\mathbf{R}^2)$, $d\mu = w(x)\,d\lambda(x)$, and if $w \in Ap(\theta)$ and $Ap(\mathbf{R}^2)$ then

$$\left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_{L^p_\mu} \le c \|f\|_{L^p_\mu}$$

where $c = c(\mu, p)$ is independent of f.

Proof. Since $T_k = H_{k+1} - H_k$ this follows immediately from Theorem 9.

References

1. A.P. Calderon, *Inequalities for the maximal function relative to a metric*, Studia Math., vol. 57 (1976), pp. 297–306.
2. A. Cordoba and C. Fefferman, *A weighted norm inequality for singular integrals*, Studia Math., vol. 57 (1976), pp. 97–101.
3. ______, *On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis*, Proc. Nat. Acad. Sci., vol. 74 (1977), pp. 423–425.
4. C. Fefferman, *The multiplier problem for the ball*, Ann. of Math., vol. 94 (1972), pp. 330–336.
5. C. Fefferman and E.M. Stein, *Some maximal inequalities*, Amer. J. Math., vol. 1 (1971), pp. 107–115.
6. ______, *H^p spaces of several variables*, Acta. Math., vol. 129 (1972), pp. 137–193.
7. J. Garcia-Cuerva, *An extrapolation theorem in the theory of A_p weights*, Proc. Amer. Math. Soc., vol. 187 (1983), pp. 422–426.
8. B. Jawerth, *Weighted inequalities for maximal operators: linearization, localization and factorization*, Amer. J. Math., vol. 108 (1986), pp. 361–414.
9. D. Krug, *Weighted norm inequalities for multiplier operators*, Ph.D. Thesis, Indiana University, 1986.
10. D. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces*, Trans. Amer. Math. Soc., vol. 259 (1980), pp. 235–254.
11. D. Kurtz and R. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Am. Math. Soc., vol. 255 (1979), pp. 343–361.
12. K.-C. Lin, Thesis, U.C.L.A., 1984.
13. B. Muckenhoupt, *Weighted norm inequalities for classical operators*, Proc. Symposia Pure Math., vol. XXXV (1), 1979, pp. 69–84.
14. J.L. Rubio de Francia, *Weighted norm inequalities and vector valued inequalities*, Springer Lecture Notes in Math., vol. 909, Springer Verlag, N.Y., pp. 86–101.
15. ______, *Factorization theory and A_p weights*, Amer. J. Math., vol. 106 (1984), pp. 533–547.
16. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton, 1970.
17. A. Torchinsky, *Real variable methods in harmonic analysis*, Academic Press, San Diego, Calif., 1986.

Northern Kentucky University
Highland Heights, Kentucky