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1-ARY FUNCTIONS AND THE F.C.P.

BY

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Let T be a countable, complete 1st order theory with no finite models. As usual, we assume that all models of T are elementary substructures of some big model U. In [K], Keisler proposed the following definition: T is said to satisfy the finite cover property (f.c.p.) if there exists a formula $\varphi(\bar{v}, \bar{w})$ of the language of T such that, for every $m \in \omega$, there are $n \in \omega, \bar{a}_0, \dots, \bar{a}_n \in U$ such that $n \geq m$,

$$\models \neg\left(\exists \bar{v} \bigwedge_{k \leq n} \varphi(\bar{v}, \bar{a}_k)\right)$$

but, for all $l \leq n$,

$$\models \exists \bar{v} \bigwedge_{k \leq n, k \neq l} \varphi(\bar{v}, \bar{a}_k).$$

To define what is a theory without the f.c.p. is now an exercise as trivial as useful; for, the $\neg$f.c.p. is a property much richer in implications than the f.c.p. For instance, a theory T without the f.c.p. is stable (and some examples of the use of the $\neg$f.c.p. in stability theory can be found in Shelah's book [S]); on the other hand, Poizat discovered some meaningful connections between the $\neg$f.c.p. and the properties of the theory of nice pairs of models of T [P].

Here we are interested in the problem of studying the f.c.p. for theories of a 1-ary function. Several papers have already been devoted to the model theory of 1-ary functions, especially in the context of Vaught's Conjecture (see [M1], [M2], [Mi]). In particular, we studied classification theory for these functions in [T], we only recall here that they are superstable. The aim of this paper is to classify the theories T of a 1-ary function f which do not satisfy the f.c.p. First let us give some examples concerning this matter.

1. If T is categorical in $\aleph_0$ or in $\aleph_1$, then T does not satisfy the f.c.p. (in fact, in general, any stable $\aleph_0$-categorical theory, as well as any $\aleph_1$-categorical theory, fails to have the f.c.p., see [K] and [BK]).

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2. Let T be the theory of a 1-ary function f such that:
For every $a \in U$ and $n \in \omega - \{0\}$, $f^n(a) \neq a$;
For every $a \in U$, there are infinitely many $b \in U$ satisfying $f(b) = a$.

Then T is neither $\aleph_0$-categorical nor $\aleph_1$-categorical; however T does not have the f.c.p.

3. Consider now the theory T_0 of a 1-ary function f such that, for every $n \in \omega - \{0\}$, there is $a \in U$ satisfying:

$f(a) = a$;
There are exactly n elements $b \in U$ such that $f(b) = a$, $b \neq a$;
For all b such that $f(b) = a$ and $b \neq a$, $f^{-1}(b) = \varnothing$.

Let T be any completion of T_0; then T has the f.c.p.

Our main result is that a theory T of a 1-ary function f does not have the f.c.p. if and only if T satisfies the conditions $P_n (n \in \omega - \{0\})$ below. However, before stating these conditions, we need to introduce the following notions.

DEFINITION . Let $a \in U$. Then

$$k(a) = \begin{cases} \min\{k \in \omega : k > 0, f^k(a) = a\} & \text{if such a } k \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to see that, for every $a \in U$:

If $k(a) < \infty$, then, for all $k \in \omega$, $f^k(a) = a$ if and only if $k(a) | k$;
If $k(a) < \infty$, then $k(f(a)) = k(a)$ (in particular, if $k(f(a)) = \infty$, then $k(a) = \infty$, too);
If $k(a) < \infty$, $f(x) = a$ and $x \neq f^{k(a)-1}(a)$, then $k(x) = \infty$.

DEFINITION . Let $a \in U, n \in \omega - \{0, 1\}$. Then $\tau_n(a) = \{x \in U$: either $x = a$ or there is $m \in \omega$ such that $0 < m < n, f^m(x) = a$ and $f^{m-1}(x) \neq f^{k(a)-1}(a)$ when $k(a) < \infty\}$.
One can easily prove that, for every $a \in U$ and $n \in \omega - \{0, 1\}$,

$$\tau_{n+1}(a) = \{a\} \cup \bigcup_x \tau_n(x)$$

where x ranges over the preimages of a in f different from $f^{k(a)-1}(a)$ when $k(a) < \infty$. Furthermore, if $f(x) = f(y) = a$, $x \neq y$ and $x, y \neq f^{k(a)-1}(a)$

when $k(a) < \infty$, then

$$a \notin \tau_n(x), \quad \tau_n(x) \cap \tau_n(y) = \varnothing.$$

Notice that in general $\tau_n(a)$ is not a structure of the language for f, as $\tau_n(a)$ contains a, but it does not include $f(a)$ except for the case $f(a) = a$ namely $k(a) = 1$. Nevertheless we shall consider below the "isomorphism type" of $\tau_n(a)$ in the sense we are going to explain here. For every a, $a' \in U$, we shall say that $\tau_n(a)$ is isomorphic to $\tau_n(a')$,

$$\tau_n(a) \simeq \tau_n(a'),$$

if $k(a) = k(a')$ and there exists a partial isomorphism of U having domain $\tau_n(a)$ and range $\tau_n(a')$, namely a bijection g of $\tau_n(a)$ onto $\tau_n(a')$ such that, for all $x, y \in \tau_n(a)$, $f(x) = y$ if and only if $f(g(x)) = g(y)$ (in particular $g(a) = a'$). Clearly $\simeq$ is an equivalence relation; then the isomorphism type of $\tau_n(a)$ will mean the equivalence class of $\tau_n(a)$ with respect to $\simeq$.

We can state now P_1.

(P_1) There exists $N \in \omega$ such that, for every $a \in U$, $f^{-1}(a)$ has either $\leq N$ or infinitely many elements.

Let us list some consequences of P_1.

(i) For every $a \in U$, the isomorphism type of $\tau_2(a)$ is given by $k(a)$ and a cardinal number among $0, 1, \ldots, N$, card U specifying the power of $\{x \colon f(x) = a,\ x \neq f^{k(a)-1}(a) \text{ if } k(a) < \infty\}$.

(ii) For every $k \in \omega - \{0\}$ or $k = \infty$, there are only finitely many isomorphism types of structures $\tau_2(a)$ with $k(a) = k$.

(iii) For every $a \in U$, let $\vartheta_{2,a}$ be the formula

$$\exists! m(a) w(f(w) = v \wedge w \neq f^{k(a)-1}(v))$$

if $k(a) < \infty$ and

$$\exists! m(a) w(f(w) = v)$$

otherwise, where $m(a)$ denotes the power of

$$\{x \colon f(x) = a, x \neq f^{k(a)-1}(a) \text{ if } k(a) < \infty\},$$

so that

$$m(a) \in \{0, 1, \ldots, N, \text{card } U\},$$

and $\exists!$ card U abridges $\exists > N$. Then, for every $a, a' \in U$,

$$\tau_2(a) \simeq \tau_2(a')$$

if and only if $k(a) = k(a')$ and $\vDash \vartheta_{2,a}(a')$ or, if you prefer, if and only if $k(a) = k(a')$ and $m(a) = m(a')$.

Now let $n \in \omega - \{0, 1\}$.

(P_n) For every $b \in U$ with $k(b) = \infty$, there is $H = H(\tau_n(b)/\simeq)$ such that, for all $a \in U$,

$$\{x: f(x) = a, x \neq f^{k(a)-1}(a) \text{ if } k(a) < \infty, \tau_n(x) \simeq \tau_n(b)\}$$

has either $\leq H$ or infinitely many elements.

Let $n \in \omega - \{0, 1\}$ and assume that P_m holds for every $m \in \omega$ with $1 \leq m \leq n$. Then an easy induction argument shows the following consequences, generalizing the ones of the case $n = 1$.

(i) For all $a \in U$, the isomorphism type of $\tau_{n+1}(a)$ is given by $k(a)$ and by the function of the (finite) set of invariants of isomorphism types of structures $\tau_n(b)$ with $b \in U$, $k(b) = \infty$, into the set of cardinals $\leq$ card U such that, for every $b \in U$ satisfying $k(b) = \infty$, the image of the corresponding invariant is the power of

$$\{x: f(x) = a, x \neq f^{k(a)-1}(a) \text{ if } k(a) < \infty, \tau_n(x) \simeq \tau_n(b)\}$$

(and hence belongs to $\{0, 1, \ldots, H(\tau_n(b)/\simeq), \text{card } U\}$).

In fact, assume $\tau_{n+1}(a) \simeq \tau_{n+1}(a')$. Then $k(a) = k(a')$ and there is a partial isomorphism g mapping $\tau_{n+1}(a)$ onto $\tau_{n+1}(a')$; in particular $g(a) = a'$ and, for every x such that $f(x) = a$, $x \neq f^{k(a)-1}(a)$ if $k(a) < \infty$, $g(x) = x'$ satisfies $f(x') = a'$, $x' \neq f^{k(a')-1}(a')$ if $k(a') = k(a) < \infty$. It follows that $\tau_n(x) \simeq \tau_n(x')$. In fact $k(x) = k(x') = \infty$ and, for all $y \in \tau_{n+1}(a)$, if $y' = g(y)$, then

$y \in \tau_n(x)$ iff there is $s < n$ such that $f^s(y) = x$
iff there is $s < n$ such that $f^s(y') = x'$
iff $y' \in \tau_n(x')$;
hence $g \restriction_{\tau_n(x)}$ is a partial isomorphism of $\tau_n(x)$ onto $\tau_n(x')$. In particular, for every $b \in U$ such that $k(b) = \infty$,
card$\{x: f(x) = a, x \neq f^{k(a)-1}(a)$ if $k(a) < \infty, \tau_n(x) \simeq \tau_n(b)\}$ = card$\{x': f(x') = a', x' \neq f^{k(a')-1}(a')$ if $k(a') < \infty, \tau_n(x') \simeq \tau_n(b)\}$.
Conversely suppose that $a, a' \in U$ satisfy $k(a) = k(a')$ and
card$\{x: f(x) = a, x \neq f^{k(a)-1}(a)$ if $k(a) < \infty, \tau_n(x) \simeq \tau_n(b)\}$ = card$\{x': f(x') = a', x' \neq f^{k(a')-1}(a')$ if $k(a') < \infty, \tau_n(x') \simeq \tau_n(b)\}$

for every $b \in U$ with $k(b) = \infty$. By recalling that

$$\tau_{n+1}(a) = \{a\} \dot{\cup} \dot{\bigcup_x} \tau_n(x)$$

(where $f(x) = a, x \neq f^{k(a)-1}(a)$ if $k(a) < \infty$) and similarly for a', one can easily build a partial isomorphism of $\tau_{n+1}(a)$ onto $\tau_{n+1}(a')$.

(ii) There are at most finitely many isomorphism types of structures $\tau_{n+1}(a)$ with $a \in U, k(a) = \infty$.

(iii) For every $a \in U$, let $\vartheta_{n+1,a}$ be the formula

$$\bigwedge_b \exists! m(n,b,a)\, w \left(f(w) = v \wedge w \neq f^{k(a)-1}(v) \wedge \vartheta_{n,b}(w)\right)$$

if $k(a) < \infty$, or

$$\bigwedge_b \exists! m(n,b,a)\, w \left(f(w) = v \wedge \vartheta_{n,b}(w)\right)$$

otherwise, where b ranges over the elements of U satisfying $k(b) = \infty$—or, more precisely, $\tau_n(b)/\simeq$ ranges over the corresponding isomorphism types, that are finitely many—and, for each b with $k(b) = \infty$,

$$\begin{aligned} m(n,b,a) &= \operatorname{card}\{x : f(x) = a, x \neq f^{k(a)-1}(a) \text{ when } \\ &\qquad\qquad k(a) < \infty, \tau_n(x) \simeq \tau_n(b)\} \\ &\in \{0, 1, \ldots, H(\tau_n(b)/\simeq), \operatorname{card} U\} \end{aligned}$$

(as before $\exists!$ card U abbreviates $\exists > H(\tau_n(b)/\simeq)$). Then, for all $a, a' \in U$,

$$\tau_{n+1}(a) \simeq \tau_{n+1}(a')$$

if and only if $k(a) = k(a')$ and $\vDash \vartheta_{n+1,a}(a')$, or, if you prefer, if and only if $k(a) = k(a')$ and $m(n,b,a) = m(n,b,a')$ for every b with $k(b) = \infty$.

THEOREM 1. *If T fails to have the f.c.p., then T satisfies P_n for all $n \in \omega - \{0\}$.*

Proof. Assume towards a contradiction that there is $n \in \omega - \{0\}$ such that P_n does not hold. Let n be minimal with this property. If $n = 1$, then, for every $m \in \omega$, there exists $a \in U$ admitting $\geq m$ but finitely many preimages; hence T has the f.c.p. (consider the formula $\varphi(v,w)$: $v \neq w \wedge f(v) = f(w)$).

Let now $n > 1$. Then there is $b \in U$ such that $k(b) = \infty$ and, for all $n \in \omega$, there is $a \in U$ admitting $\geq m$ but finitely many preimages x such that $x \neq f^{k(a)-1}(a)$ when $k(a) < \infty$ and $\tau_n(x) \simeq \tau_n(b)$ (namely $k(x) = \infty$ and $\models \vartheta_{n,b}(x)$). But in this case T admits the f.c.p. owing to the formula

$$\varphi(v,w): v \neq w \wedge f(v) = f(w) \wedge \vartheta_{n,b}(v) \wedge \vartheta_{n,b}(w)$$

(in fact, even if $k(a) < \infty$, there is at most one preimage $a' = f^{k(a)=1}(a)$ of a such that $k(a') < \infty$ and $\models \vartheta_{n,b}(a')$).

THEOREM 2. *If T satisfies P_n for all $n \in \omega - \{0\}$, then T fails to have the f.c.p.*

We tacitly assume from now on that T satisfies P_n for all $n \in \omega - \{0\}$.

LEMMA 1. *For all $a, a' \in U$ satisfying $k(a) = k(a')$, and $n \in \omega - \{0, 1\}$, if $\models \vartheta_{n+1,a}(a')$, then $\models \vartheta_{n,a}(a')$.*

Proof. We proceed by induction on n.

Let $n = 2$, and suppose $\models \vartheta_{3,a}(a')$. Then, for all $b \in U$ with $k(b) = \infty$, $m(2, b, a) = m(2, b, a')$. But in this case

$$m(a) = \sum_b m(2,b,a) = \sum_b m(2,b,a') = m(a'),$$

and hence $\models \vartheta_{2,a}(a')$.

Now let $n > 2$ and assume $\models \vartheta_{n+1,a}(a')$. Then, for all $b \in U$ with $k(b) = \infty$, $m(n, b, a) = m(n, b, a')$. Let x satisfy $f(x) = a$, $x \neq f^{k(a)-1}(a)$ when $k(a) < \infty$. Then $k(x) = \infty$ and, for every b with $k(b) = \infty$,

$$\models \vartheta_{n-1,b}(x)$$

if and only if there is c such that $k(c) = \infty$, $\models \vartheta_{n-1,b}(c)$ and $\models \vartheta_{n,c}(x)$. In fact, if $\models \vartheta_{n-1,b}(x)$, then we can put $c = x$.

Conversely suppose that there exists c as claimed, then we have $\models \vartheta_{n-1,c}(x)$ and, consequently, as $k(c) = k(x) = k(b) = \infty$,

$$\tau_{n-1}(x) \simeq \tau_{n-1}(c) \simeq \tau_{n-1}(b);$$

but then $\models \vartheta_{n-1,b}(x)$. Of course, for every c, c' with $k(c) = k(c') = \infty$, if $\models \vartheta_{n,c}(x) \wedge \vartheta_{n,c'}(x)$, then $\tau_n(c) \simeq \tau_n(c')$; hence, for all b as above,

$$m(n-1,b,a) = \sum_{k(c)=\infty,\ \models \vartheta_{n-1,b}(c)} m(n,c,a).$$

Similarly for a'. But this clearly suffices to prove our claim.

LEMMA 2. *Let $a, x' \in U$ satisfy $k(a) = \infty$, $k(x') = k(f(a))$,*

$$\models \vartheta_{n, f(a)}(x') \quad \text{for all } n \in \omega - \{0, 1\}.$$

Then there is $a' \in U$ such that $f(a') = x'$, $a' \neq f^{k(x')-1}$ when $k(x') < \infty$, $\models \vartheta_{n,a}(a')$ for all $n \in \omega - \{0, 1\}$ (and similarly in any ω-saturated model of T containing x').

Proof. First notice that $k(a) = \infty$ implies $a \neq f^h(a)$ for all $h \in \omega - \{0\}$. We have to show that the set

$$\{f(v) = x', v \neq f^{k(x')-1}(x'), \vartheta_{n,a}(v) : n \in \omega - \{0, 1\}\}$$

($\{f(v) = x', \vartheta_{n,a}(v)$: $n \in \omega - \{0, 1\}\}$ when $k(x') = \infty$, but for simplicity we will ignore this case, which can be handled in a similar way) is satisfiable. Since U is very saturated (but ω-saturated is enough), it suffices to show that this set is finitely satisfiable, and hence that, for all $n \in \omega - \{0, 1\}$,

$$\{f(v) = x', v \neq f^{k(x')-1}(x'), \vartheta_{2,a}(v), \ldots, \vartheta_{n,a}(v)\}$$

is satisfiable. Lemma 1 reduces the problem to the satisfiability of

$$\{f(v) = x', v \neq f^{k(x')-1}(x'), \vartheta_{n,a}(v)\}$$

for every $n \in \omega - \{0, 1\}$; in fact, if $f(c) = x'$ but $c \neq f^{k(x')-1}(x')$, then $k(c) = \infty = k(a)$, and hence $\models \vartheta_{n,a}(c)$ implies $\models \vartheta_{i,a}(c)$ for any i such that $2 \leq i \leq n$. On the other hand

$$\models \exists w \big(f(w) = x' \wedge w \neq f^{k(x')-1}(x') \wedge \vartheta_{n,a}(w)\big)$$

if and only if $m(n, a, x') = m(n, a, f(a)) > 0$ and hence if and only if

$$\models \exists w \big(f(w) = f(a) \wedge w \neq f^{k(f(a))-1}(f(a)) \wedge \vartheta_{n,a}(w)\big);$$

but this formula is true (take $w = a$).

DEFINITION. Let $\bar{a} = (a_0, \ldots, a_t)$ be a sequence of elements of U. The f-type of $\bar{a}$ is the subset of $tp(\bar{a}|\varnothing)$ of the formulas of the kind

$$f^h(v_i) = f^m(v_j), \quad f^h(v_i) \neq f^m(v_j)$$

with $h, m \in \omega$, $i, j \leq t$, or of the kind

$$\vartheta_{n, f^h(a_i)}(f^h(v_i))$$

with $n, h \in \omega$, $n \geq 2$ and $i \leq t$.

One can easily see that, for any $a, a' \in U$, the following propositions are equivalent:
(i) For all $h \in \omega$, $k(f^h(a)) = k(f^h(a'))$;
(ii) For all $h, m \in \omega$, $f^h(a) = f^m(a)$ iff $f^h(a') = f^m(a')$.

Hence, if $\bar{a}, \bar{a}'$ have the same f-type, then, for every $h \in \omega$ and $i \leq t$, $k(f^h(a_i)) = k(f^h(a_i'))$.

In the following, when $\bar{a} = (a_0, \ldots, a_t)$, $\bar{a}' = (a_0', \ldots, a_t')$ are two sequences of the elements of U, and $a \in \bar{a}$ (for instance $a = a_i$ with $i \leq t$), then a' will denote the element of $\bar{a}'$ corresponding to a (namely $a' = a_i'$).

LEMMA 3. *Let $\bar{a}, \bar{a}'$ satisfy the same f-type, and let x be such that:*
There are $s \in \omega$, $a \in \bar{a}$ such that $f(x) = f^s(a)$;
For all $q \in \omega$ and $\alpha \in \bar{a}$, $x \neq f^q(\alpha)$.
Then there is $x' \in U$ such that:
$f(x') = f^s(a')$;
For all $q \in \omega$ and $\alpha' \in \bar{a}'$, $x \neq f^q(\alpha')$;
For all $n \in \omega - \{0, 1\}$, $\models \vartheta_{n,x}(x')$
(and similarly in any ω-saturated model of T containing $\bar{a}'$).

Proof. First notice that $k(x) = \infty$; in fact, if $k(f^s(a)) < \infty$, then

$$x \neq f^{k(f^s(a))-1}(f^s(a)):$$

We have to show that the set

$$\{f(v) = f^s(a')\} \cup \{v \neq f^q(\alpha') : q \in \omega, \alpha' \in \bar{a}'\}$$
$$\cup \{\vartheta_{n,x}(v) : n \in \omega - \{0,1\}\}$$

is satisfiable. As U is very saturated (but ω-saturated is enough), it suffices to prove that this set is finitely satisfiable, and even that, for all $h, n \in \omega$ such that $n \geq 2$ and $h \geq k(f^s(a))$ if $k(f^s(a)) < \infty$, the set

$$\{f(v) = f^s(a')\} \cup \{v \neq f^q(\alpha') : q \leq h, \alpha' \in \bar{a}'\} \cup \{\vartheta_{n,x}(v)\}$$

is satisfiable (recall that, if $f(x') = f^s(a')$ and $x' \neq f^{k(f^s(a'))-1}(f^s(a'))$, then

$k(x') = \infty = k(x)$, hence $\models \vartheta_{n,x}(x')$ implies $\models \vartheta_{i,x}(x')$ for any i with $2 \leq i \leq n$). Let r be the power of

$$\{f^q(\alpha'): q \leq h, \alpha' \in \bar{a}', \models \vartheta_{n,x}(f^q(\alpha')), f(f^q(\alpha')) = f^s(a'),$$
$$f^q(\alpha') \neq f^{k(f^s(a'))-1}(f^s(a'))\}.$$

As a, a' have the same f-type, r is also the power of

$$\{f^q(\alpha): q \leq h, \alpha \in \bar{a}, \models \vartheta_{n,x}(f^q(\alpha)), f(f^q(\alpha)) = f^s(a),$$
$$f^q(\alpha) \neq f^{k(f^s(a))-1}(f^s(a))\}.$$

Moreover

$$\models \exists w\left(f(w) = f^s(a') \wedge \bigwedge_{q \leq h, \alpha' \in \bar{a}'} w \neq f^q(\alpha') \wedge \vartheta_{n,x}(w)\right)$$

if and only if $r < m(n, x, f^s(a')) = m(n, x, f^s(a))$, and hence if and only if

$$\models \exists w\left(f(w) = f^s(a) \wedge \bigwedge_{q \leq h, \alpha \in \bar{a}} w \neq f^q(\alpha) \wedge \vartheta_{n,x}(w)\right)$$

and this formula is true (it suffices to take $w = x$).

LEMMA 4. *Let $\bar{a}, \bar{a}' \in U$ have the same f-type, $h \in \omega - \{0\}$, $x \in U$ be such that:*
There are $s \in \omega$ and $a \in \bar{a}$ satisfying $f^h(x) = f^s(a)$;
For any $q \in \omega$ and $\alpha \in \bar{a}$, $f^{h-1}(x) \neq f^q(\alpha)$.
Then there is $x' \in U$ such that:
$f^h(x') = f^s(a')$;
$f^{h-1}(x') \neq f^q(\alpha')$ for all $q \in \omega$ and $\alpha' \in \bar{a}'$;
x' and x have the same f-type.
And similarly in any ω-saturated model of T containing $\bar{a}'$.

Proof. First notice that $k(f^i(x)) = \infty$ for all $i < h$. We proceed by induction on h (the case $h = 0$ is trivial).

First let $h = 1$. Then it suffices to apply Lemma 3; in fact $k(x) = k(x') = \infty$, and $\models \vartheta_{n,x}(x')$ for every $n \in \omega - \{0, 1\}$; moreover, if $i > 0$, then $f^i(x) = f^{s+i-1}(a)$ and $f^i(x') = f^{s+i-1}(a')$ so that, as a, a' have the same f-type, it follows that $k(f^i(x)) = k(f^i(x'))$, and $\models \vartheta_{n,f^i(x)}(f^i(x'))$ for every $n \in \omega - \{0, 1\}$.

$h \Rightarrow h + 1$. Let $y = f(x)$. Then $f^h(y) = f^s(a)$, $f^{h-1}(y) \neq f^q(\alpha)$ for any $q \in \omega$ and $\alpha \in \bar{a}$; in particular $k(y) = \infty$. By the induction hypothesis, there is $y' \in U$ satisfying $f^h(y') = f^s(a')$, $f^{h-1}(y') \neq f^q(\alpha')$ for all $q \in \omega$ and $\alpha' \in \bar{a}'$, y' admits the same f-type as y. In particular $k(y') = \infty$, $\models \vartheta_{n,y}(y')$ for every $n \in \omega - \{0, 1\}$. It follows from Lemma 2 that there is $x' \in U$ such that $f(x') = y'$ (so that $f^{h+1}(x') = f^s(a')$, $f^h(x') \neq f^q(\alpha')$ for all $q \in \omega, \alpha' \in \bar{a}'$), and $\models \vartheta_{n,x}(x')$ for every $n \in \omega - \{0, 1\}$. Furthermore $k(x') = k(x) = \infty$. This clearly implies that x, x' have the same f-type.

LEMMA 5. *For all $\bar{a}, \bar{a}' \in U$, $\bar{a} \equiv \bar{a}'$ if and only if $\bar{a}, \bar{a}'$ have the same f-type.*

Proof. ($\Rightarrow$) This is trivial.

($\Leftarrow$) It suffices to show that $\bar{a}, \bar{a}'$ correspond to each other in an infinite back-and-forth. Hence assume that $\bar{a}, \bar{a}'$ have the same f-type. We claim that, for every x, there is x' such that $(\bar{a}, x), (\bar{a}', x')$ have the same f-type (in a similar way one can show that, for every x', there is x such that $(\bar{a}, x), (\bar{a}', x')$ have the same f-type).

Case 1. There are $h, s \in \omega, a \in \bar{a}$ such that $f^h(x) = f^s(a)$. Let h be minimal with this property. If $h = 0$, then we are done, as it suffices to pick $x' = f^s(a')$. Then assume $h > 0$. By Lemma 4, as $\bar{a}, \bar{a}'$ have the same f-type and $f^h(x) = f^s(a)$ but $f^{h-1}(x) \neq f^q(\alpha)$ for all $q \in \omega$ and $\alpha \in \bar{a}$, there exists $x' \in U$ satisfying:

$f^h(x') = f^s(a')$;
$f^{h-1}(x') \neq f^q(\alpha')$ for all $q \in \omega$ and $\alpha' \in \bar{a}'$;
x, x' have the same f-type.

Let us show that $(\bar{a}, x)$ and $(\bar{a}', x')$ satisfy our claim. It suffices to prove that, for all $j, l \in \omega$ and $\alpha \in \bar{a}$,

$$f^l(x) = f^j(\alpha) \text{ if and only if } f^l(x') = f^j(\alpha').$$

Assume $f^l(x) = f^j(\alpha)$. Then $l \geq h$, hence

$$f^{l-h+s}(a) = f^l(x) = f^j(\alpha),$$

and consequently

$$f^l(x') = f^{l-h+s}(a') = f^j(\alpha').$$

Conversely, if $f^l(x') = f^j(\alpha')$, then again we have $l \geq h$, and, by proceeding as before, we get $f^l(x) = f^j(\alpha)$.

Case 2. For all $h, s \in \omega$ and $a \in \bar{a}$, $f^h(x) \neq f^s(a)$.

We need find an element $x' \in U$ satisfying:

For all $h, s \in \omega$ and $a' \in \bar{a}'$, $f^h(x') \neq f^s(a')$ (namely $x' \nsim a'$ for all $a' \in \bar{a}'$ —we denote here by $\sim$ the equivalence relation such that, for all $c, c' \in U$, $c \sim c'$ if and only if there are $i, j \in \omega$ satisfying $f^i(c) = f^j(c')$ [T]);

x' admits the same f-type as x;

(Then $(\bar{a}, x), (\bar{a}', x')$ have the same f-type.)

Suppose towards a contradiction that, for every $x' \in U$, if x' satisfies the same f-type as x, then there is $a' \in \bar{a}'$ such that $x' \sim a'$. In particular, there is $a' \in \bar{a}'$ such that $x \sim a'$. Let $h \in \omega$ be minimal such that there are $a' \in \bar{a}', s \in \omega$ such that $f^h(x) = f^s(a')$. Without loss of generality $a' = a'_0$. By using Lemma 4 if $h > 0$ and a trivial argument otherwise, we find a''_0 such that:

$f^h(a''_0) = f^s(a_0)$;

$f^{h-1}(a''_0) \neq f^q(a)$ for all $q \in \omega$ and $a \in \bar{a}$ (when $h > 0$);

a''_0, x have the same f-type.

In particular $a''_0 \sim a_0 \nsim x$, $a''_0 \nsim a'_0$. There is $a' \in \bar{a}'$ such that $a''_0 \sim a'$, and a' cannot equal a'_0. Let $h \in \omega$ be minimal such that there are $s \in \omega, a' \in \bar{a}'$ such that $f^h(a''_0) = f^s(a')$. With no loss of generality $a' = a'_1$ (hence $a'_1 \nsim a'_0$, $a_1 \nsim a_0$, $a'_1 \sim a''_0 \sim a_0$). As above we can find a''_1 such that:

$f^h(a''_1) = f^s(a_1)$;

$f^{h-1}(a''_1) \neq f^q(a)$ for all $q \in \omega$ and $a \in \bar{a}$ (when $h > 0$);

a''_1 admits the same f-type as a''_0 and x.

Then $a''_1 \sim a_1$ (and hence $a''_1 \nsim x, a'_0$), while $a''_1 \nsim a'_1$ (otherwise $a_1 \sim a''_1 \sim a'_1 \sim a''_0 \sim a_0$, contradicting $a_1 \nsim a_0$).

We can repeat this procedure to define a''_j inductively for all j with $1 \leq j \leq t$; in fact, at stage j, we can assume

$x \nsim a_0 \nsim a_1 \nsim \cdots \nsim a_j$,

$a'_0 \nsim a'_1 \nsim \cdots \nsim a'_j$,

$x \sim a'_0$

and, for all $s < j$,

$a_s \sim a''_s \sim a'_{s+1}$,

$a''_s \nsim a'_0, \ldots, a'_s$

(where we use the notation "$a \nsim b \nsim c \ldots$" to mean that $a, b, c, \ldots$ are mutually inequivalent modulo $\sim$) and deduce that there exists $a''_j \sim a_j$ such that a''_j satisfies the same f-type as x. Furthermore $a''_j \nsim a'_0, \ldots, a'_j, x$ and there is $a' \in \bar{a}'$ such that $a' \sim a''_j$, and, when $j < t$, we can assume without loss of generality that $a' = a'_{j+1}$. But, at stage t, this gets a contradiction. Then an element x' as claimed must exist.

Proof of Theorem 2. First notice that, if $k \in \omega - \{0\}$, then $\{a \in U: k(a) = k\}$ can be defined by a unique formula of our language, while, if $k = \infty$, then we have to expect to need an infinite set of formulas for defining

$\{a \in U\colon k(a) = k\}$; in the following let us denote this formula, or this set of formulas respectively, by $k(v) = k$.

Let T^* be the theory of the pairs (M', M) of models of T satisfying $M \lneq M'$ and the conditions (i) and (ii) below.

(i) Let $b \in U$ with $k(b) = \infty, n \in \omega - \{0, 1\}$. Then, for all $h \in \omega, T^*$ contains:

"For every $y \in M$, if there are infinitely many $x \in M$ satisfying $f(x) = y$ and $\models \vartheta_{n,b}(x)$, then there are $> h$ elements $x \in M' - M$ such that $f(x) = y$ and $\models \vartheta_{n,b}(x)$".

It is clear that, for every $h \in \omega$, the previous proposition can be expressed by a suitable 1st order sentence of the language for pairs of models of T.

(ii) Let $b \in U, n, s \in \omega, n \geq 2$. Let $s' \leq s + 1$ be such that, for every $j \leq s, k(f^j(b)) = \infty$ if and only if $j < s'$ (possibly $s' = 0$; in this case $k(f^j(b)) < \infty$ for all $j \leq s$). Assume that T contains the following sentences: for all $q \in \omega$,

$$\exists w\Big(\bigwedge_{j\leq s} \vartheta_{n,f^j(b)}(f^j(w)) \wedge \bigwedge_{s'\leq j\leq s} k(f^j(w)) = k(f^j(b)) \\ \wedge \bigwedge_{0<l\leq q,\, j<s'} f^l(f^j(w)) \neq f^j(w)\Big)$$

and, for all $h, q \in \omega$,

$$\forall v_0 \cdots \forall v_h \exists w\Big(\bigwedge_{i\leq h,\, j\leq s} \vartheta_{n,f^j(b)}(f^j(v_i)) \\ \longrightarrow \bigwedge_{j\leq s} \vartheta_{n,f^j(b)}(f^j(w)) \wedge \bigwedge_{s'\leq j\leq s} k(f^j(w)) = k(f^j(b)) \\ \wedge \bigwedge_{0<l\leq q,\, j<s'} f^l(f^j(w)) \neq f^j(w) \wedge \bigwedge_{i\leq h,\, l, m\leq q} f^m(w) \neq f^l(v_i)\Big).$$

Notice that to assume that T satisfies the previous sentences is the same as to require that U-as well as any ω-saturated model of T-contains infinitely many pairwise $\nsim$ elements satisfying

$$\vartheta_{n,f^j(b)}(f^j(v)), \quad k(f^j(v)) = k(f^j(b)) \quad \text{for all } j \leq s.$$

Then T^* includes the following sentences: for all $q \in \omega$,

$$\exists w\Big(\bigwedge_{j \le s} \vartheta_{n, f^j(b)}(f^j(v)) \wedge \bigwedge_{s' \le j \le s} k(f^j(v)) = k(f^j(b)) \\ \wedge \bigwedge_{0 < l \le q,\, j < s'} f^l(f^j(w)) \ne f^j(w) \wedge \bigwedge_{l \le q} f^l(w) \notin M \Big)$$

and, for all $h, q \in \omega$,

$$\forall v_0 \cdots \forall v_h \exists w\Big(\bigwedge_{i \le h,\, j \le s} \vartheta_{n, f^j(b)}(f^j(v_i)) \\ \longrightarrow \bigwedge_{j \le s} \vartheta_{n, f^j(b)}(f^j(w)) \wedge \bigwedge_{s' \le j \le s} k(f^j(w)) = k(f^j(b)) \\ \wedge \bigwedge_{0 < l \le q,\, j < s'} f^l(f^j(w)) \ne f^j(w) \wedge \bigwedge_{l \le q} f^l(w) \notin M \wedge \bigwedge_{i \le h,\, l, m \le q} \\ f^l(w) \ne f^m(v_i)\Big).$$

Notice that this is equivalent to the assumption that in every ω-saturated model (M', M) of T^* there are infinitely many pairwise $\not\sim$ elements that are $\not\sim$ to M and satisfy

$$\vartheta_{n, f^j(b)}(f^j(v)), k(f^j(v)) = k(f^j(b)) \quad \text{for all } j \le s.$$

We claim that the theory T^* we have just now introduced equals the theory T' of nice pairs of models of T. Recall that a pair (M', M) of models of T is said to be nice if M is ω_1-saturated, and, for every $\bar{a} \in M'$, any type in T over $M \cup \bar{a}$ is realized in M'. We point out also that, if T is the theory of a 1-ary function, then the theory T' of nice pairs of models of T is complete since T is superstable (see [P]). The proof of our claim requires three steps.

Step 1. Every nice pair (M', M) of models of T satisfies T^*. In fact we have the following.

(i) Let $b \in U$ with $k(b) = \infty, n \in \omega - \{0, 1\}, y \in M$, and assume that there exist infinitely many elements $x \in M$ satisfying $f(x) = y$, $\models \vartheta_{n,b}(x)$. Let $\{a_0, \ldots, a_h\}$ be a finite (possibly empty) subset of $M' - M$ whose elements satisfy $f(v) = y \wedge \vartheta_{n,b}(v)$. Then

$$\{f(v) = y\} \cup \{\vartheta_{n,b}(v)\} \cup \{v \ne d\colon d \in M \cup \{a_0, \ldots, a_h\}\}$$

can be enlarged to a type over $M \cup \{a_0, \ldots, a_h\}$, and this type must be realized in M'.

(ii) can be shown in a similar way.

Step 2. Every ω_1-saturated model of T^* is a nice pair. In fact, let (M', M) be an ω_1-saturated model of T. In particular M is ω_1-saturated. Hence it suffices to show that, if $\bar{a} \in M'$ and p is a 1-type over $M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}$ (in T), then p is realized in M'. With no loss of generality we can assume that p is not algebraic, otherwise our claim is trivially true.

Case 1. There are $h \in \omega - \{0\}, b \in M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}$ such that p contains $f^h(v) = b$ and

$$f^{h-1}(v) \neq d \quad \text{for all } d \in M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}.$$

Then p is defined by the previous formulas together with the f-type of x where x is any realization of p (this follows from Lemma 5 and the remark that the f-type of x determines the f-type of $x \cup \bar{c}$ for any $\bar{c} \in M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}$). Notice that, for every $x \vDash p$, if $k(b) < \infty$, then $f^{h-1}(x) \neq f^{k(b)-1}(b)$, hence $k(f^j(x)) = \infty$ for every $j < h$. Fix $x \vDash p$. We claim that:

There is $c \in M'$ such that $f(c) = b, c \notin M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}$ and, for all $n, j \in \omega$ with $n \geq 2$, $k(f^j(c)) = k(f^{j+h-1}(x))$, $\vDash \vartheta_{n, f^{j+h-1}(x)}(f^j(c))$.

Subcase 1. For all $n \in \omega - \{0, 1\}$, there exist infinitely many elements realizing $f(v) = b \wedge \vartheta_{n,f}h - 1_{(x)}(v)$.

Then there are infinitely many elements of $M' - M$ realizing

$$f(v) = b \wedge \vartheta_{n,f}h - 1_{(x)}(v)$$

(this is obvious if $b \notin M$, and follows from (i) if $b \in M$). On the other hand, $\{f^k(a): k \in \omega, a \in \bar{a}\}$ contains only finitely many elements satisfying this formula. In fact, let $a \in \bar{a}$. If there exists at most one $s \in \omega$ such that $f^s(a) = b$, then there is at most one $k \in \omega$ such that $f(f^k(a)) = b$ ($k = s - 1$ provided that $s > 0$). Otherwise, let s be the minimal natural number such that $f^s(a) = b$. Then $k(b) < \infty$, and, for all $k \in \omega, f^k(a) = b$ if and only if $k \equiv s \bmod k(b)$, and, consequently, $f(f^k(a)) = b$ if and only if $k + 1 \equiv s \bmod k(b)$. Then there are at most two elements of the form $f^k(a)$ with $k \in \omega$ satisfying $f(v) = b$, as, if $k, k' \in \omega, k, k' \geq s$ and $f(f^k(a)) = f(f^{k'}(a)) = b$, then $k + 1 \equiv k' + 1 \bmod k(b)$ and hence $k \equiv k' \bmod k(b)$, so that $f^k(a) = f^{k'}(a)$.

It follows that

$$\{f(v) = b\} \cup \{\vartheta_{n, f^{h-1}(x)}(v)\} \cup \{v \notin M\} \cup \{v \neq f^k(a): k \in \omega, a \in \bar{a}\}$$

can be realized in (M', M). By using the ω-saturation of (M', M) and Lemma

1, we obtain that there is $c \in M'$ satisfying

$$\{f(v) = b\} \cup \{\vartheta_{n, f^{h-1}(x)}(v) : n \in \omega - \{0,1\}\} \cup \{v \notin M\} \cup \{v \neq f^k(a) : k \in \omega, a \in \bar{a}\}.$$

As $f(c) = b$ but $c \neq f^{k(b)-1}(b)$ if $k(\dot{b}) < \infty$, then

$$k(c) = \infty = k(f^{h-1}(x)).$$

Moreover, for every $j \in \omega - \{0\}$, $f^j(c) = f^{j-1}(b) = f^{j+h-1}(x)$, hence

$$k(f^j(c)) = k(f^{j+h-1}(x)),$$

and, for all $n \in \omega - \{0, 1\}$, $\models \vartheta_{n, f^{j+h-1}(x)}(f^j(c))$.

Subcase 2. There is $n \in \omega - \{0, 1\}$ such that $f(v) = b \wedge \vartheta_{n, f^{h-1}(x)}(v)$ admits only finitely many realizations.

Then all these realizations belong to M'. As $f^{h-1}(x)$ satisfies the previous formula, $f^{h-1}(x) \in M'$ and we can assume $c = f^{h-1}(x)$.

This completes the proof of the claim. Let us come back to the problem of finding an element of M' realizing p. If $h = 1$, then we are done (c works). So assume $h > 1$. Then c and $f^{h-1}(x)$ satisfy the same f-type; furthermore $f^{h-2}(x) \neq f^q(f^{h-1}(x))$ for all $q \in \omega$ as $k(f^{h-2}(x)) = \infty$. Hence, by using Lemma 4 and the fact that M' is ω_1-saturated and contains c, we can find $x' \in M'$ such that:

$f^{h-1}(x') = c$ (and then $f^h(x) = b$, $f^{h-1}(x') \notin M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}$);
x' has the same f-type as x.

Then $x' \models p$.

Case 2. for all $h \in \omega$ and $b \in M \cup \{f^k(a): k \in \omega, a \in \bar{a}\}$, p contains $f^h(v) \neq b$.

As above, p is defined by these formulas together with the f-type of x where $x \models p$. Let $n, s \in \omega, n \geq 2$. As M is ω_1-saturated, there exist infinitely many pairwise $\nsim$ elements of M satisfying

$$\vartheta_{n, f^j(x)}(f^j(v)), k(f^j(v)) = k(f^j(x))$$

for all $j \leq s$. In fact, define s' as above, and let $\{x_0, \ldots, x_h\}$ be a finite, possibly empty, subset of M whose elements are pairwise $\nsim$ and satisfy the foregoing set of formulas; then

$$\begin{aligned}&\left\{\vartheta_{n,f^j(x)}(f^j(v)) : j \le s\right\} \cup \left\{k(f^j(v)) = k(f^j(x)) : s' \le j \le s\right\}\\ &\quad \cup \left\{f^l(f^j(v)) = f^j(v) : j < s', 0 < l \in \omega\right\}\\ &\quad \cup \left\{f^l(v) \neq f^m(x_i) : i \le h, l, m \in \omega\right\}\end{aligned}$$

is finitely satisfiable in M (in fact it is satisfied by x), hence it is satisfiable in M. Then (ii) provides infinitely many pairwise $\not\sim$ elements of M' which are $\not\sim$ to M and satisfy

$$\begin{aligned}&\left\{\vartheta_{n,f^j(x)}(f^j(v)) : j \le s\right\} \cup \left\{k(f^j(v))k(f^j(x)) : s' \le j \le s\right\}\\ &\quad \cup \left\{f^l(f^j(v)) \neq f^j(v) : j < s', 0 < l \in \omega\right\}.\end{aligned}$$

In particular there is $y \in M'$ such that y satisfies this set and $y \not\sim M \cup \bar{a}$. Then there is $x' \in M'$ such that $x' \not\sim M \cup \bar{a}$ and, for all $j, n \in \omega$ with $n \ge 2, \models \vartheta_{n,f^j(x)}(f^j(x'))$ and $k(f^j(x')) = k(f^j(x))$; in fact, it suffices to notice that the set

$$\begin{aligned}&\left\{f^l(v) \notin M : l \in \omega\right\} \cup \left\{f^l(v) \neq f^k(a) : l, k \in \omega, a \in \bar{a}\right\}\\ &\quad \cup \left\{\vartheta_{n,f^j(x)}(f^j(v)) : j, n \in \omega, n \ge 2\right\} \cup \left\{k(f^j(v)) = k(f^j(x)) : j \in \omega\right\}\end{aligned}$$

is finitely satisfiable as every subset of the kind

$$\begin{aligned}&\left\{f^l(v) \notin M : l \in \omega\right\} \cup \left\{f^l(v) \neq f^k(a) : l, k \in \omega, a \in \bar{a}\right\}\\ &\quad \cup \left\{\vartheta_{m,f^j(x)}(f^j(v)) : j \le s, 2 \le m \le n\right\} \cup \left\{k(f^j(v)) = k(f^j(x)) : j \le s\right\}\end{aligned}$$

(with $n, s \in \omega, n \ge 2$), hence every finite subset, is satisfiable (use the previous remarks and Lemma 1; recall that, if y is as above, then in particular $k(f^j(y)) = k(f^j(x))$ for all $j \le s$).

Step 3. $T^* = T'$. In fact, it follows from the Step 1 that $T^* \subseteq T'$. On the other hand, let (M', M) be a model of T and (N', N) be an ω_1-saturated elementary extension of (M', M); then $(N', N) \models T^*$, and hence the second step implies that (N', N) is a nice pair. Consequently $(N', N) \models T'$, and $(M', M) \models T'$, too. Then $T' \subseteq T^*$.

We can now conclude the proof of the theorem, as the second step ensures that every ω_1-saturated model of T' is a nice pair (in fact, this is true for T^*), and this implies that T does not have the f.c.p. (see [P], Theorem 6).

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REFERENCES

[BK] J. BALDWIN and D. KUEKER, *Ramsey quantifiers and the finite cover property*, Pacific J. Math., vol. 90 (1980), pp. 11–19

[CK] C.C. CHANG and H. KEISLER, *Model theory*, North Holland, Amsterdam, 1973.

[K] H. KEISLER, *Ultraproducts which are not saturated*, J. Symbolic Logic, vol. 32 (1967), pp. 23–47

[M1] L. MARCUS, *Minimal models of theories of one function symbol*, Israel J. Math., vol. 18 (1974), pp. 117–131

[M2] ______, *The number of countable models of a theory of one unary function*, Fund. Math., vol. 108 (1980), pp. 171–181

[Mi] A. MILLER, *Vaught's conjecture for theories of one unary operation*, Fund. Math., vol. 111 (1981), pp. 135–141

[P] B. POIZAT, *Paires de structures stables*, J. Symbolic Logic, vol. 48 (1983), 239–249

[S] S. SHELAH, *Classification theory and the number of non-isomorphic models*, North Holland, Amsterdam, 1978.

[T] C. TOFFALORI, *Classification theory for a 1-ary function*, Illinois J. Math., vol. 35 (1991), pp. 1–26.

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