ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 3, Fall 1991

FACTORIZATION OF SOLUTIONS OF CONVOLUTION EQUATIONS II

BY

GIUSEPPE MARINO, PAOLAMARIA PIETRAMALA AND DANIELE CARLO STRUPPA

0. Let $C^\infty(\mathbf{R}^n)$, be the vector space of complex valued C^∞ functions on $\mathbf{R}^n$. It is well known (from the fundamental Principle of Ehrenpreis for the case $n > 1$ and more easily, from the classical Euler exponential polynomial representation of solutions of ordinary differential equations, for the case $n = 1$) that if the partial differential equation

$$Q(D)f = 0 \tag{0.1}$$

is such that $Q \in \mathbf{C}[z_1, \dots, z_n]$ can be factored as $Q = Q_1 \cdot Q_2$, with Q_1 and Q_2 relatively prime, then every C^∞ solution of (0.1) can be written as $f = f_1 + f_2$, with $Q_i(D)f_i = 0$, $i = 1, 2$.

The natural extension of the previous result to convolution equations in the space $H(\mathbf{C})$ of entire functions was obtained by V.V. Napalkov [9]. This last result has been successively extended by the authors [6], for a class of spaces of which both $H(\mathbf{C})$ and $C^\infty(\mathbf{R})$ are particular cases, under natural hypotheses on the convolutors, without mentioning the Fundamental Principle, but employing specific properties of suitable spaces of entire functions satisfying certain growth conditions.

Recent results (Berenstein-Struppa [1], Meril-Struppa [7], Morzhakov [8]) obtained for convolution operators acting on the space $H(\Omega)$ of holomorphic functions on a convex domain Ω of $\mathbf{C}$, now allow us to extend the result of [6] to this space.

In Section 1, we give the basic definitions for the rest of the paper, while in Section 2 we adapt the well known Hörmander's L^2-theory to the spaces of Fourier-Borel transform of analytic functionals with prescribed carrier; a corona-like theorem is obtained (Theorem 2.1). The desired factorization results are finally obtained in Section 3.

Received September 1, 1989.
1980 Mathematics Subject Classification (1985 Revision) Primary 30D15; Secondary 46F15

1. In this section, we fix the notations that will be used in the paper.

Let $H(\Omega)$ denote the space of holomorphic functions on an open convex set Ω in the complex plane $\mathbf{C}$ with the topology of uniform convergence on compact subsets of Ω.

Let K be a convex compact set in $\mathbf{C}$ such that $\Omega + K \subseteq \Omega$ and consider an analytic functional $\mu \in H'(\mathbf{C})$ carried by K. Then it is well known that μ acts as a convolutor on $H(\Omega + K)$ by defining

$$\begin{aligned} \mu *: H(\Omega + K) &\to H(\Omega) \\ & f \mapsto (\mu * f)(z) := \langle \mu, \zeta \mapsto f(z+\zeta) \rangle, \quad z \in \Omega, \quad \zeta \in K. \end{aligned}$$

The properties of this convolution operator are reflected in the properties of the Fourier-Borel transform $\hat{\mu}$ of μ, defined as

$$\hat{\mu}(z) := \langle \mu, \zeta \mapsto \exp(z \cdot \zeta) \rangle$$

We recall that, via the Fourier-Borel transform, we have the following topological isomorphisms:

$$\begin{aligned} H'(\Omega) \cong \widehat{H'}(\Omega) = \{ F \in H(\mathbf{C}) : \exists A > 0, \exists T \subseteq \Omega \text{ compact convex such that} \\ |F(Z)| \leq A \exp(H_T(z)), \forall z \in \mathbf{C} \}, \end{aligned}$$

$$\begin{aligned} H'(\Omega + K) \cong \widehat{H'}(\Omega + K) = \{ F \in H(\mathbf{C}) : \exists A > 0, \\ \exists T \subseteq \Omega + K \text{ compact convex such that} \\ |F(z)| \leq A \exp(H_T(z)), \forall z \in \mathbf{C} \}, \end{aligned}$$

$$\begin{aligned} H'(K) \cong \widehat{H'(K)} = \{ F \in H(\mathbf{C}) : \forall \varepsilon > 0 \, \exists A_\varepsilon > 0 \\ \text{such that } |F(z)| \leq A_\varepsilon \exp(H_K(z) + \varepsilon |z|) \} \end{aligned}$$

where $H_M(z) := \sup_{\zeta \in M} \operatorname{Re}(z \cdot \zeta)$ is the supporting function of a compact convex set M.

The adjoint map

$$(\mu *)' : H'(\Omega) \to H'(\Omega + K)$$

defined by

$$\langle (\mu *)' \alpha, f \rangle := \langle \alpha, \mu * f \rangle \, \forall f \in H(\Omega + K), \forall \alpha \in H'(\Omega),$$

induces, via Fourier-Borel transform, a continuous multiplication operator

$$T_{\hat{\mu}} : \widehat{H'}(\Omega) \to \widehat{H'}(\Omega + K),$$

defined by

$$T_{\hat{\mu}}(\hat{\alpha})(z) := \hat{\mu}(z)\hat{\alpha}(z), \forall z \in \mathbf{C}.$$

Let us now recall the usual notion of function of completely regular growth which will be needed in the sequel.

DEFINITION 1.1. Let $f(z)$ be an entire function of exponential type. We say that f is of completely regular growth if, for every $\vartheta \in [0, 2\pi]$, the limit

$$\lim_{r \to \infty} \frac{\ln|f(re^{i\vartheta})|}{r}$$

exists when r goes to infinity by taking on all positive values except possibly for a set E_ϑ of zero relative measure. This set can be taken to be the same for all values of ϑ.

The property of functions of completely regular growth we are interested in, is expressed by the following theorem, due to Morzhakov, [8]:

THEOREM 1.1. *Let $\hat{\mu}$ be the Fourier-Borel transform of an analytic functional $\mu \in H'(\mathbf{C})$ carried by a compact K and let $\Omega \subseteq \mathbf{C}$ be open and convex set. Then the convolution operator $\mu *: H(\Omega + K) \to H(\Omega)$ is surjective if and only if $\hat{\mu}$ is a function of completely regular growth.*

Remark 1.1. If μ is slowly decreasing (in the sense of definition 1 of Berenstein-Struppa [1]), then $\mu *$ is surjective and therefore $\hat{\mu}$ is of completely regular growth. It follows, in particular, that an exponential polynomial is always of completely regular growth. Another example in which the surjectivity of $\mu *$ follows, is when $K = \{0\}$, i.e., when $\mu *$ is a differential operator of infinite order. In this case $\hat{\mu}$ is of infraexponential type (i.e., $\forall \varepsilon > 0 \, \exists A_\varepsilon > 0$ such that $|\hat{\mu}(z)| \le A_\varepsilon \exp(\varepsilon |z|)$) and the closure of $\hat{\mu} H'(\Omega)$ follows from a variation of Lindelöf theorem. More generally, if $\hat{\mu}$ is the product of an exponential polynomial and of a function of infraexponential type, then $\mu *$ is surjective.

2. In this section, we adapt the well known results of Hörmander [4], to the case of the spaces $\widehat{H'(K)}$. We refer the reader to [4] for the proofs of Lemmas 2.1, 2.2 and 2.3.

LEMMA 2.1. *If h belongs to $\widehat{H'(K)}$, then also all of its constant coefficients derivatives belong to $\widehat{H'(K)}$.*

LEMMA 2.2. *Let h be a measurable function on $\mathbf{C}$ such that $\partial h/\partial \bar{z} = 0$, and let $\Omega \subseteq \mathbf{C}$ be an open set. If there exists a compact set $T \subset \Omega$ such that*

$$\int_{\mathbf{C}} |h|^2 \exp(-2H_T)\, dm < \infty,$$

for m the Lebesgue measure, then $h \in \widehat{H'}(\Omega)$

LEMMA 2.3. *If g is a measurable function such that*

$$\int_{\mathbf{C}} |g|^2 \exp(-2H_T)\, dm < \infty$$

for some compact set T, then there exist a function h and a compact set T_1 such that

$$\int_{\mathbf{C}} |h|^2 \exp(-2H_{T_1})\, dm < \infty$$

and $\bar{\partial} h = g$.

LEMMA 2.4. *Let $g_i = \bar{\hat{\mu}}_i/M^2$, $M := (|\hat{\mu}_1|^2 + |\hat{\mu}_2|^2)^{1/2}$, with $\mu_i \in H'(\mathbf{C})$, μ_i carried by a compact set K. Suppose that there exist a compact set $T \subset \Omega$ and a constant $A > 0$ such that*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-(1/2)H_T(z)) \quad \textit{for every } z \in \mathbf{C}.$$

Then there exists a compact set $T_1 \subset \Omega + K$ such that

$$\int_{\mathbf{C}} |g_i|^2 \exp(-2H_{T_1})\, dm < \infty.$$

Proof. From the hypothesis,

$$M^2 = |\hat{\mu}_1|^2 + |\hat{\mu}_2|^2 \geq \tfrac{1}{2}\big(|\hat{\mu}_1| + |\hat{\mu}_2|\big)^2 \geq \tfrac{1}{2}A^2 \exp(-H_T).$$

Then

$$|g_i| = |\hat{\mu}_i|/M^2 \leq b_\varepsilon \exp\big(H_K + H_{B_\varepsilon}\big)/M^2 \leq C_\varepsilon \exp\big(H_K + H_T + H_{B_\varepsilon}\big)$$

from which we deduce that

$$|g_i|^2 \leq C_\varepsilon^2 \exp\big(2\big(H_K + H_T + H_{B_\varepsilon}\big)\big),$$

where B_ε is the ball $|z| \le \varepsilon$. Let ε be sufficiently small in such a way that $K + T + 2B_\varepsilon =: T_1$ is contained in $\Omega + K$. It follows that

$$\int_{\mathbf{C}} |g_i|^2 \exp(-2H_{T_1})\, dm \le C^2 \int_{\mathbf{C}} \exp(-2H_{B_\varepsilon})\, dm < \infty.$$

Remark 2.2. The proof of the previous lemma shows that the result is also true if there exist a compact set $T \subset \Omega$ and $A > 0$ such that $2T \subset \Omega$ and

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \ge A \exp(-H_T(z)),\ z \in \mathbf{C}$$

LEMMA 2.5. *With the hypotheses of Lemma* 2.4 (*or Remark* 2.2) *let S be a compact set in Ω and $h \in C^\infty(\mathbf{C})$ such that*

$$\int_{\mathbf{C}} |h|^2 \exp(-2H_S)\, dm < \infty.$$

Then there exists a compact set $S_1 \subset \Omega$ such that

$$\int_{\mathbf{C}} |g_1 + h\hat{\mu}_2|^2 \exp(-2H_{S_1})\, dm < \infty.$$

Proof. From

$$|g_1 + h\hat{\mu}_2|^2 \le |g_1|^2 + |h\hat{\mu}_2|^2 + 2|g_1 h\hat{\mu}_2|$$

it follows that

$$\int_{\mathbf{C}} |g_1 + h\hat{\mu}_2|^2 \exp(-2H_{S_1})\, dm \le I_1 + I_2 + I_3 \quad \text{for every } S_1 \subset \Omega \text{ compact},$$

where

$$I_1 = \int_{\mathbf{C}} |g_1|^2 \exp(-2H_{S_1})\, dm,$$

$$I_2 = \int_{\mathbf{C}} |h\hat{\mu}_2|^2 \exp(-2H_{S_1})\, dm,$$

$$I_3 = 2\int_{\mathbf{C}} |g_1 h\hat{\mu}_2| \exp(-2H_{S_1})\, dm.$$

Hence it is sufficient to show the convergence of integrals I_1, I_2, I_3 for some compact S_1 depending of S. Let us set $S_2 = S + K + 2B_\varepsilon$ to show the convergence of I_2. Indeed,

$$\begin{aligned}
&\int_{\mathbf{C}} |h\hat{\mu}_2|^2 \exp(-2H_{S_2})\, dm \\
&\quad \le A_\varepsilon^2 \int_{\mathbf{C}} |h|^2 \exp\big(2(H_K + H_{B_\varepsilon} - H_K - H_S - 2H_{B_\varepsilon})\big)\, dm \\
&\quad = A_\varepsilon^2 \int_{\mathbf{C}} |h|^2 \exp(-2H_S - 2H_{B_\varepsilon})\, dm \\
&\quad \le A^2 \int_{\mathbf{C}} |h|^2 \exp(-2H_S)\, dm \\
&\quad < \infty.
\end{aligned}$$

Now, let $S_1 = T_1 \cup S_2$, where T_1 is the compact set of Lemma 2.4.Then

$$\int_{\mathbf{C}} |h\hat{\mu}_2|^2 \exp(-2H_{S_1})\, dm \le \int_{\mathbf{C}} |h\hat{\mu}_2|^2 \exp(-2H_{S_2})\, dm < \infty,$$

$$\int_{\mathbf{C}} |g_1|^2 \exp(-2H_{S_1})\, dm \le \int_{\mathbf{C}} |g_1|^2 \exp(-2H_{T_1})\, dm < \infty.$$

Finally, the convergence of integral I_3 is assured from the Cauchy-Schwarz formula

$$\tfrac{1}{2} I_3 \le (I_1 I_2)^{1/2}. \qquad \blacksquare$$

We can now prove a corona-like statement for $\widehat{H'}(\Omega)$ (as Hörmander pointed out in [4], this kind of statement is, however, weaker than the corona theorem).

THEOREM 2.1. *Let $T \subset \Omega$ be a compact set such that $2T \subset \Omega$. In addition we suppose that either*

(i) $4T \subset \Omega$ *and there exists* $A > 0$ *such that*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \ge A \exp(-H_T(z)), \quad z \in \mathbf{C},$$

or

(ii) *there exists* $B > 0$ *such that*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \ge B \exp((-1/2) H_T(z)), \quad z \in \mathbf{C}.$$

Then $1 \in \hat{\mu}_1 \widehat{H'}(\Omega) + \hat{\mu}_2 \widehat{H'}(\Omega)$. *In particular,* Ω *contains the origin of* $\mathbf{C}$.

Proof. Suppose that (i) verified (we proceed similarly if (ii) is verified). Let $g_i = \overline{\hat{\mu}}_i/M^2$. Then, for any $h \in C^\infty(\mathbf{C})$, one has

$$\hat{\mu}_1(g_1 + h\hat{\mu}_2) + \hat{\mu}_2(g_2 - h\hat{\mu}_1) = 1$$

and the theorem is proved if we can find $h \in C^\infty(\mathbf{C})$ such that

$$(g_1 + h\hat{\mu}_2) \in \widehat{H'(\Omega)} \quad \text{and} \quad (g_2 - h\hat{\mu}_1) \in \widehat{H'(\Omega)}.$$

To do this, from Lemma 2.2, it is sufficient to show that

$$\bar{\partial}(g_1 + h\hat{\mu}_2) = \bar{\partial}(g_2 - h\mu_1) = 0,$$

i.e.,

(a) $\bar{\partial}h = -(\bar{\partial}g_1)/\hat{\mu}_2 = (\bar{\partial}g_2)/\hat{\mu}_1$

and

(b) $\int_{\mathbf{C}}|g_1 + h\hat{\mu}_2|^2 \exp(-2H_{S_1})\,dm < \infty$, $\int_{\mathbf{C}}|g_2 - h\hat{\mu}_1|^2 \exp(-2H_{S_1})\,dm < \infty$ for some compact $S_1 \subset \Omega$. On the other hand, from Lemma 2.5, (b) is satisfied if

(b′) $\int_{\mathbf{c}}|h|^2 \exp(-2H_S)\,dm < \infty$ for some compact set $S \subset \Omega$.

The existence of h which satisfies (a) and (b′) follows from Lemma 2.3 once we prove that

$$\int_{\mathbf{C}}|(\bar{\partial}g_2)/\hat{\mu}_1|^2 \exp(-2H_{T_1})\,dm < \infty \text{ for some compact set } T_1 \subset \Omega.$$

Now

$$\bar{\partial}g_2 = \frac{\overline{\sum_{j=1}^2 \hat{\mu}_j\left(\hat{\mu}_j\,\partial\hat{\mu}_2 - \hat{\mu}_2\,\partial\hat{\mu}_j\right)}}{M^4} = \frac{\overline{\hat{\mu}_1\left(\hat{\mu}_1\,\partial\hat{\mu}_2 - \hat{\mu}_2\,\partial\hat{\mu}_1\right)}}{M^4},$$

and therefore

$$\left|\frac{\bar{\partial}g_2}{\hat{\mu}_1}\right| \leq \frac{|\hat{\mu}_1\,\partial\hat{\mu}_2| + |\hat{\mu}_2\,\partial\hat{\mu}_1|}{M^4}.$$

Since $\partial\hat{\mu}_1, \partial\hat{\mu}_2$ belong to $\widehat{H'}(K)$ (see Lemma 2.1) and $M^{-4} \leq B\exp(-4H_T)$, for some $B > 0$, setting $T_1 := 2K + 4T + 4B_\varepsilon$ (with ε sufficiently small in such a way that $T_1 \subset \Omega$), we have

$$\int_{\mathbf{C}}|(\bar{\partial}g_2)/\hat{\mu}_1|^2 \exp(-2H_{T_1})\,dm \leq B'\int_{\mathbf{C}} \exp(4H_K + 4H_{B_\varepsilon} + 8H_T - 2H_{T_1})\,dm$$

$$= B'\int_{\mathbf{C}} \exp(-4H_{B_\varepsilon})\,dm < \infty \qquad \blacksquare$$

Remark 2.3. From the convexity of Ω and from the fact that the convex hull of a compact set of $\mathbf{C}$ is also a compact set, it follows that every compact set which appear in the previous lemmas and in Theorem 2.1 can be replaced by a compact *convex* set.

The following corollary was essentially proved in [9], and is an obvious consequence of [4], [9]. We only quote it as a motivation for our next, and final section.

COROLLARY 2.1. *Let $\mu, \mu_1, \mu_2 \in H'(\mathbf{C})$ be carried by the same compact convex set K, and suppose that $\hat{\mu} = \hat{\mu}_1 \cdot \hat{\mu}_2$ and that $\hat{\mu}_1$ and $\hat{\mu}_2$ do not have common zeroes. Then the following statements are equivalent.*

(1) *Every solution $f \in H(\mathbf{C})$ of the equation $\mu * f = 0$ has the representation $f = f_1 + f_2$, where f_i is a solution of equation $\mu_i * f = 0$, $i = 1, 2$.*

(2) *There exist $c_1 > 0$ and $c_2 > 0$ such that*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq c_1 \exp(-c_2|z|), \quad z \in \mathbf{C}.$$

(3) *For every $c > 0$ there exist a compact set $T \subset \mathbf{C}$ and $A > 0$ such that*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-cH_T(z)), \quad z \in \mathbf{C}.$$

Proof. (1) implies (2). See [9].

(2) implies (3). Indeed (3) is satisfied with $A = c_1$ and $T = B_{(c_2/c)}$.

(3) implies (1). From Theorem 2.1 we have $1 = \hat{\mu}_1 \hat{a}_1 + \hat{\mu}_2 \hat{a}_2$, for some $\hat{a}_1, \hat{a}_2 \in \widehat{H'(\mathbf{C})}$, and so $\delta = \mu_1 * a_1 + \mu_2 * a_2$, δ being the Dirac measure. Hence

$$f = f * \delta = f * \mu_1 * a_1 + f * \mu_2 * a_2 = f_1 + f_2$$

is the desired decomposition. ■

3. In this section we shall establish an analog of Corollary 2.1 in the case in which $\mathbf{C}$ is replaced by an open convex set $\Omega \subseteq \mathbf{C}$. We assume some normalizing hypotheses as follows:

(i) $\mu_1, \mu_2, \mu \in H'(\mathbf{C})$ are analytic functionals whose minimal carriers are, respectively, three compact convex sets K_1, K_2, K.

(ii) $K = K_1 + K_2$.

(iii) $K_1, K_2 \subseteq K$.

(iv) $\hat{\mu} = \hat{\mu}_1 \cdot \hat{\mu}_2$.

(v) $\hat{\mu}_1$ and $\hat{\mu}_2$ have no common zeroes.

It is easy to construct many examples of such a situation. Before doing so, however, two remarks are necessary (although probably obvious):

Remark 3.1. Conditions (ii) and (iv) imply that both K_1 and K_2 contain the origin of **C**. Indeed if, say, $0 \notin K_2$, then we could find a small ball B around the origin such that

$$B \cap K_2 = \varnothing, \text{ and therefore } (nB) \cap (nK_2) = \varnothing \text{ for any } n.$$

Also, by (ii) and (iii), $K_1 \subseteq K = K_1 + K_2 \subseteq K_1 + K_2 + K_2 \subseteq \cdots \subseteq K_1 + nK_2$ for any n; if we now take n_0 such that

$$K_1 \subseteq n_0 B,$$

we see that

$$K_1 \cap (K_1 + 2n_0 K_2) = \varnothing$$

which is clearly a contradiction.

Remark 3.2. The important consequence of this fact is that (keeping into account the assumptions of section 1) we deduce that

$$\Omega + K \subseteq \Omega \subseteq \Omega + K_i \subseteq \Omega + K;$$

i.e., $\Omega = \Omega + K = \Omega + K_2 = \Omega + K$. Therefore, in what follows, we shall not distinguish anymore between these sets.

Let us now provide some meaningful examples of sets (Ω, K, K_1, K_2) which satisfy all these conditions.

Example 3.1. Let Ω be any open convex set, and take $K = K_1 = K_2 = \{0\}$. Then all of the above conditions are trivially satisfied and the example is still of mathematical interest, as in this case $\mu_1 *, \mu_2 *, \mu *$ define differential operators of infinite order (in striking contract with what would have happened in the C^∞ case); one can also use this example to provide factorization of hyperfunction solutions of such operators (by taking appropriate boundary values of holomorphic functions).

Example 3.2. Take $\Omega = \Pi_+$ to be the open upper half plane. Then several different examples can be constructed by taking three positive numbers a, b, c, and setting

$$K_1 = [-a, b] \times i[0, c], \quad K_2 = [-b, a] \times i[0, c]$$

and therefore

$$K = [-a-b, a+b] \times i[0, 2c].$$

It is clear that other, less regular, examples can be obtained. The reader may notice that examples of this kind may appear when dealing with generalizations of the Fabry gap Theorem [1].

Example 3.3. In some recent generalizations of [1] to the case of faster growths, [11], the necessity arises of considering Ω to be an open convex cone with vertex the origin. So, if

$$\Omega = \{z \in C: \tfrac{1}{2}\pi - \vartheta < \arg z < \tfrac{1}{2}\pi + \vartheta\} = \Gamma_\vartheta$$

for some $\vartheta \in (0, \pi/2)$, one can take, for K_i, sets such as, for $i = 1, 2$,

$$K_i = \{z \in \mathbf{C}: \tfrac{1}{2}\pi - \vartheta_i \leq \arg z \leq \tfrac{1}{2}\pi + \vartheta_i\} \cup \{0\}$$

for any $\vartheta_1, \vartheta_2 < \vartheta$. Let us finally notice that both Examples 3.2 and 3.3 can be easily adapted to $\Omega = \Pi_+ - i\delta$, $\Omega = \Gamma_\vartheta - i\delta$.

Let us now consider, under the hypotheses which we just mentioned, the convolution equations

$$\mu_1 * f = 0, \mu_2 * f = 0, \mu * f = 0,$$

defined for $f \in H(\Omega)$, and denote, respectively, by W_1, W_2, W the closed subspaces of their solutions in $H(\Omega)$. As it is well known, $W_i^\perp \subseteq H'(\Omega)$, $i = 1, 2$, is the set of all functionals vanishing on W_i. Since $H(\Omega)$ is a Frechét space, the space W_i' is the quotient space $W_i' = H'(\Omega)/W_i^\perp$ (Dieudonné-Schwartz [3]). We observe that $(\widehat{H'(\Omega)W^\perp})$ is isomorphic to $\widehat{H'(\Omega)}/\hat{W}_i^\perp$ and moreover it is straightforward to verify that

$$W_i^\perp = \overline{(\mu_i *)'(H'(\Omega))}.$$

It follows that if $\hat{\mu}_i$ is of completely regular growth, then

$$\hat{W}_i^\perp = \hat{\mu}_i \widehat{H'(\Omega)}$$

since in this case $\hat{\mu}_i \widehat{H'(\Omega)}$ is closed.

Remark 3.3. It is easy to see that $\mu_2 *$ is a linear and continuous operator from W_1 to W_1. Hence, $\mu_2 *$ induces an operator on the quotient space

$\widehat{H'(\Omega)}/\hat{W}_1^{\perp}$ isomorphic to $\hat{W}_1'$, namely

$$\tilde{T}_{\hat{\mu}_2}\colon \hat{W}_1' \to \hat{W}_1'$$

defined by

$$\tilde{T}_{\hat{\mu}_2}([\hat{\alpha}]) := [\hat{\mu}_2 \cdot \hat{\alpha}].$$

The image of this operator is therefore

$$\operatorname{Im}\left(\tilde{T}_{\hat{\mu}_2}\right) = \hat{W}_2^{\perp}/\hat{W}_1^{\perp}.$$

Then, from the definition of the topology of quotient spaces, it follows immediately that $\operatorname{Im}(\tilde{T}_{\hat{\mu}_2})$ is closed in $\widehat{H'(\Omega)}/\hat{W}_1^{\perp}$ if and only if $\hat{W}_1^{\perp} + \hat{W}_2^{\perp}$ is closed in $\widehat{H'(\Omega)}$.

We provide now the main results of this section.

THEOREM 3.1. *Let $\hat{\mu}_2$ be a function of completely regular growth. Then every solution $f \in H(\Omega)$ of equation $\mu * f = 0$ has the representation $f = f_1 + f_2$ where f_i is a solution of equation $\mu_i * f = 0$, $i = 1, 2$ if and only if* $\operatorname{Im}(\mu_2 * |_{W_1}) = W_1$.

Proof. Let $g \in W_1$. From the surjectivity of $\mu_2 *$ it follows that there exists $h \in H(\Omega)$ such that $\mu_2 * h = g$. We see that $h \in W$. Indeed,

$$\mu * h = (\mu_1 * \mu_2) * h = \mu_1 * (\mu_2 * h) = \mu_1 * g = 0.$$

From the hypotheses it follows that $h = h_1 + h_2$, $h_i \in W_i$, hence

$$g = \mu_2 * h = \mu_2 * (h_1 + h_2) = \mu_2 * h_1.$$

Conversely, let $f \in W$. Then $\mu * f = 0$, i.e., $(\mu_1 * \mu_2) * f = 0$. Let $g := \mu_2 * f$, so that we have $g \in W_1$. Then, from the hypothesis, there exists $f_1 \in W_1$ such that $\mu_2 * f_1 = g$, i.e., $\mu_2 * (f - f_1) = 0$, from which $f - f_1 =: f_2 \in W_2$. ∎

We now look for some alternate conditions which will improve the factorization result, and which follow from functional analytic arguments.

PROPOSITION 3.1. $\operatorname{Im}(\mu_2 * |_{W_1}) = W_1$ *if and only if* $\operatorname{Im}(\mu_2 * |_{W_1})$ *is closed in W_1.*

Proof. Let $\operatorname{Im}(\mu_2 * |_{W_1})$ be closed in W_1. Let E be the linear span of the space of solutions of the equation $\mu_1 * f = 0$ which are of the form $z^k e^{bz}$.

We know [5] that E is dense in W_1. By using standard arguments one can prove that there exists a solution $h \in E$ of the equation $\mu_2 * h = g$ for every $g \in E$. From this and from the assumption that $\operatorname{Im}(\mu_2 * |_{W_1})$ is closed in W_1, it follows that the equation $\mu_2 * h = g$ has a solution h in W_1 for every g in W_1. ■

PROPOSITION 3.2. *$\operatorname{Im}(\mu_2 * |_{W_1})$ is closed in W_1 if and only if $\hat{W}_1^{\perp} + \hat{W}_2^{\perp}$ is closed in $\widehat{H'(\Omega)}$.*

Proof. As we have outlined in Remark 3.3, $\hat{W}_1^{\perp} + \hat{W}_2^{\perp}$ is closed in $\widehat{H'(\Omega)}$ if and only if $\operatorname{Im}(\hat{T}_{\hat{\mu}_2})$ is closed in W_1. Moreover, $\operatorname{Im}(\tilde{T}_{\hat{\mu}_2})$ is closed in W_1 if and only if $\operatorname{Im}((\mu_2 *)')$ is closed in $H'(\Omega)/W_1^{\perp}$ since the operator $\tilde{T}_{\hat{\mu}_2}$ is generated by the operator $(\mu_2 *)'$ via Fourier-Borel transform, (Dieudonné-Schwartz, [3]). Finally, $\operatorname{Im}((\mu_2 *)')$ is closed in $H'(\Omega)/W_1^{\perp}$ if and only if $\operatorname{Im}(\mu_2 * |_{W_1})$ is closed in W_1. ■

As it is well known, one can deduce from the Spectral Synthesis Theorem in $\widehat{H'(\Omega)}$, that

$$\hat{\mu}_1\widehat{H'(\Omega)} + \hat{\mu}_2\widehat{H'(\Omega)} \text{ is closed in } \widehat{H'(\Omega)}$$

if and only if

$$\hat{\mu}_1\widehat{H'(\Omega)} + \hat{\mu}_2\widehat{H'(\Omega)} = \widehat{H'(\Omega)}$$

We therefore return to the study of the ideal generated by $(\hat{\mu}_1, \hat{\mu}_2)$ in $\widehat{H'(\Omega)}$:

PROPOSITION 3.3. *Suppose that $0 \in \Omega$. If $\hat{\mu}_1\widehat{H'(\Omega)} + \hat{\mu}_2\widehat{H'(\Omega)} = \widehat{H'(\Omega)}$ then there exist a compact set $T \subset \Omega$ and $A > 0$ such that*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z)) \quad \textit{for every } z \in \mathbf{C}.$$

Proof. From the hypotheses, it follows that $1 = \hat{\mu}_1 f_1 + \hat{\mu}_2 f_2$, for some $f_1, f_2 \in \widehat{H'(\Omega)}$. Hence

$$\begin{aligned} 1 &\leq |\hat{\mu}_1|\,|f_1| + |\hat{\mu}_2|\,|f_2| \leq |\hat{\mu}_1| A_1 \exp(H_{T_1}) + |\hat{\mu}_2| A_2 \exp(H_{T_2}) \\ &\leq \max(A_1, A_2)\exp(H_{T_1 \cup T_2})\big(|\hat{\mu}_1| + |\hat{\mu}_2|\big) \end{aligned}$$

from which the proposition follows. ■

Remark 3.4. The reader will notice that this result gives a partial converse of Theorem 2.1; it is however interesting to notice what happens when Ω does not contain the origin, which is a case of frequent interest.

THEOREM 3.2. *Let $\Omega \subseteq \mathbf{C}$ be an open convex set such that $\Omega + \Omega \subseteq \Omega$, and let $K \subseteq \Omega$ be a convex compact set. Let $\mu_1, \mu_2 \in H'(\mathbf{C})$ be carried by K. Then if*

$$\hat{\mu}_1 \widehat{H'(\Omega)} + \hat{\mu}_2 \widehat{H'(\Omega)} = \widehat{H'(\Omega)}$$

there exist a compact $T \subset \Omega$, and positive constants A, B such that, on $\mathbf{C}$

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z) - B|z|).$$

Conversely, if

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z))$$

then

$$\hat{\mu}_1 \widehat{H'}(\Omega) + \hat{\mu}_2 \widehat{H'}(\Omega) = \widehat{H'}(\Omega).$$

Proof. First we notice that if $\Omega + \Omega \subseteq \Omega$ and if $0 \in \Omega$, then $\Omega = \mathbf{C}$. Since the situation for $\Omega = \mathbf{C}$ is well known, we can assume $\Omega \neq \mathbf{C}$ and therefore $0 \notin \Omega$. Let therefore $g \in \widehat{H'(\Omega)}$ be a function with no zeroes. Then, for some compact $S \subset \Omega$, and some positive constants A, B,

$$|g(z)| \leq A \exp(H_S(z)) \leq A \exp B|z|.$$

One can therefore apply the minimum modulus theorem to prove that, for some positive constants C, D,

$$|g(z)| \geq C \exp(-D|z|).$$

By noticing that, for some $f_1, f_2 \in \widehat{H'(\Omega)}$, we have

$$g = f_1 \hat{\mu}_1 + f_2 \hat{\mu}_2,$$

we immediately get the result as in Proposition 3.3. To prove the second part of this theorem, on the other hand, it suffices to follow the arguments of the proof of Theorem 2.1, with $g_i = g \cdot \overline{\hat{\mu}}_i / M^2$. ■

We can therefore state and prove the following conclusion.

THEOREM 3.3. *Suppose $\hat{\mu}_1, \hat{\mu}_2$ are of completely regular growth and suppose that every solution $f \in H(\Omega)$ of the equation $\mu * f = 0$ has the representation $f = f_1 + f_2$ with $\mu_i * f = 0$, $i = 1, 2$. Then:*

(i) *If $0 \in \Omega$, there exist a compact set $T \subset \Omega$ and $A > 0$ such that, on* $\mathbf{C}$,

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z));$$

(ii) *If $\Omega + \Omega \subseteq \Omega$, $0 \notin \Omega$, there exist a compact set $T \subset \Omega$ and $A, B > 0$ such that, on* $\mathbf{C}$,

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z) - B|z|).$$

Proof. The result follows immediately from the previous propositions and Theorems 3.1, 3.2 since from the hypothesis that $\hat{\mu}_1, \hat{\mu}_2$ are of completely regular growth, we have $\hat{W}_i^{\perp} = \widehat{\mu_i H'(\Omega)}$. ■

As a partial converse we have:

THEOREM 3.4. *Suppose $\hat{\mu}_1, \hat{\mu}_2$ are of completely regular growth. Then:*

(i) *If $0 \in \Omega$ and, for some compact $T \subset \Omega$ such that $4T \subset \Omega$, and some $A > 0$,*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z)), \quad z \in \mathbf{C},$$

then every solution $f \in H(\Omega)$ of $\mu * f = 0$ factors as

$$f = f_1 + f_2, \qquad f_i \in H(\Omega), \qquad \mu_i * f_i = 0.$$

(ii) *If $0 \notin \Omega$ but $\Omega + \Omega \subseteq \Omega$ and, for some compact $T \subset \Omega$ and some $A, B > 0$,*

$$|\hat{\mu}_1(z)| + |\hat{\mu}_2(z)| \geq A \exp(-H_T(z) - B|z|),$$

then every solution $f \in H(\Omega)$ of $\mu * f = 0$ factors as before.

Proof. Once again, this is a consequence of the Propositions of this section, as well as of Theorems 2.1 and 3.2. ■

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University of Calabria
Calabria, Italy
George Mason University
Fairfax, Virginia