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CLOSED-FORM SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS VIA QUASI-SOLUTIONS I

BY

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Dedicated to the fond memory of Allen Shields

1. Introduction

For many PDE's, closed form (or *explicit*) solutions are so hard to come by that *any* examples are valuable in themselves. This paper expounds a new method that finds closed-form solutions for several non-linear PDE's, including the Klein-Gordon, eikonal and (non-parametric) minimal surface equations. In Part II, to be published separately, the method will be used to get some new results on separation of variables in some of the PDE's of mathematical physics.

In principle, the method applies to *any* PDE, but requires some luck or special ingenuity in practice. Symbolic computation on electronic computers has been a big help with the often lengthy and complicated computations.

Two established methods of obtaining closed form solutions are the symmetry method of Sophus Lie et al. (see [DRE]), and the method of inverse scattering, of Kruskal, Lax, et al. (See [ZAS]).

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The method is to find so-called quasi-solutions of a given equation P. They are shown to be actual solutions of a higher-order equation (or system) P^*. Precisely because P^* has higher order, it is easier to find solutions of P^* than of P. (Indeed, every solution of P is already a solution of P^*.) The quasi-solutions are found by luck and guesswork—often by trying functions of a particularly simple form. From these quasi-solutions, one goes back to find actual solutions, by solving some ODE's. Here is a simple example.

Suppose we want to find explicit solutions to Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{1.1}$$

in the (x, y) plane. We write $u = \varphi(v)$ (at least locally) and call v a quasi-solution of (1.1). That is, v is functionally dependent on some harmonic function. We then say that v is quasi-harmonic. Another way to describe this is to say that the level lines of v are the same as the level lines of some harmonic function u (but the scaling of the function-values is different). Since

$$\begin{aligned} &u_x = \varphi'(v)v_x, \qquad u_y = \varphi'(v)v_y, \\ &u_{xx} = \varphi''(v)v_x^2 + \varphi'(v)v_{xx}, \qquad u_{yy} = \varphi''(v)v_y^2 + \varphi'(v)v_{yy}, \end{aligned} \tag{1.2}$$

we get

$$\varphi'(v)(v_{xx} + v_{yy}) + \varphi''(v)\left(v_x^2 + v_y^2\right) = 0, \tag{1.3}$$

$$\frac{\varphi''(v)}{\varphi'(v)} = -\frac{v_{xx} + v_{yy}}{v_x^2 + v_y^2} := K(x, y). \tag{1.4}$$

Hence K is a function (at least locally) of v, which leads to the necessary and sufficient condition

$$J\left(v, \frac{v_{xx} + v_{yy}}{v_x^2 + v_y^2}\right) = 0, \tag{1.5}$$

where J is the Jacobian determinant. This equation reduces to

$$\begin{aligned} &\left(v_x^2 + v_y^2\right)\left[(v_{xxx} + v_{xyy})v_y - (v_{xxy} + v_{yyy})v_x\right] \\ &\quad -2(v_{xx} + v_{yy})\left[(v_xv_{xx} + v_yv_{xy})v_y - (v_xv_{xy} + v_yv_{yy})v_x\right] = 0. \end{aligned} \tag{1.6}$$

In summary, for v to be a quasi-solution of (1.1) it is necessary and sufficient that it satisfy (1.6). Now, by hook or by crook, find a solution v of (1.6). (For example, by searching through low-degree polynomials in (x, y), we find that $v = x^2 + y^2$ works.) For this v, compute K and solve the ODE (1.4) for φ, which is easy enough. (In the above example, $v = x^2 + y^2$, we arrive at $u = a\log(x^2 + y^2) + b$, where a and b are any constants. Not exactly a new function, but a good illustration of the method.)

In Part II of this paper, to be published separately, these ideas will be used to obtain new results on separation of variables in some of the PDE's of classical mathematical physics.

For more difficult (and particularly non-linear) PDE's, the same ideas can be carried out, albeit with some modifications. The rest of this paper is mainly concerned with three examples of this, namely the Klein-Gordon equation

$$u_{xt} = f(u), \tag{1.7}$$

the p-eikonal equation

$$u_x^p + u_y^p = 1 \tag{1.8}$$

(the case $p = 2$ is the usual eikonal equation) and the non-parametric minimal surface equation

$$\left(1 + u_y^2\right)u_{xx} - 2u_x u_y u_{xy} + \left(1 + u_x^2\right)u_{yy} = 0. \tag{1.9}$$

2. The Klein-Gordon equation and the p-eikonal equation

The Klein-Gordon equation, for $u = u(x, t)$, is

$$u_{xt} = f(u). \tag{2.1}$$

The case $f(u) = \sin u$ is the Sine-Gordon equation—its solutions represent surfaces of constant negative Gaussian curvature in $\mathbf{R}^3$. We now describe all solutions u of (2.1) having the form $u(x, t) = \varphi(A(x) + B(t))$. This is a model for our method, because the computations are so simple in this case.

THEOREM 2.1. *Let $\varphi(s)$ satisfy the ordinary differential equation*

$$\varphi''(s) = ke^{\lambda s} f(\varphi(s)) \tag{2.2}$$

where $\lambda \neq 0$ is a constant. Then

$$A(x) = \frac{1}{\lambda}\log(\lambda x + a) + \gamma, \tag{2.3}$$

$$B(t) = \frac{1}{\lambda}\log(\lambda t + \beta) + \delta \tag{2.4}$$

where α, β, γ, δ are constants such that

$$k = \exp[-\lambda(\gamma + \delta)]. \tag{2.5}$$

Then

$$u = \varphi(A(x) + B(t)) \tag{2.6}$$

satisfies the Klein-Gordon equation (2.1). *Conversely, if $A(x)$ and $B(t)$ are any real-analytic functions for which there exists a real-analytic function φ of one variable (assuming that none of φ'', A'', and B'' vanishes identically), then there must be a constant $k \neq 0$ such that φ satisfies* (2.2) *for some constant $\lambda \neq 0$, and $A(x)$ and $B(t)$ must be given by* (2.3) *and* (2.4), *for some constants α, β, γ, δ for which* (2.5) *holds.*

Proof. We first prove the "conversely" part. All arguments of $\varphi, \varphi', \varphi'', \ldots$ are supposed to be $A(x) + B(t)$, all arguments of $A, A', A'', \ldots$ are supposed to be x, and all arguments of $B, B', B'', \ldots$ are supposed to be t. We have

$$u_x = \varphi' A', \quad u_{xt} = \varphi'' A' B', \tag{2.7}$$

so that (2.1) and (2.6) give

$$\varphi'' A' B' = f(\varphi) \tag{2.8}$$

or

$$A' B' = f(\varphi)/\varphi''. \tag{2.9}$$

Thus, $A'B'$ is functionally dependent on $A + B$, so that

$$J(A'B', A + B) = 0 \tag{2.10}$$

where J is the Jacobian determinant

$$J(v, w) = \begin{vmatrix} v_x & v_t \\ w_x & w_t \end{vmatrix}. \tag{2.11}$$

Evaluating (2.10) we get

$$A'' B'^2 - A'^2 B'' = 0, \tag{2.12}$$

and hence

$$A''/A'^2 = B''/B'^2. \tag{2.13}$$

Since we have a function of x equal to a function of t,

$$A''/A'^2 = B''/B'^2 = \mu \tag{2.14}$$

where μ is a constant, with $\mu \neq 0$ because of our hypothesis about A'' not vanishing. We have

$$A' = -\frac{1}{\mu x + a} \tag{2.15}$$

which we write as

$$A' = \frac{1}{\lambda x + \alpha}, \tag{2.16}$$

and hence (2.3) follows. In the same way, we get (2.4). Also

(2.17)

$$\varphi''(A + B)/f(\varphi(A + B)) = \frac{1}{(\lambda x + \alpha)(\lambda t + \beta)} = k \exp\{\lambda(A + B)\}.$$

(2.18) Writing $s = A + B$,

we have

$$\varphi''(s)/f(\varphi(s)) = k \exp \lambda s, \tag{2.19}$$

which is equivalent to (2.2).

In the other direction, let A, B, and φ satisfy (2.3), (2.4), and (2.2), and let (2.5) hold. Then by (2.7), with $u = \varphi(A + B)$,

$$u_{xt} = \varphi'' A'B', \tag{2.20}$$

$$u_{xt} = ke^{\lambda(A+B)} f(\varphi(A + B))A'B' = f(\varphi(A + B)) = f(u), \tag{2.21}$$

and the theorem is proved.

The p-eikonal equation is

$$u_x^p + u_y^p = 1. \tag{2.22}$$

Note that the case $p = 2$ is the eikonal equation (see [GAR]). Pursuing the method of quasi-solutions, we put

$$u = \varphi(v) \tag{2.23}$$

to get

$$\varphi'(v)^p[v_x^p + v_y^p] = 1, \tag{2.24}$$

which implies that $v_x^p + v_y^p$ is a function of v, so that, taking the Jacobian, we have

$$J = \begin{vmatrix} (v_x^p + v_y^p)_x & (v_x^p + v_y^p)_y \\ v_x & v_y \end{vmatrix} = 0. \tag{2.25}$$

We try a multiplicative separation of variables

$$v = A(x)B(y). \tag{2.26}$$

(Note that in [GAR], for the eikonal equation, an *additive* separation of variables in the *original* equation was used.) Then (2.25) becomes

$$\begin{vmatrix} A'^{p-1}A''B^p + A^{p-1}A'B'^p & A'^pB^{p-1}B' + A^pB'^{p-1}B'' \\ A'B & AB' \end{vmatrix} = 0. \tag{2.27}$$

We then have

$$\frac{A'^{p-2}A''}{A^{p-1}} - \frac{A'^p}{A^p} = \frac{B'^{p-2}B''}{B^{p-1}} - \frac{B'^p}{B^p}. \tag{2.28}$$

Since the left-hand side of (2.28) is a function of x alone, and the right-hand side a function of y alone, we have

$$\frac{A'^{p-2}A''}{A^{p-1}} - \frac{A'^p}{A^p} = \lambda \tag{2.29}$$

$$\frac{B'^{p-2}B''}{B^{p-1}} - \frac{B'^p}{B^p} = \lambda, \tag{2.30}$$

for some constant λ (the *same* constant in (2.29) and (2.30)). Now (2.29) (and (2.30)) can be integrated explicitly. Rewrite (2.29) as

$$\left(\frac{A'}{A}\right)^{p-2} \frac{A''A - A'^2}{A^2} = \lambda, \tag{2.31}$$

and set

$$\theta = \frac{A'}{A}, \qquad \theta' = \frac{A''A - A'^2}{A^2}. \tag{2.32}$$

Then (2.31) becomes

$$\theta^{p-2}\theta' = \lambda. \tag{2.33}$$

Skipping some steps, we get

$$A = k \exp\left(\frac{(\bar{\lambda} x + \bar{c})^{p/(p-1)}}{\bar{\lambda} p (p-1)^{p/(1-p)}} \right), \tag{2.34}$$

$$B = \bar{k} \exp\left(\frac{(\bar{\lambda} y + \bar{\bar{c}})^{p/(p-1)}}{\bar{\lambda} p (p-1)^{p/(1-p)}} \right), \tag{2.35}$$

where $k, \bar{k}, \bar{\lambda}, \ldots, \bar{\bar{c}}$ are constants. From (2.24), we get

$$\varphi'(v)^p [A'^p B^p + A^p B'^p] = 1. \tag{2.36}$$

Now

$$A'^p B^p = A^p B^p (\bar{\lambda} x + \bar{c})^{p/(p-1)} (p-1)^{p/(p-1)}, \tag{2.37}$$

$$A^p B'^p = A^p B^p (\bar{\lambda} y + \bar{\bar{c}})^{p/(p-1)} (p-1)^{p/(p-1)} \tag{2.38}$$

and

$$\log AB = \log v = \frac{(\bar{\lambda} x + \bar{c})^{p/(p-1)} + (\bar{\lambda} y + \bar{\bar{c}})^{p/(p-1)}}{\bar{\lambda} p (p-1)^{p/(1-p)}} + \log k\bar{k}. \tag{2.39}$$

Hence

(2.40)

$$(\bar{\lambda} x + \bar{c})^{p/(p-1)} + (\bar{\lambda} y + \bar{\bar{c}})^{p/(p-1)} = \bar{\lambda} p (p-1)^{p/(1-p)} [\log v - \log k\bar{k}],$$

from which we get

$$\varphi'(v) = \frac{1}{(\bar{\lambda} p)^{1/p} [\log v - (\log k\bar{k})]^{1/p}} \frac{1}{v} \tag{2.41}$$

and therefore

$$\varphi(v) = \frac{1}{\bar{\lambda}} \left\{ (\bar{\lambda} x + c)^{p/(p-1)} + (\bar{\lambda} y + \bar{\bar{c}})^{p/(p-1)} \right\}^{(p-1)/p}, \tag{2.42}$$

which can finally be rewritten as

(2.43) $$u = \varphi(v) = \left[(x - x_0)^{p/(p-1)} + (y - y_0)^{p/(p-1)}\right]^{(p-1)/p} + c.$$

It is easy enough to check that (2.43) is indeed a solution of (2.22).

3. The minimal surface equation

We consider the minimal surface equation (non-parametric form)

(3.1) $$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.$$

For some historical background on this equation, see [BAR], [DAR, First Part, Book III, Chapter I], and [OSS].

For a (real-analytic) function $v = v(x, y)$, let

(3.2) $$\begin{aligned} R &= v_x^2 + v_y^2 \\ S &= v_{xx} + v_{yy} \\ T &= v_{xx}v_y^2 - 2v_x v_y v_{xy} + v_{yy}v_x^2. \end{aligned}$$

We suppose that $v_x^2 + v_y^2$ never vanishes (locally). If we take $u = \varphi(v)$, where φ is real-analytic, we get

(3.3)
$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = \varphi''(v)R + \varphi'(v)S + \varphi'^3(v)T.$$

Let $w = w(x, y)$ be some (locally) real-analytic function, not constant, that is independent of v in the sense that

(3.4) $$(\operatorname{grad} v) \cdot (\operatorname{grad} w) = 0.$$

For a differentiable function F, define

(3.5) $$F^{\#} = \begin{vmatrix} F_x & F_y \\ v_x & v_y \end{vmatrix} = J(F, v).$$

Then, aside from the factor

(3.6) $$\Delta = \begin{vmatrix} w_x & w_y \\ v_x & v_y \end{vmatrix},$$

which is locally, non-vanishing, $F^{\#}$ is the same as $\partial F/\partial w$.

Let W be the Wronskian matrix

$$W = \begin{bmatrix} R & S & T \\ R_w & S_w & T_w \\ R_{ww} & S_{ww} & T_{ww} \end{bmatrix}. \tag{3.7}$$

Let $W^{\#}$ be the matrix

$$W^{\#} = \begin{bmatrix} R & S & T \\ R^{\#} & S^{\#} & T^{\#} \\ R^{\#\#} & S^{\#\#} & T^{\#\#} \end{bmatrix}. \tag{3.8}$$

We note that, as long as $\Delta \neq 0$, W is singular exactly where $W^{\#}$ is singular.

The next theorem gives a necessary and sufficient condition that a function v be a quasi-solution of the minimal surface equation. In case it *is* a quasi-solution, a method is provided for finding the associated solution $u = \varphi(v)$

THEOREM 1. *Let $v(x, y)$ be a non-constant real-analytic function. With the preceding notation, suppose that*

$$\det W^{\#} = 0. \tag{3.9}$$

Now $W^{\#}$ has rank 2 if and only if one or more of the cases I–VI *below holds.* (*Actually, since $R \neq 0$, cases* III *and* IV *form a complete list for rank* 2 *already, but the other cases might be more amenable to computation.*) *Also, $W^{\#}$ has rank* 1 *if and only if case* VII *holds. Then in each of the first six cases, A, B, and C are functions of v alone, and we write them as $A[v]$, $B[v]$, and $C[v]$. Moreover,*

$$AR + BS + CT = 0. \tag{3.10}$$

If v is a quasi-solution of (3.1), *then* (3.9) *holds, and in cases* I–VI, *we have*

$$2AC = B\frac{dC}{dv} - C\frac{dB}{dv} \tag{3.11}$$

and

$$C/B > 0. \tag{3.12}$$

In these cases I–VI, $u = \varphi(v)$ *is an actual solution of* (3.1), *where*

$$\varphi(z) = \pm \int \sqrt{C[z]/B[z]}\, dz. \tag{3.13}$$

Conversely, if v is any function that satisfies (3.9), (3.11), *and* (3.12) (*in any one of cases* (I–VI)), *then v must be a quasi-solution of* (3.1).

In case VII, v *must be a quasi-solution of* (3.1) *and* $u = \varphi(v)$ *is an actual solution of* (3.1), *where*

$$(3.14)\quad \varphi = \int\left\{\exp\left(2\int\frac{S}{R}[z]\right)\left[2\int\left(\frac{T}{R}[z]\exp\left(-2\int\frac{S}{R}[z]\right)\right)+a\right]^{-1/2}\right\}+b,$$

where a and b are any constants. (*Note that in this case, both* S/R *and* T/R *are functions of* v *alone.*)

The seven cases are:

(I) $S \neq 0$ *and* $(T/S)^{\#} \neq 0$.

(II) $S \neq 0$ *and* $(R/S)^{\#} \neq 0$.

(III) $R \neq 0$ *and* $(S/R)^{\#} \neq 0$.

(IV) $R \neq 0$ *and* $(T/R)^{\#} \neq 0$.

(V) $T \neq 0$ *and* $(R/T)^{\#} \neq 0$.

(VI) $T \neq 0$ *and* $(S/T)^{\#} \neq 0$.

(VII) $R \neq 0$ *and* $(S/R)^{\#} = 0$ *and* $(T/R)^{\#} = 0$.

The quantities A, B, *and* C *are defined in the corresponding cases* I–VI *by*:

(I*) $\quad A = -1, \quad B = \dfrac{R}{S} - \dfrac{(R/S)^{\#}}{(T/S)^{\#}}\dfrac{T}{S}, \quad C = \dfrac{(R/S)^{\#}}{(T/S)^{\#}}.$

(II*) $\quad A = \dfrac{(T/S)^{\#}}{(R/S)^{\#}}, \quad B = \dfrac{T}{S} - \dfrac{(T/S)^{\#}}{(R/S)^{\#}}\dfrac{R}{S}, \quad C = -1.$

(III*) $\quad A = \dfrac{(T/R)^{\#}}{(S/R)^{\#}}, \quad B = \dfrac{T}{R} - \dfrac{(T/R)^{\#}}{(S/R)^{\#}}\dfrac{S}{R}, \quad C = -1.$

(IV*) $\quad A = \dfrac{S}{R} - \dfrac{(S/R)^{\#}}{(T/R)^{\#}}\dfrac{T}{R}, \quad B = -1 \quad C = \dfrac{(S/R)^{\#}}{(T/R)^{\#}}.$

(V*) $\quad A = \dfrac{(S/T)^{\#}}{(R/T)^{\#}}, \quad B = -1, \quad C = \dfrac{S}{T} - \dfrac{(S/T)^{\#}}{(R/T)^{\#}}\dfrac{R}{T}.$

(VI*) $\quad A = -1, \quad B = \dfrac{(R/T)^{\#}}{(S/T)^{\#}}, \quad C = \dfrac{R}{T} - \dfrac{(R/T)^{\#}}{(S/T)^{\#}}\dfrac{S}{T}.$

Proof of Theorem 1. Let us call case a the union of cases I–VI and let case b be case VII. First we observe that cases a and b are exhaustive and mutually exclusive. Now in case b, we have rank $W^{\#} = 1$, since then $S/R = \alpha(v)$ and $T/R = \beta(v)$ because

$$(S/R)^{\#} = 0 \Leftrightarrow \frac{\partial}{\partial w}(S/R) = 0, \qquad (T/R)^{\#} = 0 \Leftrightarrow \frac{\partial}{\partial w}(T/R) = 0.$$

To show that in case a, rank $W^{\#} = 2$, let us suppose, for example, that we are in case V, so that $T \neq 0$ and $(R/T)^{\#} \neq 0$. But $W^{\#}$ contains the 2×2 submatrix

$$\begin{bmatrix} R & T \\ R^{\#} & T^{\#} \end{bmatrix},$$

which is non-singular because $(R/T)^{\#} \neq 0$.

Of cases I–VI, we will do only case V in detail. The others of these cases go the same way. From (3.9), we get

$$\begin{vmatrix} R & S & T \\ R_w & S_w & T_w \\ R_{ww} & S_{ww} & T_{ww} \end{vmatrix} = 0. \tag{3.15}$$

Since $T \neq 0$, we can divide by T and use the proof of [STO, Lemma 6.3.1] to get

$$\begin{vmatrix} R/T & S/T & 1 \\ (R/T)_w & (S/T)_w & 0 \\ (R/T)_{ww} & (S/T)_{ww} & 0 \end{vmatrix} = 0, \tag{3.16}$$

or

$$(R/T)_w(S/T)_{ww} - (R/T)_{ww}(S/T)_w = 0, \tag{3.17}$$

which is equivalent to

$$\frac{\partial}{\partial w}\left(\frac{(S/T)_w}{(R/T)_w}\right) \neq 0. \tag{3.18}$$

(Since we are in case V, division by $(R/T)_w$ is possible.) Hence

$$\frac{(S/T)_w}{(R/T)_w} = \alpha(v)$$

is a function of v alone. Consequently

$$\frac{S}{T} = \alpha(v)\frac{R}{T} + \beta(v), \tag{3.19}$$

where $\beta(v)$ is a function of v alone. We have

$$\beta(v) = \frac{S}{T} - \frac{(S/T)_w}{(R/T)_w}\frac{R}{T}. \tag{3.20}$$

$$\frac{(S/T)_w}{(R/T)_w} = \frac{(S/T)^{\#}}{(R/T)^{\#}}, \tag{3.21}$$

and so, from (3.19), we have

$$\frac{(S/T)^{\#}}{(R/T)^{\#}}R - S + \left[\frac{S}{T} - \frac{(S/T)^{\#}}{(R/T)^{\#}}\frac{R}{T}\right]T = 0. \tag{3.22}$$

This is obvious, of course, but it is not obvious that the coefficients of R, S, T in (3.22) depend on v alone. Note that these coefficients are just the A, B, C of case V*.

Now suppose that v is a quasi-solution of (3.1). This means that $u = \varphi(v)$ is an actual solution of (3.1), for some function φ. From (3.3), this is equivalent to

$$\varphi''(v)R + \varphi'(v)S + \varphi'(v)^3 T = 0. \tag{3.23}$$

Compare (3.23) and (3.22). Let $\mathscr{G}$ be the field of all quotients of real-analytic functions of x and y (i.e., of v and w), let $\mathscr{F}$ be the subfield of those functions in $\mathscr{G}$ that depend on v alone, and let $V = \mathscr{G}^3$ be the vector space over $\mathscr{G}$ of all triples $\langle L, M, N\rangle$ of functions L, M, N in $\mathscr{G}$. Consider the vector $\langle R, S, T\rangle$ in V. Then (3.22) says that the vector $\langle A[v], B[v], C[v]\rangle$ is orthogonal to $\langle R, S, T\rangle$, and (3.23) says that the vector $\langle \varphi''[v], \varphi'[v], \varphi'[v]^3\rangle$ is orthogonal to $\langle R, S, T\rangle$. Suppose there were two linearly independent vectors $\langle A_1(v), B_1(v), C_1(v)\rangle$ and $\langle A_2(v), B_2(v), C_2(v)\rangle$ which were orthogonal to $\langle R, S, T\rangle$. Then we would have

$$A_1(v)R(v,w) + B_1(v)S(v,w) + C_1(v)T(v,w) = 0,$$
$$A_2(v)R(v,w) + B_2(v)S(v,w) + C_2(v)T(v,w) = 0.$$

If B_1 or B_2 is zero, then $(R/T)^{\#}$ would be zero because R/T would be a function of v alone. If B_1 or B_2 is not zero, then we may cross-multiply by B_2, B_1 and eliminate the S-term above to see that $(R/T)^{\#}$ would again be 0. But we are in case V where $(R/T)^{\#} \neq 0$ by hypothesis. Therefore $\langle A, B, C\rangle$ and $\langle \varphi''[v], \varphi'[v], \varphi'^3[v]\rangle$ are linearly dependent over $\mathscr{F}$. This means that there exists a function $\lambda(v) \in \mathscr{F}$ such that

(3.24)

$$\varphi''(v) = \lambda(v)A[v], \qquad \varphi'(v) = \lambda(v)B[v], \qquad \varphi'^3(v) = \lambda(v)C[v],$$

from which we get

$$\frac{\varphi''(v)}{\varphi'(v)} = \frac{A[v]}{B[v]} \tag{3.25}$$

and

$$\varphi'^2(v) = \frac{C[v]}{B[v]}. \tag{3.26}$$

Since (aside from the trivial case) $\varphi \neq$ const, we have (3.12) and (3.13). From (3.26), we get

$$2\log|\varphi'(v)| = \log|C[v]| - \log|B[v]|,$$

and consequently

$$2\frac{\varphi''(v)}{\varphi'(v)} = \frac{C'[v]}{C[v]} - \frac{B'[v]}{B[v]}. \tag{3.27}$$

Comparing this with (3.25), we get (3.11). This proves the first (direct) part of Theorem 1, in cases I–VI.

Let us now prove the converse part in cases I–VI. Suppose that we are in case V and that v is a function that satisfies (3.9), (3.11), and (3.12). As above, we have (3.10), where A, B, C are given by V*, and we know that A, B, C are functions of v alone. Let $\varphi(z)$ be defined by (3.13). We must prove that $u = \varphi(z)$ is a solution of (3.1). Now $\varphi'^3/\varphi' = C/B$ and $\varphi''/\varphi' = A/B$, from (3.11) and (3.13). Thus

(3.28)
$$\varphi''R + \varphi' S + \varphi'^3 T = \frac{1}{\varphi'}\left(\frac{A}{B}R + S + \frac{C}{B}T\right) = \frac{1}{B\varphi'}(AR + BS + CT) = 0.$$

But by (3.3), this is equivalent to $u = \varphi(v)$ being a solution of (3.1), and the theorem is proved in cases I–VI.

In case VII, we have $R \neq 0$ and $(S/R)^\# = 0$ and $(T/R)^\# = 0$, so that $B = T/R$ and $C = -S/R$ are functions of v alone. (Notice that if $S \neq 0$, then the conjunction $(R/S)^\# = 0$ and $(T/S)^\# = 0$ is equivalent to the conjunction $(S/R)^\# = 0$ and $(T/R)^\# = 0$, with a similar statement in the case $T \neq 0$, so there is no point in introducing additional cases.) Recall that φ being a quasi-solution of (3.1) is equivalent to

$$R\varphi'' + S\varphi' + T\varphi'^3 = 0, \tag{3.3}$$

or

$$\varphi'' + \frac{S}{R}\varphi' + \frac{T}{R}\varphi'^3 = 0. \tag{3.29}$$

Let $\Phi = \varphi'$, so that (3.29) is equivalent to

$$\Phi' + \frac{S}{R}\Phi + \frac{T}{R}\Phi^3 = 0, \tag{3.30}$$

which is a *Bernoulli* equation. Hence, on dividing it by Φ^3, it becomes a *linear* equation in $\Psi = 1/\Phi^2$. Solving this equation and substituting back, we get (3.14).

4. Legendre transformations

It is possible to apply Legendre transformations (see [COH]) to the minimal surface equation (3.1). We refer to [BJÖ] for a detailed exposition—we summarize the relevant parts of that paper. Legendre applied a real Legendre (hodograph) transformation (see [LEG]) to get a linear equation instead of (3.1), but Björling applied a complex one to get a simpler (in some respects), linear equation, for which he was able to find the "general solution" in terms of two arbitrary functions of one variable.

Notation. For the original minimal surface equation (3.1), we still use x and y for the independent variables, but we now use z (instead of u) for the dependent variable. Then (3.1) becomes

$$\left(1 + z_y^2\right)z_{xx} - 2z_xz_yz_{xy} + \left(1 + z_x^2\right)z_{yy} = 0. \tag{4.1}$$

Now we take a real Legendre transformation, using ξ and η for the independent variables and ω for the dependent variable, with the relation

$$\begin{gathered}\omega(\xi,\eta) + z(x,y) = x\xi + y\eta,\\ \xi = z_x, \qquad x = \omega_\xi, \qquad \eta = z_y, \qquad y = \omega_\eta,\end{gathered} \tag{4.2}$$

so that, in terms of the new variables, (4.1) becomes

$$(1 + \xi^2)\omega_{\xi\xi} + 2\xi\eta\omega_{\xi\eta} + (1 + \eta^2)\omega_{\eta\eta} = 0. \tag{4.3}$$

It turns out that it is easier to find quasi-solutions of (4.3) (and hence actual solutions) than it is of (4.1). But even easier than (4.3) is the equation of Björling that arises from the following complex Legendre transformation.

We choose new independent variables α and β and a new dependent variable w as follows. First, we replace (x, y) by (u, v) where

$$(4.4)\qquad \begin{aligned} u &= x + iy, & x &= \frac{u+v}{2}, \\ v &= x - iy, & y &= \frac{u-v}{2i}. \end{aligned}$$

Then we have

$$(4.5)\qquad w(\alpha, \beta) + z(x, y) = p_1 u + q_1 v,$$

$$\alpha = \frac{p_1}{1 + \sqrt{1 + 4p_1 q_1}}, \qquad p_1 = \frac{2\alpha\beta}{\alpha + \beta}, \qquad u = \frac{\partial w}{\partial p_1},$$

$$\beta = \frac{p_1}{1 - \sqrt{1 + 4p_1 q_1}}, \qquad q_1 = -\frac{1}{2(\alpha + \beta)}, \qquad v = \frac{\partial w}{\partial q_1}.$$

Then the equation equivalent to (4.1) is

$$(4.6)\qquad w_{\alpha\beta} + \frac{2\alpha}{\alpha^2 - \beta^2} w_\alpha - \frac{2\beta}{\alpha^2 - \beta^2} w_\beta = 0.$$

Incidentally, the general solution of (4.6), as found by Björling, is

$$(4.7)\qquad w = \frac{\varphi(\alpha)}{\alpha + \beta} + \frac{2\psi(\beta)}{\alpha + \beta} - \frac{\alpha - \beta}{2(\alpha + \beta)}[\varphi'(\alpha) - 2\psi'(\beta)],$$

where φ and ψ are "arbitrary" functions of one variable. The trouble with this, as with our own methods when applied to (4.6) is that solutions of (4.6), when transformed back into solutions of (4.1), may give non-real (i.e., complex) solutions of (4.1), which have a certain rococo interest, but are not to the point.

Lászlo Lempert has suggested to the author the following device for getting real solutions to (4.1) from complex solutions to (4.6). Namely, transform (4.6) to (4.3) (in principle, by going from (4.6) to (4.1) and then from (4.1) to (4.3).) Now, (4.3) is linear, and the coefficients are real, so we may take the real (or imaginary part) $\tilde{\omega}$ of ω to get a solution $\tilde{\omega}$ of

$$(4.3')\qquad (1 + \xi^2)\tilde{\omega}_{\xi\xi} + 2\xi\eta\tilde{\omega}_{\xi\eta} + (1 + \eta^2)\tilde{\omega}_{\eta\eta} = 0.$$

Finally, we perform the transformation (4.2) to get a real solution to (4.1). In a moment, we will do this in a more direct way. What happens in practice (see the next section) is that it is easy to get many explicit (complex) solutions to (4.6), and that (at least on a computer), the procedure described above for getting real solutions of (4.1) goes smoothly, *except* at the final point where we have

$$x = x(\xi, \eta) \qquad (4.8)$$
$$y = y(\xi, \eta),$$

and we must invert to find

$$\xi = \xi(x, y) \qquad (4.9)$$
$$\eta = \eta(x, y).$$

In practice, so far, the explicit form of (4.8) is reasonably complicated, and there seems to be no practical way to actually perform the inversion to get (4.9).

Now for the details of the direct implementation of Lempert's suggestion. From (4.2) and (4.5), we get

$$w(\alpha, \beta) - \omega(\xi, \eta) = (x\xi + y\eta) - (p_1 u + q_1 v). \qquad (4.10)$$

We suppose we have an explicit expression for $w(\alpha, \beta)$. We wish to express (4.10) entirely in terms of ξ and η. Now

$$\alpha = \frac{\frac{1}{2}\xi + \frac{1}{2i}\eta}{1 + \sqrt{1 + \xi^2 + \eta^2}}, \qquad \beta = \frac{\frac{1}{2}\xi + \frac{1}{2i}\eta}{1 - \sqrt{1 + \xi^2 + \eta^2}}. \qquad (4.11)$$

To see this, we note that, from [BJÖ], $p_1 = \partial z/\partial u$, $q_1 = \partial z/\partial v$ so that

$$p_1 = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad (4.12)$$
$$q_1 = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

But, from (4.4),

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \qquad \frac{\partial y}{\partial u} = \frac{1}{2i} \qquad (4.13)$$
$$\frac{\partial x}{\partial v} = \frac{1}{2}, \qquad \frac{\partial y}{\partial v} = -\frac{1}{2i},$$

$$p_1 = \frac{1}{2}z_x + \frac{1}{2i}z_y \qquad (4.14)$$
$$q_1 = \frac{1}{2}z_x - \frac{1}{2i}z_y$$
$$p_1 q_1 = \frac{1}{4}\left(z_x^2 + z_y^2\right).$$

Now, again from [BJÖ],

$$\alpha = \frac{p_1}{1 + \sqrt{1 + 4p_1q_1}}, \qquad \beta = \frac{p_1}{1 - \sqrt{1 + 4p_1q_1}}. \tag{4.15}$$

From these formulas, (4.11) follows directly from (4.5).

Now we need x and y (or equivalently, u and v) in terms of ξ and η. By (4.11), it would be enough to get u and v in terms of α and β. But from (4.5)

$$u = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial p_1} + \frac{\partial w}{\partial \beta}\frac{\partial \beta}{\partial p_1}, \qquad v = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial q_1} + \frac{\partial w}{\partial \beta}\frac{\partial \beta}{\partial q_1}. \tag{4.16}$$

Now from (15), we compute $\partial\alpha/\partial p_1$, $\partial\alpha/\partial q_1$, $\partial\beta/\partial p_1$, $\partial\beta/\partial q_1$. The results are

$$\frac{\partial \alpha}{\partial p_1} = \frac{\left(1 + \sqrt{1 + \xi^2 + \eta^2}\right) - \frac{1}{2}(\xi^2 + \eta^2)\left(\sqrt{1 + \xi^2 + \eta^2}\right)^{-1}}{\left(1 + \sqrt{1 + \xi^2 + \eta^2}\right)^2}, \tag{4.17}$$

$$\frac{\partial \alpha}{\partial q_1} = -\frac{2}{\left(1 + \sqrt{1 + \xi^2 + \eta^2}\right)^2}\frac{1}{\sqrt{1 + \xi^2 + \eta^2}}\left(\frac{1}{2}\xi + \frac{1}{2i}\eta\right)^2, \tag{4.18}$$

$$\frac{\partial \beta}{\partial p_1} = \frac{\left(1 - \sqrt{1 + \xi^2 + \eta^2}\right) + \frac{1}{2}(\xi^2 + \eta^2)\left(\sqrt{1 + \xi^2 + \eta^2}\right)^{-1}}{\left(1 - \sqrt{1 + \xi^2 + \eta^2}\right)^2}, \tag{4.19}$$

$$\frac{\partial \beta}{\partial q_1} = \frac{2\left(\frac{1}{2}\xi + \frac{1}{2i}\eta\right)^2}{\left(1 - \sqrt{1 + \xi^2 + \eta^2}\right)^2\sqrt{1 + \xi^2 + \eta^2}}. \tag{4.20}$$

The remaining quantities on the right-hand side of (4.16) are $\partial w/\partial\alpha$ and $\partial w/\partial\beta$. But w is an explicit function of α and β, and hence so are these partial derivatives. Then we can use (4.11) to get them explicitly in terms of ξ and η. Hence, from (4.16) and the above remarks, we have u and v as explicit functions of ξ and η. Thus, we have an explicit version of (4.8). We invert (4.8) to get (4.9). Then we use (4.5) (after replacing $w(\alpha, \beta)$ by its real part $\tilde{w}(\alpha, \beta)$) to get $\tilde{z}(x, y)$ explicitly, which will be a real solution of (4.1).

We repeat what we said above about carrying out these computations in actual cases. Despite apparent complexities, everything goes well until we get (4.8). To invert (4.8) to get (4.9), though, in the cases we have tried, seems impossible in practice, at least with the presently available machines and programs.

5. Computer implementation

In this section we present a summary of what the computer did in some special cases, in implementing the methods described in §§3, 4. At the University of Illinois, this computation was done by Byoung Keum and Daniel Lee, who were then graduate students, using Mathematica. The author did some further computation on Macsyma while attending the Workshop on Symbolic Computation at the Institute for Mathematics and its Applications in Minneapolis. He takes this opportunity to thank Keum and Lee for all their help, and the IMA for the use of its facilities. The implementation was quite direct, except for the following scheme introduced by Daniel Lee for making certain functional dependencies explicit. Without some such procedure, the method would not work in practice.

In a typical situation for this paper, (see (1.4) and (1.5)) we have v as a function of x and y and K as a function of x and y and we are sure that K is a function of v because the Jacobian $J(v, K)$ vanishes identically. But we want a closed-form expression for K as a function of v. From $v =$ function(x, y), we get $y =$ function(x, v). Using this, we rewrite $K =$ function(x, y) as $K =$ function(x, v). We solve this for x to get $x =$ function(K, v). Then using this in $y =$ function(x, v) above, we get $y =$ function(K, v). Now, using both x and y as functions of K and v, we rewrite $K =$ function(x, y) above to get $K =$ function(K, v). We solve this last equation for K as a function of v. This procedure works most of the time, but other times the steps cannot be carried out in closed form.

First of all, in the direct method (without Legendre transformations) of §3, one would naturally try (even without prior knowledge) $v = x^2 + y^2$ and $v = y/x$, and they lead directly to the catenoid (see [BAC]) and the helicoid (see [BAC]), respectively. If one starts with $v = \cos y/\cos x$ (which probably requires some outside inspiration), it leads to the actual solution $u = \log(\cos y/\cos x)$, which is Scherk's surface (see [BAC]). The other choices of v that we tried just didn't satisfy the conditions of Theorem 1, so they were no good.

When we made the Legendre transformation (4.5), we found some quasi-solutions of (4.6) that led us back to the helicoid and the catenoid at the end. A recurring quasi-solution of (4.6) was β/α (occurring also as β^2/α^2, β^3/α^3, etc.) which led to the actual solutions of (3.1) below (written as they came off the computer):

(5.1)
$$u = \frac{27x^2 - 2x^4 - 8ix^3y + 27y^2 + 12x^2y^2 + 8ixy - 2y^4}{18x^2 + 36ixy - 18y^2}$$

(5.2)
$$u = -\log\left\{\frac{2x - 2iy + \left[-16\left(1 - \dfrac{2iy}{x+iy}\right) + (-2x+2iy)^2\right]^{1/2}}{-2x + 2iy + \left[-16\left(1 - \dfrac{2iy}{x+iy}\right) + (-2x+2iy)^2\right]^{1/2}}\right\}.$$

The quasi-solution $\alpha - \beta$ of (4.6) leads to an actual *complex* solution (5.3) of (3.1) that is certainly complicated, but perhaps there are ways of simplifying it.

$$
\begin{aligned}
(5.3)\quad u = -\frac{1}{24}\Bigg(\Bigg[&-1 - 2x + 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2} \\
&-\Bigg[-1 + 2x + 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2}\Bigg)^3 \\
&-(x^2+y^2)\Bigg/\Bigg\{(x+iy)\Bigg(\Bigg[-1 - 2x - 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2} \\
&\qquad +\Bigg[-1 + 2x + 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2}\Bigg)\Bigg\} \\
&+(x+iy)\Bigg[-1 - 2x - 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2} \\
&\times\Bigg[-1 + 2x + 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2} \\
&\div\Bigg\{\Bigg[-1 - 2x - 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2} \\
&\qquad +\Bigg[-1 + 2x + 2iy + \frac{2iy}{x+iy}\Bigg]^{1/2}\Bigg\}.
\end{aligned}
$$

We now describe our one attempt to computer-implement the method at the end of §4 for getting *real* solutions of (3.1) from solutions of (4.6). In brief, we started with the quasi-solution $\alpha^2 + \beta^2 - 4\alpha\beta$ of (4.6), which led to the actual complex solution (5.1). Everything went well until we arrived at

$$
(5.4)\qquad
\begin{aligned}
x &= \frac{3\eta^5 + 2\eta^3 + (-3\xi^4 - 6\xi^2)\eta}{\eta^6 + 3\xi^2\eta^4 + 3\xi^4\eta^2 + \xi^6} \\
y &= -\frac{3\xi\eta^4 + 6\xi\eta^2 - 3\xi^5 - 2\xi^3}{\eta^6 + 3\xi^2\eta^4 + 3\xi^4\eta^2 + \xi^6}.
\end{aligned}
$$

These equations must be inverted to solve for ξ and η as functions of x and y, and there seems to be no practical way to do this.

Finally, we mention our first attempt to find quasi-solutions of 3.1, namely to let v be the cubic

$$v = x^3 + ax^2y + bxy^2 + cy^3,$$

and then to find the conditions on a, b, c that make $\det W^{\#} = 0$. There was no trouble in getting the machine to do this for us, but unfortunately the conditions took seven pages of printout, and they were so complicated that there was no apparent way to interpret them, so we abandoned this line of attack.

As with many of our computations, maybe faster machines and better programs will someday make these computations feasible when they are presently not feasible.

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