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QUATERNION L-VALUE CONGRUENCES AND GOVERNING FIELDS OF S-CLASS GROUPS

BY

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0. Introduction

The goal of Galois Module Theory is to describe the algebraic structure of modules acted on by Galois group rings. A fundamental result was M.J. Taylor's proof (cf. [F3] for a full discussion) that the ring of integers in a tamely ramified extension of number fields is a free Galois module if and only if a certain *analytic* invariant, constructed from root numbers of Artin L-functions, is trivial. Noticing similarities between the above setting and that of J. Tate's approach to the Stark conjectures in [T], T. Chinburg in [Ch1] conjectured a similar relationship for the Galois module structure of certain S-units. His proof in [Ch2] of this conjecture for a certain family of quaternion extensions (which are the first technically interesting case) relied upon establishing the existence of a governing field for the variation of the structure of the S-class group when S is the set of ramified primes of these extensions. By different techniques, Chinburg in [Ch3] was able to find L-value congruences for a subset of the fields considered in [Ch2] which by our results lead to a precise governing field.

We give a precise governing field for the variation of the Galois module structure of the S-class group for all of the quaternion extensions considered in [Ch2]. Using G. Gras's analytic genus theory, we proceed to give a precise governing field in the context of a previously unstudied family of quaternion extensions. This new result suggests that one now study the algebraic structure for these extensions.

Our approach uses congruence techniques to replace longer classfield theoretic arguments showing that a particular set of primes generates the 2-Sylow subgroup of the ideal classgroup of an extension. That stronger congruences then determine the existence of the governing fields follows from an observation of [Ch2], as discussed in our Section 6.

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1. Results

We study those N which are pure quaternion extensions of $\mathbf{Q}$ and which have complex multiplication. That is, the Galois group of the extension $N/\mathbf{Q}$ is isomorphic to the quaternion group of order 8 and N has a (unique) subfield F such that (i) a finite prime p of $\mathbf{Q}$ ramifies to N if and only if p ramifies to F, and (ii) F is totally real and N totally complex. The existence of the quaternion extensions in all of the cases to which we refer follows from the results of Fröhlich in [F1].

THEOREM I. *Let*

$$F = \mathbf{Q}(\sqrt{pr}, \sqrt{q})$$

for primes $p \equiv r \equiv -q \equiv 3 \bmod 4$ *with Legendre symbols*

$$\left(\frac{q}{p}\right) = \left(\frac{q}{r}\right) = -1.$$

Let N be the unique complex pure quaternion extension of $\mathbf{Q}$ containing F. Let $m = m_1 m_2 \cdots m_n$ be a product of rational primes, such that $\left(\frac{m_i}{pr}\right) = 1$ and $\left(\frac{m_i}{q}\right) = -1$. Let $l \equiv 1 \bmod 4$ be a rational prime distinct from the m_i such that $\left(\frac{l}{pr}\right) = 1$ and $\left(\frac{l}{q}\right) = -1$. Let $m_0 = m$ if $m \equiv 1 \bmod 4$ and let $m_0 = pm$ if $m \equiv 3 \bmod 4$. Let $N[m_0 l]$ be the unique complex quaternion extension of $\mathbf{Q}$ containing F and ramified over exactly p, r, q, l, and the prime divisors of m. Let $V[ml]$ be the unique irreducible 2-dimensional representation of the Galois group of $N[m_0 l]$ over $\mathbf{Q}$. Let $L(s, V[ml])$ be the Artin L-function of $V[ml]$. Then $L(0, V[ml])$ is a rational integer such that

(i) *$L(0, V[ml])$ is exactly divisible by 2^{4+n} and*

(ii) *$L(0, V[ml])/2^{4+n} \bmod 4\mathbf{Z}$ is constant for m fixed.*

Let i be a primitive fourth root of unity. Let p and q be two rational primes. Let s be the order of the image of p in $((\mathbf{Z}/q\mathbf{Z})^*)/((\mathbf{Z}/q\mathbf{Z})^*)^4$. We define the quartic symbol $\left(\frac{p}{q}\right)_4$ to be i^s. Note that this is well-defined if $\left(\frac{p}{q}\right) = 1$, where $\left(\frac{p}{q}\right)$ is the Legendre symbol for p with respect to q.

THEOREM II. *Let*

$$F = Q(\sqrt{p}, \sqrt{q})$$

for primes $p \equiv q \equiv 5 \bmod 8$ *such that*

$$\left(\frac{p}{q}\right) = 1, \quad \left(\frac{p}{q}\right)_4 = -\left(\frac{q}{p}\right)_4 = -1.$$

Let $l \equiv 1 \bmod 4$ be a rational prime such that $-\left(\frac{l}{q}\right) = \left(\frac{l}{p}\right) = 1$ *and* $\left(\frac{l}{p}\right)_4 = \left(\frac{p}{l}\right)_4 = 1$. *Let $N[l]$ be the unique complex quaternion extension of* $\mathbf{Q}$ *containing F and ramified exactly over p, q and l. Let $V[l]$ be the unique irreducible 2-dimensional representation of the Galois group of $N[l]$ over* $\mathbf{Q}$. *Then $L(0, V[l])$ is a rational integer with*

(i) *$L(0, V[l])$ exactly divisible by* 2^4 *and*
(ii) *$L(0, V[l])/2^4$* $\bmod 4\mathbf{Z}$ *constant.*

The proof of Theorem II is made difficult by the small number of primes dividing the conductors of V and $V[l]$. We have used G. Gras's analytic genus theory to overcome this difficulty.

We let H_8 be the quaternion group of order eight and $\mathbf{Z}[H_8]$ be its integral group ring. Let $Cl(\mathbf{Z}[H_8])$ be the finite torsion subgroup of the Grothendieck group of finitely-generated $\mathbf{Z}[H_8]$-modules of finite projective dimension. Suppose $N[d]$ is a complex quaternion extension of $\mathbf{Q}$ and F its biquadratic subextension. Let $S[d]$ be the set of ramified and archimedean places of $N[d]$. Let $S'[d]$ be the places of F determined by $S[d]$. Now suppose that we have a family of twists $N[dl]$ quaternion over $\mathbf{Q}$, containing F. The $S[dl]$-class group of $N[dl]$, $Cl_{S[dl]}N[dl]$, is the ideal class group of $N[dl]$ modulo the primes determined by the finite places of $S[dl]$. Let $f_{S[d]}(l)$ be the class of $Cl_{S[dl]}N[dl]$ in $Cl(\mathbf{Z}[H_8])$. Let K be the maximal abelian extension of F to which all of the elements of $S'[d]$ split. Let K' be the fixed field of the subgroup of $Gal(K/\mathbf{Q})$ generated by elements whose orders are powers of primes congruent to 1 or 7 mod 8. Let $H_{S[d]}$ be the fixed field of the maximal 2-power order subgroup of $Gal(K'/\mathbf{Q})$.

Recall that a governing field in the sense of H. Cohn and J. Lagarias [CL] for a function f on a set of rational primes A to some set B is a finite Galois extension H of $\mathbf{Q}$ such that $f(l)$ is determined by the Frobenius conjugacy class $Frob_{H/\mathbf{Q}}(l)$ whenever l is unramified in H. We call H a minimal governing field for $f(l)$ if no proper sub-extension of $H/\mathbf{Q}$ is also a governing field for $f(l)$. Theorem I and Theorem II provide the precise congruence results for the determination of minimal governing fields for the variation of the Galois module structure in each of these families.

COROLLARY I. *Under the hypotheses of Theorem* I, $H_{S[d]}$ *is a minimal governing field for* $f_{S[d]}(l)$.

COROLLARY II. *Under the hypotheses of Theorem* II, H_S *is a minimal governing field for* $f_{S(l)}$.

Each of the above results depends upon the generation of the 2-Sylow subgroup of the class group of a quaternion N by the ramified primes of N above $\mathbf{Q}$. That this does not hold for all N quaternion over $\mathbf{Q}$, even when N

is pure and complex and F, the biquadratic subextension, has odd class number is shown by the following proposition.

PROPOSITION I. *Let*

$$F = \mathbf{Q}(\sqrt{pq}, \sqrt{pr}, \sqrt{qr})$$

for $p \equiv q \equiv r \equiv 3 \bmod 4$ *such that*

$$\left(\frac{p}{qr}\right) = \left(\frac{p}{qr}\right) = \left(\frac{r}{pq}\right) = -1.$$

The class number of F is odd. There exists a unique complex pure quaternion extension N over $\mathbf{Q}$ *which contains F. The three finite primes of F which ramify to N generate a subgroup of order* 4 *of the class group of N; however, the class number of N is divisible by* 8.

Section 2 presents necessary background material for the proofs of our theorems. Part A sharpens and extends results on congruences of L-functions over $\mathbf{Q}$ via an application of G. Gras's [G] main theorem. Part B presents results related to real quadratic number fields. We first give tables of some of the ray class group characters to be used in the main proofs. We then prove two lemmas needed later. Lastly, we state the form of a deep result of Deligne and Ribet to be used throughout our proofs. Section 3 presents an introduction to the techniques used in the proofs of our theorems. We consider whether the ramified primes generate the 2-Sylow subgroup of the class group of our quaternion extensions. We illustrate how the reduction-of-level techniques combined with the results of Deligne and Ribet can be used to study this problem. The idea of using the Deligne and Ribet work to obtain results in genus theory is clear in Gras [G]. The use of these techniques in the present setting is new. Section 4 provides the proof for Theorem I. Section 5 presents the proof of Theorem II. Section 6 gives the proof of Corollary I and Corollary II.

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2. Congruences and characters

Part A. Results over Q

In proving our theorems about quaternion L-functions, we use the reduction-of-level technique and induction to resolve our questions to simpler objects. One of the most basic of these is the collection of quartic characters over the rationals. To study L-value congruences of these, we use a version of G. Gras's [G] main theorem. Let $\mathbf{Q}_\infty$ be the cyclotomic $\mathbf{Z}_2$-extension of $\mathbf{Q}$, i.e. the totally real subfield of $\mathbf{Q}(\zeta_{2^\infty})$ with Galois group over $\mathbf{Q}$ isomorphic to $\mathbf{Z}_2$. Let k_χ be the cyclic extension of $\mathbf{Q}$ corresponding via class field theory to a 2-power order even character χ. Gras proves the existence of L, the unique minimal abelian extension of $\mathbf{Q}$ containing both k_χ and $\mathbf{Q}_\infty$ such that

(i) L/k_χ has all of its ramification above 2 and
(ii) $Gal(L/\mathbf{Q}_\infty) \cong \oplus H_l$ for $\ell \in S$, where S is the set of primes (other than 2) which ramify in $k_\chi/\mathbf{Q}$ and $H_\ell \cong \langle h_\ell \rangle$ is the inertia subgroup of ℓ in $Gal(k_\chi/\mathbf{Q})$.

Now let J_0 be the set of primes in S which have trivial residue extension to k_χ. Gras shows that for ℓ in J_0 one can choose a lift $\left(\frac{L/\mathbf{Q}}{\ell}\right)$ to $L/\mathbf{Q}$ of the Frobenius $\left(\frac{L^{H_\ell}/\mathbf{Q}}{\ell}\right)$ such that

$$\chi\left(\left(\frac{L/\mathbf{Q}}{\ell}\right)\right) = \omega(\ell),$$

where ω is the odd quadratic Dirichlet character of conductor 4. Furthermore, we can pick a topological generator γ of $Gal(\mathbf{Q}_\infty/\mathbf{Q})$ within $Gal(L/\mathbf{Q})$, such that $\chi(\gamma)$ is a primitive nth root of unity for n the order of χ. With the above choices made, for all $\ell \in S$ let σ_ℓ be defined by

$$\left(\frac{L/\mathbf{Q}}{\ell}\right) = \gamma\sigma_\ell.$$

Gras proves that the equation

$$c_L \alpha_G = \sum_{S \supseteq J} \alpha_J \prod_{\ell \in S \setminus J} (1 - \ell^{-1}\sigma_\ell) \prod_{\ell \in J} (1 - h_\ell) \tag{2.1}$$

in the group ring $\mathbf{Z}_2[Gal(L/\mathbf{Q}_\infty)]$ is solvable for the α_J when $c_L = |G|^{-1}\prod_{\ell \in S}(1 - \ell^{-1})$ and α_G is the sum of the elements of the group $G = Gal(L/\mathbf{Q}_\infty)$. Let m be the maximal ideal of $\mathbf{Q}_2(\chi)$. Gras's main theorem [G; (0.3)] shows membership of $\chi(\alpha_{J_0})$ in m (which is independent of any choices involved in finding the α_J) to determine the m-divisibility of $L(s, \chi\omega)$. We illustrate this technique with an application.

Lemma 2.1. *If $q \equiv 5 \bmod 8$ and λ_q is an odd quartic Dirichlet character of conductor q, then*

$$L(0, \lambda_q) \in (1 + i)\mathbf{Z}_2[i] \setminus 2\mathbf{Z}_2[i].$$

Proof. Here $\lambda_q\omega$ is an even character, of order 4, and of conductor $4q$. By the theorem of Gras, $\nu_{(1+i)}(2^{-1}L_2(0,\lambda_q\omega)) \ge 0$; i.e., $\nu_{(1+i)}(2^{-1}(1-\lambda_q(2))L(0,\lambda_q) \ge 0$. But, $q = 5 \mod 8$ implies that $\lambda_q(2) = \pm i$. Thus $L(0,\lambda_q) \in (1-i)\mathbf{Z}_2[i]$. We show that this is the exact 2-divisibility of $L(0,\lambda_q)$.

Here $S = J_0 = q$. Let $H_q = \langle h \rangle$, $h^4 = e$. We may assume that we have chosen h, $\left(\frac{L/Q}{\ell}\right)$, and γ such that $\chi(h) = i$, $\chi\left(\left(\frac{L/Q}{\ell}\right)\right) = 1$, and $\chi(\gamma) = i$. From Equation 2.1, $\alpha_\emptyset \equiv 1$ modulo (*Augmentation*). We find a solution (in $\mathbf{Z}_2[H_q]$) by letting $\alpha_\emptyset = 1$ and

$$\alpha_q = -14q\left[(3q+1) + 2(q+1)h + (q+3)h^2\right].$$

From this, $\chi(\alpha_{J_0})$ is a $(1+i)$-adic unit, hence $L(0,\lambda_q) \in (1+i)\mathbf{Z}_2[i] \setminus 2\mathbf{Z}_2$.

Remark. This congruence can be obtained in another manner. One may use the analytic formula:

$$(2.2)\quad L(0,\lambda) = \sum_{j=1}^{(q-1)/2} \lambda(j)(q-2j)/q \equiv \sum_{j=1}^{(q-1)/2} \lambda(j) \mod 2\mathbf{Z}_2[i]$$

Simple parity arguments now give our result.

The above lemma can be combined with the reduction-of-level method to determine congruence data for other L-values. However, this does not seem to allow us to determine $L(0,\lambda_q\tau_p)$. The small number of primes dividing the conductor hinders the approach.

PROPOSITION 2.1. *Let $\ell_i \equiv \ell_i' \equiv 5 \mod 8$ be primes for $i = 1$ to some s, with $\ell_i \ne \ell_j$ and $\ell_i' \ne \ell_j'$ for $i \ne j$. Let ψ_i and ψ_i' be 2^{t_i} order Dirichlet characters of primitive conductor ℓ_i and ℓ_i' respectively. Let $T = \max_i(t_i)$. Suppose $\Psi = \prod_{i=1}^s \psi_i$, and $\Psi' = \prod_{i=1}^s \psi_i'$ are odd characters of order 2^T and of conductor ℓ and ℓ' respectively. Suppose further that $\psi_i(\ell_j) = \psi_i'(\ell_j')$ for all $i \ne j$. Let m be the maximal ideal of $\mathbf{Q}_2(\Psi)$. Let D be the number of distinct rational primes ramifying in the extension corresponding to the character $\omega\Psi$ above the ℓ_i. Then*

(i) $(1-\Psi(2))L(0,\Psi) \equiv (1-\Psi'(2))L(0,\Psi') \equiv \mod(m^{D-1+T})\mathbf{Z}_2[\Psi]$ *and, if the $\ell_i \equiv \ell_i' \mod 2^T\mathbf{Z}$ for all i, then*

(ii) $(1-\Psi(2))L(0,\Psi) \equiv (1-\Psi'(2))L(0,\Psi') \mod(m^{D+T})\mathbf{Z}_2[\Psi]$.

Proof. For any odd Φ of 2-power order, with D defined as above, the valuation with respect to m satisfies

$$\nu_m\left(2^{-1}L_2(0,\omega\Phi)\right) \ge D - 1.$$

Hence

$$\nu_m(2^{-1}(1-\Phi(2))L(0,\Phi)) \geq D-1$$

and (i) follows.

Now let $\chi = \omega\Phi$. Gras shows that the strict inequality in the above occurs if and only if $\chi(\alpha_{J_0})$ is in m. For a given Dirichlet character α, let E_α be the corresponding cyclic extension of $\mathbf{Q}$. Define

$$K = \left(\prod_{\psi_i \text{ odd}} E_{\Psi/\psi_i\omega}\right)\prod\left(\prod_{\psi_i \text{ even}} E_{\Psi/\psi_i}\right)$$

and

$$K' = \left(\prod_{\psi'_i \text{ odd}} E_{\Psi'/\psi'_i\omega}\right)\prod\left(\prod_{\psi'_i \text{ even}} E_{\Psi'/\psi'_i}\right).$$

By our hypotheses, $Gal(K/\mathbf{Q}) \simeq Gal(K'/\mathbf{Q})$. By definition, $Gal(L/\mathbf{Q}_\infty)$ is isomorphic to the product of the inertia groups for the ℓ_i. By, say, Washington [W; Theorem 3.7], $Gal(K/\mathbf{Q})$ is also isomorphic to this product. Since L is minimal with respect to its properties of definition, $K \cup \mathbf{Q}_\infty \supseteq L$. As every non-trivial subextension of K has some prime other than 2 ramifying to it from $\mathbf{Q}$, $K \cap \mathbf{Q}_\infty = \mathbf{Q}$. Therefore, $L = K \cup \mathbf{Q}_\infty$.

We also find $L' = K' \cup \mathbf{Q}_\infty$ and from our construction, the isomorphism sending $Gal(K/\mathbf{Q})$ to $Gal(K'/\mathbf{Q})$ takes H_{ℓ_i} isomorphically to $H_{\ell'_i}$ for all i. We choose generators h_{ℓ_i} and $h_{\ell'_i}$ for these cyclic groups such that each h_{ℓ_i} is sent to $h_{\ell'_i}$. Recall, for $p \equiv 5 \bmod 8$, the order of p is U_{2^n}, the units of $\mathbf{Z}/2^n\mathbf{Z}$, is p^{n-2} for $n \geq 2$ and 1 for $n < 2$. Thus, such p are inert in $\mathbf{Q}_\infty/\mathbf{Q}$. From this, we can choose topological generators γ and γ' as in our discussion prior to Equation 2.1, and an isomorphism between $Gal(L/\mathbf{Q})$ and $Gal(L'/\mathbf{Q})$ sending γ to γ' such that each σ_{ℓ_i} goes to $u_i\sigma_{\ell'_i}$, where the u_i are units in the isomorphism of $Gal(\mathbf{Q}_\infty/\mathbf{Q})$ with $\mathbf{Z}_2$, which are trivial mod 2^T. Now when we consider Equation 2.1 in order to solve for the α_{J_0}, we may assume that $h_{\ell_i} = h_{\ell'_i}$ and $\sigma_{\ell_i} = \sigma_{\ell'_i}$ for all i. Since we have required that the $\ell_i \equiv \ell'_i \bmod 2^T\mathbf{Z}$ for all i, we see that $\chi(\alpha_{J_0}) = \chi(\alpha'_{J_0})$ and hence we are done, by Gras's theorem.

COROLLARY 2.1. *Let $p \equiv q \equiv 5 \bmod 8$ be primes. Let λ_p be a primitive quartic character* mod p *and τ_q be the quadratic character* mod q. *Then the following hold*:

(0) $L(0,\lambda_p) \in (1+i)\mathbf{Z}_2[i] \setminus 2\mathbf{Z}_2[i]$.

(1) *If* $\left(\frac{q}{p}\right) = -1$, *then* $L(0,\lambda_p\tau_q) \in 2\mathbf{Z}_2[i] \setminus 2(1+i)\mathbf{Z}_2[i]$.

(2) *If* $\left(\frac{q}{p}\right) = 1$ *and* $\left(\frac{q}{p}\right)_4 = -1$, *then* $L(0,\lambda_p\tau_q) \in 2(1+i)\mathbf{Z}_2[i] \setminus 4\mathbf{Z}_2[i]$.

(3) *If* $\left(\frac{q}{p}\right)_4 = 1$, *and* $\left(\frac{q}{p}\right)_4 = 1$, *then* $L(0,\lambda_p\tau_q) \in 4\mathbf{Z}_2[i]$.

Proof. Here $\lambda_p(2) = \pm i$, hence the $L(0, \Psi) \equiv \bmod (1+i)D\mathbf{Z}_2[i]$. By our Proposition, it suffices to consider the L-value for a single choice of p and q satisfying the given restrictions. For the (0) case, we have already seen two proofs, but now we may simply point out that $D = 1$ and observe that $L(0, \lambda_5) \in (1+i)\mathbf{Z}_2[i] \setminus 2\mathbf{Z}_2[i]$.

(1) has $D = 2$ and $L(0, \lambda_5\tau_{13})$ gives our result. (2) has $D = 3$ and $L(0, \lambda_5\tau_{29})$ gives our result. (3) has $D = 3$ and $L(0, \lambda_{13}\tau_{29})$ gives our result. It is interesting to note that although

$$L(0, \lambda_{13}\tau_{29}) \in 4\mathbf{Z}_2[i] \setminus 4(1+i)\mathbf{Z}_2[i],$$

one finds that

$$L(0, \lambda_5\tau_{101}) \in 4(1+i)\mathbf{Z}_2[i] \setminus 8\mathbf{Z}_2[i].$$

Thus we could hope to prove no stronger result in this setting.

COROLLARY 2.2. *Let $p \equiv q \equiv \ell \equiv 5 \bmod 8$ be primes. Let λ_p be a primitive quartic character* $\bmod p$, *and τ_q and τ_ℓ be the quadratic characters* $\bmod q$ *and ℓ respectively. Then the following hold.*

(A) *For* $\left(\frac{\ell}{p}\right) = \left(\frac{q}{p}\right) = 1$ *and* $\left(\frac{\ell}{q}\right) = 1$,
(1) $L(0, \lambda_p\tau_q\tau_\ell) \in 8\mathbf{Z}_2[i\}$ *if at least one of* $\left(\frac{q}{p}\right)_4$ *and* $\left(\frac{\ell}{p}\right)_4$ *equals one,*
(2) $L(0, \lambda_p\tau_q\tau_\ell) \in 4(1+i)\mathbf{Z}_2[i]/8\mathbf{Z}_2[i]$ *otherwise.*

(B) *For* $\left(\frac{\ell}{p}\right) = \left(\frac{q}{p}\right) = 1$ *and* $\left(\frac{\ell}{q}\right) = -1$,
(3) $L(0, \lambda_p\tau_q\tau_\ell) \in 8\mathbf{Z}_2[i]$ *if* $\left(\frac{q}{p}\right)_4 = 1 = -\left(\frac{\ell}{p}\right)_4$,
(4) $L(0, \lambda_p\tau_q\tau_\ell) \in 4(1+i)\mathbf{Z}_2[i]/8\mathbf{Z}_2[i]$ *if* $\left(\frac{q}{p}\right)_4 = -1$ *or* $\left(\frac{q}{p}\right)_4 = \left(\frac{\ell}{p}\right)_4 = 1$.

(C) *For* $\left(\frac{\ell}{p}\right) = -\left(\frac{q}{p}\right) = -1$,
(5) $L(0, \lambda_p\tau_q\tau_\ell) \in 4\mathbf{Z}_2[i]/4(1+i)\mathbf{Z}_2[i]$ *if* $\left(\frac{q}{p}\right)_4 = 1$ *and* $\left(\frac{\ell}{q}\right) = -1$,
(6) $L(0, \lambda_p\tau_q\tau_\ell) \in 4(1+i)\mathbf{Z}_2[i]$ *if* $\left(\frac{q}{p}\right)_4 = 1$ *and* $\left(\frac{\ell}{p}\right) = 1$,
(7) $L(0, \lambda_p\tau_q\tau_\ell) \equiv 2(1 + \left(\frac{\ell}{q}\right) \bmod 8\mathbf{Z}_2[i]$ *if* $\left(\frac{q}{p}\right)_4 = -1$,

(D) *For* $\left(\frac{\ell}{p}\right) = \left(\frac{q}{p}\right) = -1$,
(8) $L(0, \lambda_p\tau_q\tau_\ell) \in 2(1+i)\mathbf{Z}_2[i]/4\mathbf{Z}_2[i]$.

Proof. Let $\varepsilon = (\lambda_p + \mathbf{1})(\tau_q\tau_\ell + \tau_q + \tau_\ell + \mathbf{1})$, considered as an imprimitive function on $(\mathbf{Z}/pq\ell\mathbf{Z})^*$. The analytic formula, Equation (2.2), gives

$L(0, \varepsilon) \equiv 0 \bmod 8\mathbf{Z}_2[i]$. But,

(2.3)

$$\begin{aligned} L(0,\varepsilon) &= L(0,\lambda_p\tau_q\tau_\ell) + L(0,\lambda_p\tau_q\mathbf{1}_\ell) + L(0,\lambda_p\mathbf{1}_q\tau_\ell) + L(0,\lambda_p\mathbf{1}_{q\ell}) \\ &= L(0,\lambda_p\tau_q\tau_\ell) + \left(1 - \left(\frac{\ell}{p}\right)_4\left(\frac{\ell}{q}\right)\right)L(0,\lambda_p\tau_q) \\ &\quad + \left(1 - \left(\frac{q}{p}\right)_4\left(\frac{q}{\ell}\right)\right)L(0,\lambda_p\tau_\ell) \\ &\quad + \left(1 - \left(\frac{q}{p}\right)_4\right)\left(1 - \left(\frac{\ell}{p}\right)_4\right)L(0,\lambda_p), \end{aligned}$$

where we temporarily admit a sign ambiguity in the case that either ℓ or q is a non-square at p. We could now proceed case by case, via applications of Corollary 2.1.

Part B. Results in quadratic subfields

B1. Character tables

We determine certain characters for $k = \mathbf{Q}(\sqrt{p})$, one of the quadratic subfields of the $F = \mathbf{Q}(\sqrt{p}, \sqrt{q})$ of Theorem II.

For $j \in \{2, 4\}$ and P a prime of k, let $F(P, j)$ be $(O_k/P)^*/((O_k/P)^*)^j$. For P one of the infinite places of k, let $F(P, j)$ be $\{1, -1\}$. Also, let $k^*_{(f)}$ denote the non-zero elements of k which are prime to the ideal f. Let P_p be the prime of k above p and let P_q and $\bar{P}_q$ be the primes above q. We consider the homomorphism from $k^*_{(pq)}$ to

$$F(P_p, 2) \times F(P_q, 4) \times F(\bar{P}_q, 4) \times F(P_\infty, 2) \times F(\bar{P}_\infty, 2).$$

Note that we do not use the usual identification of each of the two factors related to q with

$$(\mathbf{Z}/q\mathbf{Z})^*/((\mathbf{Z}/q\mathbf{Z})^*)^4.$$

By Fröhlich [F1], we know that there exists a complex quaternionic extension N of $\mathbf{Q}$ containing F which is ramified at exactly p and q. Thus there is an odd, order 4 ray class group character, χ_2, of primitive conductor pq over $k = \mathbf{Q}(\sqrt{p})$. Indeed, let λ be a generating character of the dual of $F(P_q, 4)$, and $\bar{\lambda}$ its image under σ, the non-trivial element of $Gal(k/\mathbf{Q})$. Let the symbol $(-)$ represent the quadratic character on a group of order 2.

TABLE 2.1

$k = \mathbf{Q}(\sqrt{p})$; $p \equiv 5 \bmod 8$, $q \equiv 5 \bmod 8$, $\left(\frac{p}{q}\right) = 1$, $\left(\frac{p}{q}\right)_4 = -\left(\frac{q}{p}\right)_4 = -1$					
	$F(P_p, 2)$	$F(P_q, 4)$	$F(\bar{P}_q, 4)$	$F(P_\infty, 2)$	$F(\bar{P}_\infty, 2)$
-1:	1	ε^2	$\bar{\varepsilon}^2$	$-$	$-$
ε_k:	-1	ε	$\bar{\varepsilon}$	$+$	$-$
χ_2:	$(-)$	λ	$\bar{\lambda}^{-1}$	$(-)$	$(-)$
μ:	1	$(-)$	$(-)$	1	1
χ_1:	1	$(-)$	1	$(-)$	$(-)$
$\chi_1\mu$:	1	1	$(-)$	$(-)$	$(-)$
χ_3:	$(-)$	1	1	$(-)$	$(-)$
$\chi_3\mu$:	$(-)$	$(-)$	$(-)$	$(-)$	$(-)$
χ_q:	1	λ^{-1}	$\bar{\lambda}^{-1}$	$(-)$	$(-)$

Then χ_2 may be represented as:

	$F(P_p, 2)$	$F(P_q, 4)$	$F(\bar{P}_q, 4)$	$F(P_\infty, 2)$	$F(\bar{P}_\infty, 2)$
χ_2:	$(-)$	λ	$\bar{\lambda}^{-1}$	$(-)$	$(-)$

For σ the non-trivial element of the Galois group of $k/\mathbf{Q}$, $\chi_2^{-1} = \chi_2^{\sigma}$; as it must, as χ_2 represents a quaternionic extension over $\mathbf{Q}$. Now, any class group character evaluated at the image of a unit of k gives the value 1. Furthermore, we know that $\{-1, \varepsilon_k\}$ is a generating set for the units of k, where ε_k is a fundamental unit. By genus theory, ε_k is of norm -1. Genus theory also tells us that the class number h_k is odd and, as we have chosen our biquadraic field to have odd class number, that ε_k is a non-square at both P_q and $\bar{P}_q$. Thus we have

ε_k:	δ	ε	$(-1)(\bar{\varepsilon})^{-1}$	$+$	$-$

Here ε is the image of ε_k in $F(P_q, 4)$, and δ is in $\{1, -1\}$. Now, the product $\lambda(\varepsilon)\bar{\lambda}^{-1}(-1(\bar{\varepsilon})^{-1})$ is $\lambda(-1)$ times -1. Since $q \equiv 5 \bmod 8$, and has residue degree one to k, -1 is a square, but not a fourth power, at both P_q and $\bar{P}_q$. Therefore, $\lambda(-1) = -1$. Since $\chi_2(\varepsilon_k) = 1$, we see that δ must be -1.

We now list all of the characters of conductor dividing pq and of order 2 or 4.

Note that $\mu = (\chi_2)^2 = (\chi_q)^2$ and $\chi_q = \chi_1\chi_2\chi_3$, both of which correspond to extensions of k which are abelian over $\mathbf{Q}$.

Consider the characters of conductor dividing ℓ. By genus theory,

$$\#\left[\left(F(P_\ell, 2) \oplus F(\bar{P}_\ell, 2)/\langle -1, \varepsilon_k\rangle\right] \quad \text{divides} \quad 2h_{\mathbf{Q}(\sqrt{p}, \sqrt{\ell})}.$$

TABLE 2.2

$k = \mathbf{Q}(\sqrt{pr});\ p \equiv r \equiv -q \equiv 3 \bmod 4,\ \left(\frac{q}{p}\right) = \left(\frac{q}{r}\right) = -1$

	$F(P_p, 2)$	$F(P_r, 2)$	$F(\bar{P}_q, 4)$	$F(\bar{P}_q, 4)$	$F(P_\infty, 2)$	$F(\bar{P}_\infty, 2)$
-1:	-1	-1	ε^τ	$\bar{\varepsilon}^\tau$	$-$	$-$
ε_{pr}:	δ	$-\delta$	ε	$\bar{\varepsilon}^{-1}$	$+$	$+$
χ_2:	$(-)$	$(-)$	λ	$\bar{\lambda}^{-1}$	$(-)$	$(-)$
χ_1:	1	1	1	1	$(-)$	$(-)$
χ_3:	$(-)$	$(-)$	$(-)$	1	$(-)$	$(-)$
χ_q:	1	1	λ^{-1}	$\bar{\lambda}^{-1}$	$(-)$	$(-)$

Since $\ell \equiv 1 \bmod 4$, -1 is a square at P_ℓ and $\bar{P}_\ell$. By Galois action, ε_k is a square at P_ℓ if and only if it is at $\bar{P}_\ell$. Hence, $h_{\mathbf{Q}(\sqrt{p},\sqrt{\ell})}$ is even if and only if ε_k is a square at these primes. But, by Fröhlich [2; Theorem 5.7], $h_{\mathbf{Q}(\sqrt{p},\sqrt{\ell})}$ is even if and only if $\left(\frac{p}{\ell}\right)_4\left(\frac{\ell}{p}\right)_4 = 1$. Hence, in the setting of Theorem II, ε_k is a square at P_ℓ and $\bar{P}_\ell$, the two primes of k above ℓ. Let

$$S_\ell = \{\mathbf{1}_\ell, \psi'_\ell, \psi''_\ell, \psi_\ell\},$$

where the elements of S_ℓ are quadratic characters of primitive conductor 1, P_ℓ, $\bar{P}_\ell$, $P_l\bar{P}_\ell$, respectively.

We now give the characters used in our proof of Theorem I. We reproduce a table of Chinburg's [Ch2], established by similar methods as above.

Since ε_{pr}, the fundamental unit of $k = \mathbf{Q}(\sqrt{p\,r})$, has norm 1, the images of ε_{pr} at conjugate places are either both squares, or both non-squares. If t is a rational integer with $\left(\frac{t}{pr}\right) = 1$, then let P_t and $\bar{P}_t$ be the two primes of k above t. Let ψ_t be the quadratic character of primitive conductor $P_t\bar{P}_t$. Since the images of the units vanish under ψ_t, it is indeed a ray class character of k. Thus we define the ray class characters ψ_{m_i} for the m_i of the hypotheses of Theorem I. Let $\mathbf{1}_t$ be the trivial character of conductor t. We define

$$S_{m_i} = \{\mathbf{1}_{m_i}, \psi_{m_i}\} \quad \text{and} \quad S_m = \prod_{i=1}^{n} S_{m_i}.$$

We now discuss characters whose conductors are divided by P_ℓ or $\bar{P}_\ell$. We have ψ_ℓ as above. If $\left(\frac{\ell}{p}\right) = 1$, then Chinburg [2] shows ε_{pr} to be a square at both P_ℓ and $\bar{P}_\ell$. We then have ray class characters ψ'_ℓ and ψ''_ℓ of primitive conductor P_ℓ and $\bar{P}_\ell$, respectively. We define

$$S_\ell = \{\mathbf{1}_\ell, \psi'_\ell, \psi''_\ell, \psi_\ell\}.$$

If $\left(\frac{\ell}{p}\right) = -1$, then the primes P_ℓ and $\bar{P}_\ell$ remain inert to $k(\sqrt{-p}) = k(\sqrt{-\varepsilon_{pr}})$. Hence $-\varepsilon_{pr}$ is a non-square at both P_ℓ and $\bar{P}_\ell$. Since $\ell \equiv$

TABLE 2.3

$k = \mathbf{Q}(\sqrt{pr}),\ \left(\frac{\ell}{p}\right) = -1$								
	$F(P_p, 2)$	$F(P_r, 2)$	$F(\overline{P}_q, 2)$	$F(P_q, 2)$	$F(P_\ell, 2)$	$F(\overline{P}_\ell, 2)$	$F(P_\infty, 2)$	$F(\overline{P}_\infty, 2)$
-1:	-1	-1	1	1	1	1	$-$	$-$
ε_k:	δ	$-\delta$	-1	-1	-1	-1	$+$	$+$
ν'_ℓ:	1	1	$(-)$	1	$(-)$	1	1	1
ν''_ℓ:	1	1	1	$(-)$	1	$(-)$	1	1

1 mod 4, ε_{pr} itself is a non-square at these primes. Thus there is no even ray class character of primitive conductor P_ℓ or $\overline{P}_\ell$. We now define ν'_ℓ and ν''_ℓ as below.

We define $S'_\ell = \{\mathbf{1}_\ell, \nu'_\ell, \nu''_\ell, \mu\psi_\ell\}$.

B2. Auxiliary congruence results

It will be useful to have the following lemmas, both generalizations of techniques of Chinburg ([Ch3] and [Ch2], respectively.)

LEMMA 2.2. *Let k be a real quadratic extension of $\mathbf{Q}$, with odd class number and odd discriminant. Let f be a conductor of k, co-prime to 2. Let α be an odd quadratic character of k of (possibly imprimitive) conductor f. Let $\sharp f$ be the number of distinct finite primes of k dividing f. Furthermore, let $L(s, {}_f(\alpha))$ be the Artin L-function for the (possibly imprimitive) character ${}_f(\alpha)$ which α induces on G_f. Then $L(0, {}_f(\alpha)) \in 2^{\sharp f}\mathbf{Z}_2$.*

Proof. Let ξ be the primitive character associated to α and L be the quadratic extension of k corresponding to ξ. For a number field K, let $\zeta_K(s)$ be the usual zeta-function, h_K be the class number of K, $Reg(K)$ the regulator of K, and w_K be the number of roots of unity in K. Then

$$L(0, \xi) = \left.\frac{\zeta_L(s)}{\zeta_k(s)}\right|_{s=0} = \frac{h_L\, Reg(L) w_k}{h_k\, Reg(k) w_L}$$

Since α is odd, L/k has complex multiplication. As L/k is unramified over two, L and k have the same units up to torsion. Thus $Reg(L) = 2\, Reg(k)$. If w_L is unequal to w_k, then for some root of unity μ, $L = k(\mu)$. Since f is co-prime to 2, this μ could only be a third root of unity. Therefore, either w_L and w_k are equal, or they differ by an odd factor. Thus

$$L(0, \xi) = 2\frac{h_L}{h_k} \cdot u^{-1},$$

u a unit in $\mathbf{Z}_2$ (intersected with $\mathbf{Q}$).

Let δ be the number of primes of k dividing the conductor of ξ. L/k is ramified at these δ primes as well as at the two infinite places. Therefore, as h_k is odd and O_k^* has two generators, whose images are distinct in $F(P_\infty, 2) \times F(\bar{P}_\infty, 2)$, we find

$$\frac{2^{\delta+2}}{2^2} \Big| 2\frac{h_L}{h_k},$$

i.e.,

$$L(0, \xi) \in 2^\delta \mathbf{Z}_2.$$

But

$$L(0,{}_f(\alpha)) = \left[\prod_{Q \mid (f \setminus cond(\xi))} (1 - \alpha(Q))\right] L(0, \xi)$$
$$\in 2^{(\# f - \delta)} 2^\delta \mathbf{Z}_2 = 2^{\# f} \mathbf{Z}_2.$$

LEMMA 2.3. *Let α be a non-trivial quadratic character of a real quadratic extension $k = \mathbf{Q}(\sqrt{d})$ of $\mathbf{Q}$ with discriminant D. Suppose that α is non-Galois over $\mathbf{Q}$, i.e. the field corresponding to α is not Galois over $\mathbf{Q}$. Let f be the primitive conductor of α, and suppose that D and f are co-prime. Let $P_1, P_2, \ldots, P_n$ be distinct primes of k lying over $p_1, p_2, \ldots, p_n$, distinct rational primes. For A in $\mathbf{Q}$, let $|A|$ be the product of the prime divisors of A, each taken to the first power. Let σ be the non-trivial element of the Galois group of $k/\mathbf{Q}$. Suppose that $|Dff^\sigma|$ divides the product $p_1 p_2 \cdots p_n$ and $\alpha\alpha^\sigma(P_i) = 1$ for all P_i which do not divide f^σ. Then,*

$$\alpha(P_1 P_2 \cdots P_n) = \begin{cases} 1 & \text{if } \alpha \text{ is even}, \\ -1 & \text{otherwise}. \end{cases}$$

Proof. Let L' and L'' be the extensions of k corresponding to α and to α^σ respectively. Let L be the compositum of L' and L''. Then $L/\mathbf{Q}$ is Galois of degree 8, with non-Galois subextensions and more than one subextension of index 2, thus the Galois group of $L/\mathbf{Q}$ is isomorphic to D_8, the dihedral group of order 8. Now, D_8 has a unique cyclic subgroup of order 4. L/k is clearly not cyclic. There are two other quadratic subextensions of $L/\mathbf{Q}$. One is $\mathbf{Q}(\sqrt{p_1 \cdots p_n})$; let us call the other k'. At least one of the p_j does not ramify to k'. Let Q be a prime of k' above such a p_j. The inertia group of Q to L fixes a subfield of L to which Q is unramified. Since Q is ramified to $k(\sqrt{p_1 \cdots p_n})$, but L is unramified (except possibly at the infinite places) over $k(\sqrt{p_1 \cdots p_n})$, L/k' must not be cyclic. Thus, $L/\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ is the cyclic extension.

If α is even, then $L/\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ is unramified. Otherwise, it is unramified except at the infinite places. In this latter case, the very existence of L implies that the narrow Hilbert class field of $\mathbf{Q}(\sqrt{p_1 \cdots p_n})$, H_+, is not equal to the Hilbert class field of $\mathbf{Q}(\sqrt{p_1 \cdots p_n})$, H. But, for a quadratic extension of $\mathbf{Q}$, $[H_+ : H] \leq 2$. Therefore, $[H_+ : H] = 2$ and H_+ is the compositum of H and L. Letting B_i be the unique prime of $\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ above each p_i, we have $\prod_i B_i = (\sqrt{p_1 \cdots p_n})$ is principle, hence trivial in the class group.

But, for α non-even, $(\sqrt{p_1 \cdots p_n})$ is non-trivial in the narrow class group. As the Artin map gives an isomorphism between the narrow class group and $Gal(H_+/\mathbf{Q}(\sqrt{p_1 \cdots p_n}))$, $(\sqrt{p_1 \cdots p_n})$ has for its image an element of order two. We know that H_+ is the compositum of L and H, and the Artin map for the extension $H/\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ is trivial on $(\sqrt{p_1 \cdots p_n})$. Therefore, the Artin map for $L/\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ must take $(\sqrt{p_1 \cdots p_n})$ to an element of order two. Thus, we find that the Artin map for the extension $L/\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ takes $\prod_i B_i = (\sqrt{p_1 \cdots p_n})$ to an element of order one or two, depending on whether α is even or not. But, all of the B_i are split to $k(\sqrt{p_1 \cdots p_n})$, by our assumptions. Thus, B_i splits to L if and only if $\alpha(P_i) = 1$. Equivalently, the image of B_i under the Artin map corresponding to $L/\mathbf{Q}(\sqrt{p_1 \cdots p_n})$ is trivial if $\alpha(P_i) = 1$ and has order two otherwise. Hence, $\alpha(P_1 \cdots P_n) = 1$ if and only if α is even.

Throughout the remainder of this paper, we use the results of Deligne and Ribet in the following form.

THEOREM 2.1 (Deligne and Ribet). *Let k be a totally real field to which* 2 *is unramified. Let $r = [k : \mathbf{Q}]$. Let f be an ideal of k prime to* 2 *and G_f the ray class group of conductor f of k. Let L be a finite extension of $\mathbf{Q}_2$. Let F be the set of odd functions on G_f with values in O_L, the ring of integers of L. For any c in G_f, there is an additive functional on F, denoted $\Delta_c(0, -)$, with values in $2^r O_L$. When ε is an odd character of G_f, $\Delta_c(0, \varepsilon) = (1 - \varepsilon(c))L(0, \varepsilon)$.*

The above version of the theorem is virtually that in [Ch3], but see also [R]: remarks (1) and (2) to (2.1) as well as (3.1); also *confer* [DR].

3. Generation of the 2-Sylow subgroup of the ideal class group

In the cases which we consider, $N/\mathbf{Q}$ is a tame extension with Galois group isomorphic to H_8, the quaternion group of order 8. Let F be the unique biquadratic subextension. Assume that the class number of F, h_F, is odd. Further, assume N is totally complex, while F is totally real. Let the number of finite primes which ramify from F to N be t. Let the t primes of N lying above these be called A_1 through A_t. By Kummer theory (cf. [Ch2; pp 40–41], under our hypotheses the set of the $A_i^{h_F}$ generates a subset of the ideal class group of N isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{t-1}$. Is this the full 2-Sylow subgroup of the ideal class group of N?

Our approach is as follows. We choose a quadratic subfield of F, say k, which has odd class number. As N is cyclic over each of the quadratic fields, N/k corresponds to a ray class group character of k, say χ_2. The induction of χ_2 to $\mathbf{Q}$, $Ind_k^{\mathbf{Q}}\chi_2$, is the character of the unique irreducible 2-dimensional representation, say V, of the Galois group H_8. Thus, $L(s, \chi_2) = L(s, V)$.

It is well known that the L-function of the regular representation of a Galois extension is the zeta-function of the upper field. In the case of $N/\mathbf{Q}$, we find that this regular representation decomposes into the direct sum of the regular representation of $Gal(F/\mathbf{Q})$ with two copies of V. Hence, $L(s, \chi_2)^2 = \zeta_N(s)/\zeta_F(s)$. But, the leading coefficient of the expansion of a ζ-function of a field E at $s = 0$ is $-h_E Reg(E)/w_E$. From our choice of N and F, the image of the units of F in N generate the units of N and we find that

$$L(0, \chi_2)^2 = 2^3 h_N/h_F. \tag{3.1}$$

Now, since N/F is a ramified extension of degree two, h_F divides h_N. Since χ_2 is a quartic character, from Siegel [Si] we know that $L(0, \chi_2)$ is in $\mathbf{Q}(i)$. As it squares to a positive rational integer, $L(0, \chi_2)$ is itself a rational integer.

From Equation (3.1), the $A_i^{h_F}$ generate all of the 2-Sylow subgroup of the ideal class group of N (and hence if S includes all of the A_i, then the order of the S-class group of N is odd) if and only if $2^{3+(t-1)} \| L(0, \chi_2)^2$. Thus, we need study $L(0, \chi_2) \bmod 2^{2+[t/2]}\mathbf{Z}$.

The study of this congruence is carried out via the use of the reduction-of-level technique combined with the results of Deligne and Ribet.

Example 3.1. Let us take F and $N[m_0]$ as in Theorem I. Note that here t equals $4 + 2n$. Let $k = \mathbf{Q}(\sqrt{pr})$, and

$$f = prqm = P_p P_r P_q \bar{P}_q P_{m_1} \bar{P}_{m_1} \cdots P_{m_n} \bar{P}_{m_n}.$$

We use the notation of Table 2.2 and let

$$S = \{\chi_2, \chi_2^{-1}, \chi_q, \chi_q^{-1}, \chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}, S_{m_i} = \{\mathbf{1}_{m_i}, \psi_{m_i}\}$$

and

$$S_m = \prod_{i=1}^{n} S_{m_i}.$$

By the Deligne and Ribet theorem, for any $c \in G_f$,

$$\frac{1}{2} \sum_{\chi \in S} \sum_{\Psi \in S_m} \Delta_c(0,{}_f(\chi\Psi)) \in 2^{4+n}\mathbf{Z}_2[i]. \tag{3.2}$$

We choose c such that $\chi_2(c) \in \{\pm i\}$. Since χ_2 corresponds to a quaternion extension, the non-trivial element of $Gal(k/\mathbf{Q})$ sends χ_2 to χ_2^{-1}, but leaves the Ψ fixed, and thus

$$L(0,f(\chi_2\Psi)) = L(0, f(\chi_2^{-1}\Psi)).$$

From this,

$$\frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \Delta_c(0,{}_f(\chi\Psi)) = \sum_{\Psi \in S_m} L(0,{}_f(\chi_2, \Psi)). \tag{3.3}$$

By Lemma 2.2, we have

$$\frac{1}{2} \sum_{\chi \in \{\chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}} \sum_{\Psi \in S_m} \Delta_c(0,{}_f(\chi\Psi)) \in 2^{4+n}\mathbf{Z}_2. \tag{3.4}$$

Now, let

$$\beta_\Psi = \tfrac{1}{2}(1 - \chi_q\Psi(c))(1 - \chi_q\Psi(P_p))(1 - \chi_q\Psi(P_r))L(0,{}_{qm}(\chi_q\Psi)).$$

By our Lemma 4.1 (whose proof is independent of this section), $L(0,{}_{qm}(\chi_q\Psi))$ has value in

$$2^{n+2}\mathbf{Z}_2[i] \setminus 2^{n+2}(1+i)\mathbf{Z}_2[i].$$

Our choice of c gives $\chi_q(c) \in \{\pm i\}$. Thus

$$\mathrm{Trace}_{\mathbf{Q}[i]/\mathbf{Q}}(\beta_\Psi) \in 2^{n+3}\mathbf{Z}_2 \setminus 2^{n+4}\mathbf{Z}_2,$$

and we have

$$\frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \Delta_c(0,{}_f(\chi)) \equiv 2^3 \mod 2^4\mathbf{Z}_2 \quad \text{if } n = 0, \tag{3.5a}$$

and, since there are an even number of the β_Ψ when $n > 0$,

$$\frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\Psi \in S_m} \Delta_c(0,{}_f(\chi\Psi)) \in 2^{4+n}\mathbf{Z}_2 \quad \text{if } n > 0. \tag{3.5b}$$

Combining (3.2) through (3.5), we find that $L(0,V) \equiv 2^3 \bmod 2^4\mathbf{Z}_2$, and, with an easy complete induction argument, that

$$L(0,V[m]) \equiv 2^{3+n} \bmod 2^{4+n}\mathbf{Z}_2.$$

Hence $L(0,V) \equiv 2^3 \bmod 2^4\mathbf{Z}$ and $L(0,V[m]) \equiv 2^{3+n} \bmod 2^{4+n}\mathbf{Z}$. We conclude that the $A_i^{h_F}$ generate all of 2-Sylow subgroup of the ideal class group of the $N[m_0]$.

Example 3.2. Let us take F and N as in Theorem II. Note that here t equals 4. Let $k = \mathbf{Q}(\sqrt{p})$, and $f = pq = P_p P_q \bar{P}_q$. We use the notation of Table 2.1 and let

$$S = \{\chi_2, \chi_2^{-1}, \chi_q, \chi_q^{-1}, \chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}.$$

By the Deligne and Ribet theorem, for any $c \in G_f$,

$$\frac{1}{2}\sum_{\chi \in S} \Delta_c(0, {}_f(\chi)) \in 2^4\mathbf{Z}_2[i]. \tag{3.6}$$

By choosing c such that $\chi_2(c) \in \{\pm i\}$ (which forces $\mu(c) = -1$), we find that

$$\frac{1}{2}\sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \Delta_c(0, {}_f(\chi)) = L(0, \chi_2). \tag{3.7}$$

Unfortunately, our Lemma 2.2 is insufficient to ensure that the L-values of the quadratic characters are all congruent to zero. However, we may choose c such that $\chi_3(c) = 1$. Since $\chi_1(P_p) = \chi_1\mu(P_p) = 1$, we need only study $L(0, \chi_3\mu)$. Now, as $\chi_3\mu$ corresponds to a fourth degree Galois extension of $\mathbf{Q}$ to which only p, q and ∞ ramify (p totally ramified), this field must be the cyclic extension of $\mathbf{Q}$ corresponding to $\lambda_p\tau_q$. Thus, $L(0, \chi_3\mu) = L(0, \lambda_p\tau_q)L(0, \lambda_p^3\tau_q)$. We apply Corollary 2.2 to conclude that $L(0, \chi_3\mu) \in 2^4\mathbf{Z}_2$. Note that here we have used the results of Gras. Thus,

$$\frac{1}{2}\sum_{\chi \in \{\chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}} \Delta_c(0, {}_f(\chi)) \in 2^4\mathbf{Z}_2. \tag{3.8}$$

Now, let

$$\beta' = \tfrac{1}{2}(1 - \chi_q(c))(1 - \chi_q(P_p))L(0, \chi_q).$$

Since χ induces down to $\mathbf{Q}$ as the product $\lambda_q\tau_p * \lambda_q$, Corollary 2.3 gives

$$L(0, \chi_q) \in 4\mathbf{Z}_2[i] \setminus 4(1+i)\mathbf{Z}_2[i].$$

(This is the significant use of Gras's results.) Now,

$$\chi_q(P_p) = \left(\frac{p}{q}\right)_4 = -1.$$

By our choice of c, $\chi_q(c) \in \{\pm i\}$. Thus, $\mathrm{Trace}_{\mathbf{Q}[i]/\mathbf{Q}}(\beta') \in 2^3\mathbf{Z}_2$ and we have

$$\frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \Delta_c(0,{}_f(\chi)) \equiv 2^3 \mod 2^4\mathbf{Z}_2. \tag{3.9}$$

Combining (3.6) through (3.9), we find that $L(0,V) \equiv 2^3 \mod 2^4\mathbf{Z}_2$. Since $L(0,V)$ is a rational integer, $L(0,V) \equiv 2^3 \mod 2^4\mathbf{Z}$. We conclude that the $A_i^{h_F}$ generate all of the 2-Sylow subgroup of the ideal class group of N in this example.

Example 3.3 (Proof of Proposition I). Let us take

$$F = \mathbf{Q}\left(\sqrt{pq}, \sqrt{pr}, \sqrt{qr}\right),$$

with $p \equiv q \equiv r \equiv 3 \mod 4$ primes such that

$$\left(\frac{p}{r}\right) = \left(\frac{q}{p}\right) = \left(\frac{r}{q}\right) = \pm 1.$$

By Fröhlich [F2; Theorem 5.7], F has odd class number. By Fröhlich [F1], there is a unique complex quaternion extension N of $\mathbf{Q}$ containing F which is ramified only above the rational primes p, q and r. There is exactly one prime of F above each of these primes. Therefore, $t = 3$ and we have an order 4 subgroup of the ideal class group of N generated by the image of the ramified primes.

Let $k = \mathbf{Q}(\sqrt{pq})$. N/k is cyclic of degree four, corresponding to, say χ_2. $f = P_pP_qP_r$ is the conductor of χ_2. By genus theory, the fundamental unit of k, ε_{pq}, is totally positive. Since χ_2^2 corresponds to $F/k, \chi_2^2$ is an even quadratic character of primitive conductor P_r. Therefore, both -1 and ε_{pq} are squares at P_r. We know that -1 is not a square at either P_p or P_q, thus since $\chi_2(-1)$ must be trivial, -1 is a fourth power at P_r. We now show that ε_{pq} is a square at exactly one of P_p or P_q. If not, then there would exist an even quadratic character α, of primitive conductor P_pP_q. By its construction, α would be Galois over $\mathbf{Q}$ and P_r would be inert in the corresponding extension K_α/k. Thus $K_\alpha/\mathbf{Q}$ would be cyclic of degree 4 and ramified at exactly p and q. But, for $p \equiv q \equiv 3 \mod 4$, there exists no such extension of $\mathbf{Q}$. Thus ε_{pq} has the image asserted above. Since $\chi_2(\varepsilon_{pq})$ must be trivial, we find that ε_{pq} is a square, but not a fourth power at P_r. We may now construct the following table.

TABLE 3.1

$k = \mathbf{Q}(\sqrt{pq})$ when $F = \mathbf{Q}(\sqrt{pq}, \sqrt{pr}, \sqrt{qr})$

	$F(P_p, 2)$	$F(P_q, 2)$	$F(P_r, 4)$	$F(P_\infty, 2)$	$F(\bar{P}_\infty, 2)$
-1:	-1	-1	1	$-$	$-$
ε_k:	δ	$-\delta$	γ^2	$+$	$+$
χ_2:	$(-)$	$(-)$	λ	$(-)$	$(-)$
χ_2^2:	1	1	$(-)$	1	1
θ:	1	1	1	$(-)$	$(-)$
β:	1	1	$(-)$	$(-)$	$(-)$

Let $S = \{\chi_2, \chi_2^{-1}, \theta, \beta\}$. By the Deligne and Ribet theorem, for any $c \in G_f$,

$$\frac{1}{2} \sum_{\chi \in S} \Delta_c(0, {}_f(\chi)) \in 2^3\mathbf{Z}_2[i]. \tag{3.10}$$

By choosing c such that $\chi_2(c) \in \{\pm i\}$, we find that

$$\frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \Delta_c(0, {}_f(\chi)) = L(0, \chi_2). \tag{3.11}$$

By Lemma 2.2, we have

$$\frac{1}{2} \sum_{\chi \in \{\theta, \beta\}} \Delta_c(0, {}_f(\chi)) \in 2^3\mathbf{Z}_2[i]. \tag{3.12}$$

Therefore, $L(0, \chi_2) \in 2^3\mathbf{Z}_2[i]$ (indeed, $L(0, \chi_2) \in 2^3\mathbf{Z}$) and the ramified primes of N do not generate the 2-Sylow subgroup of the ideal class group of N.

4. The Proof of Theorem I

As Chinburg [Ch3] obtains the result we desire in the case when $n = 0$ and $\left(\frac{\ell}{p}\right) = 1$, we consider only the remaining cases. Let

$$f_m = P_p P_r P_q \bar{P}_q P_{m_1} \bar{P}_{m_1} \cdots P_{m_n} \bar{P}_{m_n} P_\ell \bar{P}_\ell$$

be a conductor for $k = \mathbf{Q}(\sqrt{pr})$. From Section 2, we have S, our set of odd characters of conductor dividing $prqO_k$, as well as the sets of even characters $S_{m_i} = \{\mathbf{1}_{m_i}, \psi_{m_i}\}$ and

$$S_\ell = \{\mathbf{1}_\ell, \psi'_\ell\, \psi''_\ell, \psi_\ell\} \text{ when } \left(\frac{\ell}{p}\right) = 1$$

and

$$S'_\ell = \{\mathbf{1}_\ell, \nu'_\ell, \nu''_\ell, \mu\psi_\ell\} \text{ when } \left(\frac{\ell}{p}\right) = -1.$$

Let $S_m = \prod_{i=1}^n S_{m_i}$. Let T_ℓ be either of S_ℓ or S'_ℓ, with clarification provided as necessary. We also set the ordered set $(\phi'_\ell, \phi''_\ell, \phi_\ell)$ to be $(\psi'_\ell, \psi''_\ell, \psi_\ell)$ if $\left(\frac{\ell}{p}\right) = 1$ and $(\nu'_\ell, \nu''_\ell, \mu\psi_\ell)$ when $\left(\frac{\ell}{p}\right) = -1$.

Let

$$h = \frac{1}{2} \sum_{\chi \in S} \sum_{\Psi \in S_m} \sum_{\psi \in T_\ell} {}_{f_m}(\chi\Psi\psi).$$

Then by the Deligne and Ribet theorem,

$$(4.1) \quad \Delta_c(0, h) \in (2^{-1}2^3 2^n 2^2)2^2 \mathbf{Z}_2[i] = 2^{6+n}\mathbf{Z}_2[i], \quad \text{for } \chi \in G_{f_m}.$$

The freedom in this approach lies in the choice of c and in the use of the reduction-of-level techniques. First, as we want to isolate $L(0, \chi_2\psi_m\phi_l)$, we choose c such that $\chi_2(c) = i$. Thus, $\mu(c) = -1$ and $\chi_q(c) \in \{\pm i\}$. Let $\chi_q(c) = i$ and $\chi_1(c) = 1$. Thus $\chi(c)$ is now determined for $\chi \in S$. Let $\psi(c) = 1$ for all $\psi \in S_m$. Let $\psi_\ell(c) = \left(\frac{\ell}{p}\right)$.

PROPOSITION 4.1.

$$\frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in \{\mathbf{1}_\ell, \psi_\ell\}} \Delta_c\left(0, {}_{f_m}(\chi\Psi\psi)\right) = \sum_{d \mid ml} 2^{\omega(d)} L\left(0, V\left[\frac{m\ell}{d}\right]\right),$$

where $\omega(d)$ is the number of distinct prime divisors of d.

Proof. Consider

$$(4.2) \quad \tfrac{1}{2}\Delta_c\left(0, {}_{f_m}(\chi_2\psi_{m_1} \cdots \psi_{m_n}\phi_\ell)\right)$$
$$= \tfrac{1}{2}\left(1 - \chi_2\psi_{m_1} \cdots \psi_{m_n}\phi_\ell(c)\right)L(0, \chi_2\psi_{m_1} \cdots \psi_{m_n}\phi_\ell)$$
$$= \tfrac{1}{2}\left(1 - \chi_2\psi_{m_1} \cdots \psi_{m_n}\phi_\ell(c)\right)L(0, V[ml]);$$

note that for $\phi_\ell = \mu\psi_\ell$, we have $\chi_2\phi_\ell = \chi_2^{-1}\psi_\ell$.

Now,

$$\left(\chi_2\psi_{m_1} \cdots \psi_{m_n}\phi_\ell\right)^\sigma = \chi_2^{-1}\psi_{m_1} \cdots \psi_{m_n}\phi_\ell, \quad \text{and} \quad (f_m)^\sigma = f_m.$$

Thus,

$$L\left(0,{}_{f_m}\left(\chi_2^{-1}\psi_{m_1}\cdots\psi_{m_n}\phi_\ell\right)\right)=L\left(0,{}_{f_m}\left(\chi_2\psi_{m_1}\cdots\psi_{m_n}\phi_\ell\right)\right).$$

Furthermore, $\chi_2(c) = \pm i$ and the $\psi(c) = \pm 1$. Therefore,

$$\text{(4.3)}\qquad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_c\left(0,{}_{f_m}\left(\chi\psi_{m_1}\cdots\psi_{m_n}\phi_\ell\right)\right)=L(0,V[m\ell]).$$

For the remaining summands, we use induction via

$$\text{(4.4)}\qquad L\left(0,{}_{f_m}\left(\chi\psi_{m_1}\cdots\psi_{m_{i-1}}\mathbf{1}_{m_i}\psi_{m_{i+1}}\phi_\ell\right)\right)$$
$$=\prod_{Q|P_{m_i}\bar{P}_{m_i}}\left(1-\chi\psi_{m_1}\cdots\psi_{m_{i-1}}\psi_{m_{i+1}}\phi_\ell(Q)\right),$$
$$L\left(0,{}_{f_{m/m_i}}\left(\chi_2\psi_{m_1}\cdots\widehat{\psi_{m_i}}\cdots\psi_{m_n}\phi_\ell\right)\right)$$
$$=2L\left(0,{}_{f_{m/m_i}}\left(\chi_2\psi_{m_1}\cdots\widehat{\psi_{m_i}}\cdots\psi_{m_n}\phi_\ell\right)\right),$$

as $\chi_2^2 = \mu$ and as $\left(\frac{m_i}{r}\right) = -1$ gives $\mu(Q) = -1$.

PROPOSITION 4.2.

$$\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_3\mu,\chi_1,\chi_1\mu\}}\ \sum_{\Psi\in S_m}\ \sum_{\psi\in T_\ell}\Delta_c\left(0,{}_{f_m}(\chi\Psi\psi)\right)\equiv 0 \mod 2^{6+n}\mathbf{Z}_2[i].$$

Proof. All of the characters in the equation are quadratic and the result follows immediately from Lemma 2.2.

PROPOSITION 4.3.

$$\frac{1}{2}\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\Psi\in S_m}\ \sum_{\psi\in T_\ell}\Delta_c\left(0,{}_{f_m}(\chi\Psi\psi)\right)\equiv 0 \mod 2^{6+n}\mathbf{Z}_2[i].$$

LEMMA 4.1. *The value* $L(0,\chi_q\psi_{m_1}\cdots\psi_{m_n})$ *is in* $2^{n+2}\mathbf{Z}_2[i]\setminus 2^{n+2}(1+i)\mathbf{Z}_2[i]$.

Proof. *Case* 1. $q \equiv 1 \bmod 8$.

Now, $Ind_k^{\mathbf{Q}}\chi_q = \tau t_p + \tau t_r$, where τ is a primitive even quartic Dirichlet character of conductor q and where t_r are the Legendre symbols mod p and mod r, respectively. For z in $\mathbf{Z}$, let $z' = z$ if $z \equiv 1 \bmod 4$ and $z' = pz$ if $z \equiv 1 \bmod 4$. Now let $t_{z'}$ be the quadratic Dirichlet character of conductor z'. Then

$$(4.5) \qquad L(0, \chi_q \psi_{m_1} \cdots \psi_{m_n}) = L(0, \tau t_r t_{m'_1} \cdots t_{m'_n}).$$

Let

$$(4.6) \quad \varepsilon_n = (\tau - \mathbf{1}_q)(t_p + \mathbf{1}_p)\Bigg(t_{m'_1} \cdots t_{m'_n} + \sum_{i=1}^{n}{}_{m'} \left(t_{m'_1} \cdots \hat{t}_{m'_i} \cdots t_{m'_n} \right) + \sum_{i=1}^{n} \sum_{j>i}{}_{m'} \left(t_{m'_1} \cdots \hat{t}_{m'_i} \cdots \hat{t}_{m'_j} \cdots t_{m'_n} \right) + \cdots + \mathbf{1}_{m'},$$

where $\hat{\alpha}$ means α is omitted from a sum, and $t_{y'}$ is as above.

Since τ is a quartic character, the $(\tau - \mathbf{1}_q)$ factor takes values in $(1 + i)\mathbf{Z}_2[i]$. The next factor clearly takes values in $2\mathbf{Z}_2$. The final factor is the sum over all characters in the group generated by the $t_{m'_i}$, thus takes values in $2^n\mathbf{Z}$. Therefore, ε_n takes values in $(i - 1)2^{n+1}\mathbf{Z}_2[i]$.

We use the analytic formula to determine

$$\begin{aligned}(4.7) \quad L(0, \varepsilon_n) &= -\phi(pqm_1 \cdots m_n)/2 - \sum_{j=1}^{pqm_1 \cdots m_n} \varepsilon_n(j) j / pqm_1 \cdots m_n \\ &= -(p-1)(q-1)(m_1 - 1) \cdots (m_1 - 1)/2 - T\end{aligned}$$

where

$$T \equiv 0 \mod 2^{n+1}(i+1)\mathbf{Z}_2[i].$$

Therefore,

$$L(0, \varepsilon_n) \equiv 0 \mod 2^{n+1}(i+1)\mathbf{Z}_2[i] \quad (n > 0),$$

as the $m_i \equiv q \equiv -p \equiv 1 \bmod 4$. Since $L(0, \varepsilon_n) = 0$ for α an even Dirichlet character, one has

$$\begin{aligned}(4.8) \quad L(0, \varepsilon_n) &= L(0, \tau t_p t_{m'_1} \cdots t_{m'_n}) + \cdots + L(0, \tau_{pm}(t_p)) \\ &\quad - \left[L(0, {}_{pqm}(t_p t_{m'_1} \cdots t_{m'_n})) + \cdots + L(0, {}_{pqm}(t_p)) \right].\end{aligned}$$

Note that

$$(4.9) \qquad L(0, {}_{pqm}(t_p)) = (1 - t_p(q)) \left(\prod_{i=1}^{n} (1 - t_p(m_i)) \right) L(0, t_p).$$

Let L be the extension of $\mathbf{Q}$ corresponding to $t_p t_{m'_1} \cdots t_{m'_j}$. Then

(4.10)

$$\zeta_L(0) = -\frac{h_L Reg(L)}{w_L} = \zeta_{\mathbf{Q}}(0) L\big(0, t_p t_{m'_1} \cdots t_{m'_j}\big) = -\tfrac{1}{2} L\big(0, t_p t_{m'_1} \cdots t_{m'_j}\big).$$

As L is imaginary, $Reg(L) = 1$.

$w_L | w_{\mathbf{Q}(\zeta_{pm'_1 \cdots m'_j})} = 2pm'_1 \cdots m'_j$, thus $2^1 \| w_L$. By genus theory, $2^j | h_L$. Therefore, $2^{j-1} | \zeta_L(0)$; hence $2^j | L(0, t_p t_{m'_1} \cdots t_{m'_j})$. Now, each inverse Euler factor has a value in $\{0, 2\}$, so induction on j gives

$$(4.11) \qquad L\big(0, \tau t_p t_{m'_1} \cdots t_{m'_j}\big) + \cdots + L\big(0, \tau_m(t_p)\big) \equiv 2^{n+1}\mathbf{Z}_2[i],$$
$$L\big(0, \tau t_p t_{m'_1} \cdots t_{m'_n}\big) \in (1+i)^{n+2}\mathbf{Z}_2[i] \setminus (1+i)^{n+3}\mathbf{Z}_2[i].$$

Chinburg [Ch3; Equation 4.9] shows $L(0, \tau t_p) \in 2\mathbf{Z}_2[i] \setminus 2(1+i)\mathbf{Z}_2[i]$. We use complete induction on n to show $L(0, \tau t_p t_{m'_1} \cdots t_{m'_n}) \in (1+i)^{n+2}\mathbf{Z}_2[i] \setminus (1+i)^{n+3}\mathbf{Z}_2[i]$.

Suppose we know our claim for all $n \le j-1$. We want to show our claim for $n = j$. We have

$$(4.12) \qquad L\big(0, \tau t_p t_{m'_1} \cdots t_{m'_j}\big) + \cdots + \prod_{i=1}^{j}\big(1 - \tau t_p(m_i)\big) L(0, \tau t_p)$$
$$\equiv 0 \mod 2^{j+1}\mathbf{Z}_2[i].$$

Since the $\tau t_p(m_i) \in \{\pm i\}$, all summands other than $L(0, \tau t_p t_{m'_1} \cdots t_{m'_j})$ are in

$$(1+i)^{j+2}\mathbf{Z}_2[i] \setminus (1+i)^{j+3}\mathbf{Z}_2[i].$$

The total number of summands is $\sum_{i=0}^{j} \binom{j}{i}$, a power of two.

Summing in pairs, we find each pair other than that including $L(0, \tau t_p t_{m'_1} \cdots t_{m'_j})$ is in $(1+i)^{j+3}\mathbf{Z}_2[i]$. If $L(0, \tau t_p t_{m'_1} \cdots t_{m'_j})$ itself were in $(1+i)^{j+3}\mathbf{Z}_2[i]$, then its partner would also be, as the total sum is in $(1+i)^{2j+2}\mathbf{Z}_2[i]$, by Equation 4.12. Thus

$$L\big(0, \tau t_p t_{m'_1} \cdots t_{m'_j}\big) \notin (1+i)^{j+3}\mathbf{Z}_2[i].$$

On the other hand, as all other summands as well as the sum itself are in $(1+i)^{j+2}\mathbf{Z}_2[i]$,

$$L\big(0, \tau t_p t_{m'_1} \cdots t_{m'_j}\big) \in (1+i)^{j+2}\mathbf{Z}_2[i].$$

By symmetry, $L(0, \tau t_r t_{m'_1} \cdots t_{m'_n})$ also has the above property, hence

$$L\big(0, \chi_q \psi_{m_1} \cdots \psi_{m_n} = L\big(0, \tau t_p t_{m'_1} \cdots t_{m'_n}\big) L\big(0, \tau t_r t_{m'_1} \cdots t_{m'_n}\big)$$
$$\in 2^{n+2}\mathbf{Z}_2[i] \setminus 2^{n+2}(1+i)\mathbf{Z}_2[i].$$

Case 2. $q \equiv 5 \bmod 8$.

Now, $Ind_k^{\mathbf{Q}}(\chi_q) = \lambda + \lambda t_{pr}$, for λ a primitive odd quartic Dirichlet character of conductor q. Hence,

$$L\big(0, \chi_q \psi_{m_1} \cdots \psi_{m_n}\big) = L\big(0, \lambda t_{m'_1} \cdots t_{m'_n}\big) L\big(0, \lambda t_{pr} t_{m'_1} \cdots t_{m'_n}\big).$$

We will work with the first factor, the determination of the second factor is virtually the same.

Let

$$(4.13) \quad \gamma_n = (\lambda - \mathbf{1}_q)\Bigg[\Big(t_{m'_1} \cdots t_{m'_n} + \sum_{i=1}^{n} {}_{m'}\big(t_{m'_1} \cdots \hat{t}_{m'_i} \cdots t_{m'_n}\big) + \sum_{i=1}^{n} \sum_{j>i} {}_{m'}\big(t_{m'_1} \cdots \hat{t}_{m'_i} \cdots \hat{t}_{m'_j} \cdots t_{m'_n}\big) + \cdots + \mathbf{1}_m\Bigg].$$

Thus, γ_n has values in $2^n(1+i)\mathbf{Z}_2[i]$.

$$\begin{aligned}(4.14) \quad L(0, \gamma_n) &= -\phi([p]qm_1 \cdots m_n)/2 \\ &\quad - \sum_{j=1}^{[p]qm_1 \cdots m_n} \gamma_n(j)j/[p]qm_1 \cdots m_n \\ &= -[p-1](q-1)(m_1-1)\cdots(m_1-1)/2 - U \\ &\qquad\qquad (\text{where } U \in 2^n(i+1)\mathbf{Z}_2[i]) \\ &\equiv 0 \mod 2^n(i+1)\mathbf{Z}_2[i],\end{aligned}$$

where factors in square brackets need be considered only when one of the m_i is congruent to 3 mod $4\mathbf{Z}$.

But,

$$\begin{aligned}(4.15) \quad L(0, \gamma_n) &= L\big(0, \lambda t_{m'_1} \cdots t_{m'_n}\big) + \cdots + L\big(0, \lambda_{m[p]}(\mathbf{1})\big) \\ &\quad - \big[L\big(0_q\big(t_{m'_1} \cdots t_{m'_n}\big)\big) + \cdots + L\big(0, {}_{qm[p]}(\mathbf{1})\big)\big] \\ &= L\big(0, \lambda t_{m'_1} \cdots t_{m'_n}\big) + \cdots + L\big(0, \lambda_{m[p]}(\mathbf{1})\big),\end{aligned}$$

as the $t_{m'_i}$ are all even characters.

Claim. $L(0, \lambda t_{m'_1} \cdots t_{m'_n}) \in (1+i)^{n+1}\mathbf{Z}_2[i] \setminus (1+i)^{n+1}\mathbf{Z}_2[i]$.

The proof for $n = 0$ we have seen via Gras's method in Corollary 2.2. Thus we use complete induction on n as above.

Chinburg [Ch3; Equation 4.14] shows that

$$L(0, \lambda t_{pr}) \in (1+i)^3\mathbf{Z}_2[i] \setminus (1+i)^4\mathbf{Z}_2[i].$$

Complete induction again gives

$$L(0, \lambda t_{pr} t_{m'_1} \cdots t_{m'_n}) \in (1+i)^{n+3}\mathbf{Z}_2[i] \setminus (1+i)^{n+4}\mathbf{Z}_2[i].$$

Hence

$$L(0, \chi_q \psi_{m_1} \cdots \psi_{m_n}) \in 2^{n+2}\mathbf{Z}_2[i] \setminus 2^{n+2}(1+i)\mathbf{Z}_2[i].$$

Proof of Proposition 4.3. Let $\delta = 0$ if $\chi_q(P_p P_q) = 1$ and $\delta = 1$ otherwise. Let $f' = f_m / P_p P_q$.

Case 1. $\phi'_\ell(P_p P_q) = 1$.

$$(4.16) \quad \frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in T_\ell} \Delta_c(0, {}_{f_m}(\chi\Psi\psi))$$

$$= \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in T_\ell} (-\chi\Psi\psi(P_p))^\delta \Delta_c(0, {}_{f'}(\chi\Psi\psi))$$

$$\left(\text{as } \chi\Psi\psi(P_p) = \pm i\right)$$

$$\equiv \sum_{\chi \in \{\chi_1, \chi_1\mu\}} \sum_{\Psi \in S_m} \sum_{\psi \in T_\ell} (-\chi\Psi\psi(P_p))^\delta \Delta_c(0, {}_{f'}(\chi\Psi\psi))$$

$$\bmod 2^{n+7}\mathbf{Z}_2[i],$$

this last from the Deligne and Ribet Theorem.

Now, this is in $2^{2n+5}\mathbf{Z}_2[i]$, by Lemma 2.2. If $n > 0$, we are done. If $n = 0$, we are interested in only the case of $\left(\frac{\ell}{p}\right) = -1$ and note that $\nu'_\ell(c) = \nu''_\ell(c)$; $\nu'_\ell\nu''_\ell(P_p) = \mu\psi_\ell(P_p) = 1$; $(\nu'_\ell)^\sigma = \nu''_\ell$; $(f')^\sigma = f'$ and $\chi^\sigma = \chi$ for $\chi \in \{\chi_1, \chi_1\mu\}$.

Thus,

$$(4.17)\qquad \sum_{\chi\in\{\chi_1,\chi_1\mu\}}\ \sum_{\psi\in\{\nu'_\ell,\nu''_\ell\}} \left(-\chi\psi(P_p)\right)^\delta \Delta_c\left(0,{}_{f'}(\chi\psi)\right)$$

$$\equiv \sum_{\chi\in\{\chi_1,\chi_1\mu\}} 2\left(-\chi\nu'_\ell(P_p)\right)^\delta \Delta_c\left(0,{}_{f'}(\chi\nu'_\ell)\right)$$

$$\equiv 0 \mod 2^6\mathbf{Z}_2 \text{ (by Lemma 2.2)}.$$

Therefore,

$$(4.18)\qquad \frac{1}{2}\sum_{\chi\in\{\chi_1,\chi_q^{-1}\}}\ \sum_{\psi\in S_\ell} \Delta_c\left(0,{}_f(\chi\psi)\right)$$

$$\equiv \sum_{\chi\in\{\chi_1,\chi_1\mu\}}\ \sum_{\psi\in\{\mathbf{1}_{q\ell},\mu\psi_\ell\}} \left(-\chi\psi(P_p)\right)^\delta \Delta_c\left(0,{}_{f'}(\chi\psi)\right) \mod 2^6\mathbf{Z}_2[i].$$

But,

$$1-\chi_1(c) = 0, \qquad 1-\chi_1\mu\psi_\ell(c) = 0,$$

$$1-\chi_1\mu(P_\ell) = 1-(-1)(-1) = 0$$

and

$$1-(\chi_1\mu)(\mu\psi_\ell)(P_q) = 0.$$

Therefore,

$$\frac{1}{2}\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\psi\in S_\ell} \Delta_c\left(0,{}_f(\chi\psi)\right) \in 2^6\mathbf{Z}_2[i]$$

in this case and we proceed with:

Case 2. $\phi'_\ell(P_pP_r) = -1$.
Let $\delta' = 0$ if $\chi_q\phi'_\ell(P_pP_r) = 1$ and $\delta' = 1$ otherwise.

(4.19)
$$\begin{aligned}
&\frac{1}{2}\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in T_\ell}\Delta_c(0,{}_{f_m}(\chi\Psi\psi))\\
&\equiv \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\phi'_\ell,\phi''_\ell\}} -\left(\chi\Psi\psi(P_p)\right)^{\delta'}\Delta_c(0,{}_{f'}(\chi\Psi\psi))\\
&\quad+\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\mathbf{1}_\ell,\phi_\ell\}} -\left(\chi\Psi\psi(P_p)\right)^{\delta}\Delta_c(0,{}_{f'}(\chi\Psi\psi))\\
&\equiv \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\phi'_\ell,\phi''_\ell\}}\left[-\left(\chi\Psi\psi(P_p)\right)^{\delta'}-\left(\chi\Psi\psi(P_p)\right)^{\delta}\right]\\
&\quad\times\Delta_c(0,{}_{f'}(\chi\Psi\psi))\\
&\quad-\sum_{\chi\in\{\chi_1,\chi_1\mu\}}\sum_{\Psi\in S_m}\sum_{\psi\in T_\ell} -\left(\chi\Psi\psi(P_p)\right)^{\delta}\Delta_c(0,f'(\chi\Psi\psi))\\
&\qquad\qquad\qquad\qquad\text{(by the Deligne and Ribet Theorem)}\\
&\equiv \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\phi'_\ell,\phi''_\ell\}}\left[-\left(\chi\Psi\psi(P_p)\right)^{\delta'}-\left(\chi\Psi\psi(P_p)\right)^{\delta}\right]\\
&\quad\times\Delta_c(0,{}_{f'}(\chi\Psi\psi)),
\end{aligned}$$

by the work in Case 1. Now, $\phi'_\ell(P_p) = \phi''_\ell(P_p)$, $\phi'_\ell(c) = \phi''_\ell(c)$ and $(\phi'_\ell)^\sigma = \phi''_\ell$ gives

(4.20)
$$\begin{aligned}
&\frac{1}{2}\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in T_\ell}\Delta_c(0,{}_{f_m}(\chi\Psi\psi))\\
&\equiv \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m} 2(-1)^{\delta'}\left(1+\chi\Psi\phi'_\ell(P_p)\right)\Delta_c(0,{}_{f'}(\chi\Psi\phi'_\ell))\\
&\equiv -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m} 2(-1)^{\delta'}\left(1+\chi\Psi\mathbf{1}_\ell(P_p)\right)\Delta_c(0,{}_{f'}(\chi\Psi\mathbf{1}_\ell))\\
&\quad-\sum_{\chi\in\{\chi_1,\chi_1\mu\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\mathbf{1}_\ell,\phi'_\ell\}} 2(-1)^{\delta'}\left(1+\chi\Psi\psi(P_p)\right)\\
&\quad\times\Delta_c(0,{}_{f'}(\chi\Psi\psi)),\\
&\equiv -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m} 2(-1)^{\delta'}\left(1+\chi\Psi\mathbf{1}_\ell(P_p)\right)\Delta_c(0,{}_{f'}(\chi\Psi\mathbf{1}_\ell))\\
&\qquad\qquad\qquad\qquad \bmod 2^{6+n}\mathbf{Z}_2[i]\ \text{(by Lemma 2.2)}.
\end{aligned}$$

Let

$$\beta_\Psi = 2(-1)^{\delta'}\big(1+\chi_q\Psi(P_p)\big)\big(1-\chi_q\Psi(c)\big)\big(1-\chi_q\Psi(P_\ell)\big) \times\big(1-\chi_q\Psi(\bar{P}_\ell)\big)L\big(0,{}_{qm}(\chi_q\Psi)\big).$$

Recalling that $L(0,\chi_q\Psi)$ is in $2^{n+2}\mathbf{Z}_2[i]\setminus 2^{n+2}(1+i)\mathbf{Z}_2[i]$, we find that $\beta_\Psi \in 2^{n+5}\mathbf{Z}_2[i]\setminus 2^{n+5}(1+i)\mathbf{Z}_2[i]$. Therefore,

$$(4.21)\quad \frac{1}{2}\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in T_\ell}\Delta_c\big(0,{}_{f_m}(\chi\Psi\psi)\big)$$
$$\equiv -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}2(-1)^{\delta'}\big(1+\chi\Psi\mathbf{1}_\ell(P_p)\big)\Delta_c\big(0,{}_{f'}(\chi\Psi\mathbf{1}_\ell)\big)$$
$$= -\sum_{\Psi\in S_m}\operatorname{Trace}_{\mathbf{Q}[i]/\mathbf{Q}}(\beta\Psi)$$
$$\equiv 0 \mod 2^{n+6}\mathbf{Z}_2[i].$$

Thus we have completed the proof of Proposition 4.3.

PROPOSITION 4.4.

$$\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\phi'_\ell,\phi''_\ell\}}\Delta_c\big(0,{}_{f_m}(\chi\Psi\psi)\big)$$
$$\equiv \begin{cases} 0 & \mod 2^{n+6}\mathbf{Z}_2[i] \quad \text{if } n>0,\\ 2^5 & \mod 2^6\mathbf{Z}_2[i] \quad \text{if } n=0 \left(\text{and } \left(\frac{\ell}{p}\right)=-1\right).\end{cases}$$

Proof.

(4.22)

$$\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\phi'_\ell,\phi''_\ell\}}\Delta_c\big(0,{}_{f_m}(\chi\Psi\psi)\big)$$
$$=\frac{1}{2}\sum_{\Psi\in S_m}\big[1-\chi_2\Psi\phi'_\ell(c)+1-\chi_2^{-1}\Psi\phi''_\ell(c)\big]L\big(0,{}_{f_m}(\chi_2\Psi\phi'_\ell)\big)$$
$$+\frac{1}{2}\sum_{\Psi\in S_m}\big[1-\chi_2\Psi\phi''_\ell(c)+1-\chi_2^{-1}\Psi\phi'_\ell(c)\big]L\big(0,{}_{f_m}(\chi_2^{-1}\Psi\phi'_\ell)\big)$$
$$=\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}L\big(0,{}_{f_m}(\chi\Psi\phi'_\ell)\big).$$

Letting $f'' = f_m/\bar{P}_\ell = P_p P_r P_q \bar{P}_q P_{m_1} \bar{P}_{m_1} \cdots P_{m_n} \bar{P}_{m_n} P_\ell$, we find that

$$(4.23)\quad \frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in \{\phi'_\ell, \phi''_\ell\}} \Delta_c(0, {}_{f_m}(\chi\Psi\psi)) = \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \left(1 - \chi\Psi\phi'_\ell(\bar{P}_\ell)\right) L(0, {}_{f''}(\chi\Psi\phi'_\ell)).$$

Choose $c' \in \{c, c^{-1}\}$ such that $\chi\Psi\phi'_\ell(\bar{P}_\ell) = \chi\Psi\phi'_\ell(c')$. We now have

$$(4.24)\quad \frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in \{\phi'_\ell, \phi''_\ell\}} \Delta_c(0, {}_{f_m}(\chi\Psi\psi)) = \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \Delta_{c'}(0, {}_{f''}(\chi\Psi\phi'_\ell)).$$

But

$$\sum_{\chi \in S} \sum_{\Psi \in \{\mathbf{1}_\ell, \phi'_\ell\}} \Delta_{c'}(0, {}_{f''}(\chi\Psi\psi)) \in 2^{n+6}\mathbf{Z}_2[i],$$

by the results of Deligne and Ribet. Now, Lemma 2.2 gives

$$(4.25)\quad \sum_{\chi \in \{\chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}} \sum_{\Psi \in S_m} \sum_{\psi \in \{\mathbf{1}_\ell, \phi'_\ell\}} \Delta_{c'}(0, {}_{f''}(\chi\Psi\psi)) \in 2^{2n+6}\mathbf{Z}_2[i].$$

Therefore,

$$(4.26)\quad \frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in \{\phi'_\ell, \phi''_\ell\}} \Delta_c(0, {}_{f_m}(\chi\Psi\psi)) \equiv - \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \Delta_{c'}(0, {}_{f''}(\chi\Psi\mathbf{1}_\ell)) - \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\Psi \in S_m} \sum_{\psi \in \{\mathbf{1}_\ell, \phi'_\ell\}} \Delta_{c'}(0, {}_{f''}(\chi\Psi\psi)).$$

LEMMA 4.2.

$$-\frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\Psi \in S_m} \Delta_{c'}(0, {}_{f''}(\chi\Psi\mathbf{1}_\ell)) \equiv 2^{4+2n}\left(1 + \phi'_\ell\left(\bar{P}_\ell \bar{P}_q\right)\right) \mod 2^{6+n}\mathbf{Z}_2[i].$$

Proof.

$$(4.27)\quad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}\Delta_{c'}\big(0,{}_{f''}(\chi\Psi\mathbf{1}_\ell)\big)$$

$$= \sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}(1-\chi\Psi(c'))(1-\chi\Psi(P_\ell))L\big(0,{}_{prqm}(\chi\Psi)\big)$$

$$= \sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}\big[(1-\chi_2\Psi(c'))(1-\chi_2\Psi(P_\ell))$$

$$+(1-\chi_2^{-1}\Psi(c'))(1-\chi_2^{-1}\Psi(P_\ell))\big]L\big(0,{}_{prqm}(\chi_2\Psi)\big)$$

But

$$\chi_2\Psi(c') = \chi_2(\bar{P}_\ell)\Psi(\bar{P}_\ell)\phi'_\ell(\bar{P}_\ell)/\phi'_\ell(c') \text{ (by our choice of } c')$$

$$= -\chi_2(P_\ell)\Psi(\bar{P}_\ell)\phi'_\ell(\bar{P}_\ell)\phi'_\ell(\bar{P}_q)$$

as $\phi'_\ell(c') \in \{\pm 1\}$ and by our initial choice of c. Therefore,

$$(4.28)\quad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}\Delta_{c'}\big(0,{}_{f''}(\chi\Psi\mathbf{1}_\ell)\big)$$

$$= \sum_{\Psi\in S_m}\big(2+2\phi'_\ell(\bar{P}_\ell)\phi'_\ell(\bar{P}_q)\big)L\big(0,{}_{prqm}(\chi_2\Psi)\big)$$

$$= \begin{cases} 4\displaystyle\sum_{\Psi\in S_m} L\big(0,{}_{prqm}(\chi_2\Psi)\big) & \text{if } \phi'_\ell(\bar{P}_\ell)\phi'_\ell(\bar{P}_q) = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Since Example 3.1 shows that $2^{n+3}\| L(0,{}_{prqm}(\chi_2\Psi))$, we have proved our lemma.

LEMMA 4.3.

$$-\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\mathbf{1}_\ell,\nu'_\ell\}}\Delta_{c'}\big(0,{}_{f''}(\chi\Psi\psi)\big)$$

$$\equiv 2^{4+2n}\big(1+\phi'_\ell(P_pP_r)\big) \mod 2^{6+n}\mathbf{Z}_2[i].$$

Proof. *Case* 1. $\phi'_\ell(P_pP_r) = 1$.
Let $f''' = f''/P_pP_r = P_q\bar{P}_qP_{m_1}\bar{P}_{m_1} \cdots P_{m_n}\bar{P}_{m_n}P_\ell$.

$$(4.29)\quad -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{1_\ell,\phi'_\ell\}}\Delta_{c'}(0,{}_{f''}(\chi\Psi\psi))$$

$$= -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{1_\ell,\phi'_\ell\}}2\big(-\chi\Psi\psi(P_p)\big)^{\delta}\Delta_{c'}(0,{}_{f'''}(\chi\Psi\psi))$$

$$\equiv \sum_{\chi\in\{\chi_1,\chi_1\mu\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{1_\ell,\phi'_\ell\}}2\big(-\chi\Psi\psi(P_p)\big)^{\delta}\Delta_{c'}(0,{}_{f'''}(\chi\Psi\psi))$$

$$\bmod 2^{6+n}\mathbf{Z}_2[i],$$

by the Deligne and Ribet theorem. When $n > 0$, we may apply Lemma 2.2 to conclude that this last is in $2^{6+n}\mathbf{Z}_2[i]$. For the case of $n = 0$, we proceed, noting that $\phi'_\ell = \nu'_\ell$.

We have $1 - \chi_1(c) = 0$, and $1 - \chi_1\mu(P_\ell) = 1 - (-1)(-1) = 0$. Thus,

(4.30)

$$-\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\psi\in\{1_\ell,\nu'_\ell\}}\Delta_{c'}(0,{}_{f''}(\chi\psi))$$

$$\equiv \sum_{\chi\in\{\chi_1,\chi_1\mu\}}2\big(-\chi\nu'_\ell(P_p)\big)^{\delta}\Delta_{c'}(0,{}_{f'''}(\chi\nu'_\ell)) \quad \bmod 2^{6+n}\mathbf{Z}_2[i]$$

$$= 2\big(-\chi_1\nu'_\ell(P_p)\big)^{\delta}(1-\chi_1\nu'_\ell(c'))\big(1-\chi_1\nu'_\ell(\bar{P}_q)\big)L\big(0,{}_{f'''}(\chi_1\nu'_\ell)\big)$$

$$+ 2\big(-\chi_1\mu\nu'_\ell(P_p)\big)^{\delta}(1-\chi_1\mu\nu'_\ell(c'))\big(1-\chi_1\mu\nu'_\ell(P_q)\big)L\big(0,{}_{f'''}(\chi_1\nu'_\ell)\big)$$

But, $\chi_1(c') = 1$, $\mu(c') = -1$, $\chi_1(\bar{P}_q) = -1$, $\mu\nu'_\ell = \psi_\ell\nu''_\ell$, and $\psi_\ell(P_q) = -1$. Thus the above equals

$$2\big(-\chi\nu'_\ell(P_p)\big)^{\delta}(1-\nu'_\ell(c'))\big(1+\nu'_\ell(\bar{P}_q)\big)L\big(0,{}_{f'''}(\chi_1\nu'_\ell)\big)$$

$$+ 2\big(-\chi\mu\nu'_\ell(P_p)\big)^{\delta}(1+\nu'_\ell(c'))\big(1-\nu'_\ell(P_q)\big)L\big(0,{}_{f'''}(\chi_1\nu'_\ell)\big)$$

and now, by our choice of c, $\nu'_\ell(c') = \nu'_\ell(\bar{P}_q)$, hence both of these summands must be zero, and we have proved our lemma in this case.

Case 2. $\phi'_\ell(P_pP_r) = -1$.

(4.31)

$$-\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\mathbf{1}_\ell,\phi'_\ell\}}\Delta_{c'}(0,_{f''}(\chi\Psi\psi))$$
$$\equiv -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\mathbf{1}_\ell,\phi'_\ell\}} 2\Big[\big(-\chi\Psi\mathbf{1}_\ell(P_p)\big)^{\delta}-\big(-\chi\Psi\mathbf{1}_\ell(P_p)\big)^{\delta'}\Big]$$
$$\times\Delta_{c'}(0,_{f'''}(\chi\Psi\psi)) + \sum_{\chi\in\{\chi_1,\chi_1\mu\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\mathbf{1}_\ell,\phi'_\ell\}}2\big(-\chi\Psi\psi(P_p)\big)^{\delta'}$$
$$\times\Delta_{c'}(0,_{f'''}(\chi\Psi\psi)) \mod 2^{6+n}\mathbf{Z}_2[i].$$

In Case 1, we showed that this second summand was congruent to zero. Letting

$$\theta_\Psi = 2\Big[\big(-\chi_q\Psi\mathbf{1}_\ell(P_p)\big)^{\delta}-\big(-\chi_q\Psi\mathbf{1}_\ell(P_p)\big)^{\delta'}\Big]$$
$$\times\big(1-\chi_q\Psi(c')\big)\big(1-\chi_q\Psi(P_\ell)\big)L\big(0,_{qm}(\chi_q\Psi)\big)$$

and noting that exactly one of δ or δ' is zero and the other 1, with $\chi_q\Psi\mathbf{1}_\ell(P_p)\in\{\pm i\}$, we find that

$$\theta_\Psi\in 2^{n+4}(1+i)\mathbf{Z}_2[i]\setminus 2^{n+5}\mathbf{Z}_2[i].$$

But then

$$\mathrm{Trace}_{\mathbf{Q}[i]/\mathbf{Q}}(\theta_\Psi)\in 2^{n+5}\mathbf{Z}_2.$$

Therefore, either $n>0$ and the sum of these $\mathrm{Trace}_{\mathbf{Q}[i]/\mathbf{Q}}(\theta_\Psi)$ are then in $2^{n+5}\mathbf{Z}_2$, or we have $n=0$ and find

$$(4.32)\qquad -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\psi\in\{\mathbf{1}_\ell,\nu'_\ell\}}\Delta_{c'}(0,_{f''}(\chi\psi))\equiv 2^5 \mod 2^6\mathbf{Z}_2[i].$$

We can now complete the proof of Proposition 4.4. By combining Lemma 4.2 and Lemma 4.3 with Equation 4.26,

$$(4.33)\qquad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\Psi\in S_m}\sum_{\psi\in\{\phi'_\ell,\phi''_\ell\}}\Delta_c(0,_{f_m}(\chi\Psi\psi))$$
$$\equiv 2^{2n+4}\big(1+\phi'_\ell(\bar P_\ell\bar P_q)\big)+2^{2n+4}\big(1-\phi'_\ell(P_pP_r)\big)$$
$$\equiv 2^{2n+4}\big(1+\phi'_\ell(P_pP_r\bar P_\ell\bar P_q)\big)$$
$$\equiv 2^{2n+5} \mod 2^{n+6}\mathbf{Z}_2[i] \text{ (by Lemma 2.3)}.$$

Combining Propositions 4.1 through 4.4, we find that

$$0 \equiv \sum_{d \mid ml} 2^{\omega(d)} L\left(0, V\left[\frac{ml}{d}\right]\right) + 0 + 0 + 2^{2n+5} \mod 2^{n+6}\mathbf{Z}_2[i].$$

Since Example 3.1 shows that all of the terms are rational integers, the congruence holds true modulo $2^{n+6}\mathbf{Z}$. As $L(0, V)$ is exactly divisible by 2^3, the case of $n = 0$ follows immediately. When $n > 0$, we solve for $L(0, V[m\ell])$ using

$$-\left(2^{\omega(d)} L\left(0, V\left[\frac{m\ell}{d}\right]\right)\right) \equiv 3\left(2^{\omega(d)} L\left(0, V\left[\frac{m\ell}{d}\right]\right)\right) \mod 2^{n+6}\mathbf{Z},$$

which follows from the divisibility results of Example 3.1.

5. The proof of Theorem II

Recall that we are now in the context of

$$F = \mathbf{Q}(\sqrt{p}, \sqrt{q}), \qquad p \equiv q \equiv 5 \mod 8, \qquad \left(\frac{p}{q}\right) = 1,$$
$$\left(\frac{p}{q}\right)_4 = \left(\frac{q}{p}\right)_4 = -1.$$

We also have $\ell \equiv 1 \mod 4$ such that

$$-\left(\frac{\ell}{q}\right) = \left(\frac{\ell}{p}\right) = 1 \quad \text{and} \quad \left(\frac{\ell}{p}\right)_4 = \left(\frac{p}{\ell}\right)_4 = 1.$$

Let us set $k = \mathbf{Q}(\sqrt{p})$. Recall from Section 2 that we have $S_\ell = \{\mathbf{1}_\ell, \psi'_\ell, \psi''_\ell, \psi_\ell\}$, the set of even characters of conductor dividing ℓ. We let S be the set of odd characters of k of conductor pq and order at most 4, given in Table 2.1. Let $f = P_p P_q \bar{P}_q P_\ell \bar{P}_\ell$. Then for all $c \in G_f$, the Deligne and Ribet theorem gives

$$\frac{1}{2} \sum_{\chi \in S} \sum_{\psi \in S_\ell} \Delta_c(0, {}_f(\chi\psi)) \in 2^6\mathbf{Z}_2[i]. \tag{5.1}$$

We will choose c so as to achieve the proof of our theorem.

Proposition 5.1. *Let $c \in G_f$ be such that $\chi_2(c) \in \{\pm i\}$ and $\psi_\ell(c) = 1$. Then*

$$\frac{1}{2} \sum_{\chi \in \{\chi_2, \chi_2^{-1}\}} \sum_{\psi \in \{\mathbf{1}_\ell, \psi_\ell\}} \Delta_c(0, {}_f(\chi\psi)) = L(0, V[\ell]) + 2L(0, V).$$

Proof. The same techniques as in the proof of Proposition 4.1 may be applied.

PROPOSITION 5.2. *For $c \in G_f$ as above and such that $\chi_3(c) = 1$,*

$$\frac{1}{2} \sum_{\chi \in \{\chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}} \sum_{\psi \in S_\ell} \Delta_c\big(0, {}_f(\chi\psi)\big) \in 2^5\mathbf{Z}_2[i] \setminus 2^6\mathbf{Z}_2[i].$$

Proof. Since $P_p = (\sqrt{p})$, we find that $\chi_1(P_p) = \mu(P_p) = \psi_\ell(P_p) = 1$. Therefore,

$$\begin{aligned}(5.2)\qquad \Delta_c\big(0, {}_f(\chi_1\mathbf{1}_\ell)\big) &= \Delta_c\big(0, {}_f(\chi_1\mu\mathbf{1}_\ell)\big)\\ &= \Delta_c\big(0, {}_f(\chi_1\psi_\ell)\big) = \Delta_c\big(0, {}_f(\chi_1\mu\psi_\ell)\big) = 0.\end{aligned}$$

Further, note that Lemma 2.3 implies that $\chi_1\psi'_\ell(P_p\overline{P}_q\overline{P}_\ell) = -1$. Thus,

$$\big(1 - \chi_1\psi'_\ell(P_p)\big)\big(1 - \chi_1\psi'_\ell(\overline{P}_q)\big)\big(1 - \chi_1\psi'_\ell(\overline{P}_l)\big) = 0.$$

Therefore,

$$\Delta_c\big(0, {}_f(\chi_1\psi'_\ell)\big) = 0.$$

Similarly,

$$(5.3)\qquad \Delta_c\big(0, {}_f(\chi_1\mu\psi'_\ell)\big) = \Delta_c\big(0, {}_f(\chi_1\psi''_\ell)\big) = \Delta_c\big(0, {}_f(\chi_1\mu\psi''_\ell)\big) = 0.$$

We have $(\psi'_\ell)^\sigma = \psi''_\ell$; also, $\psi_\ell(c) = 1$ gives $\psi'_\ell(c) = \psi''_\ell(c)$. Combining this with $(\chi_3)^\sigma = \chi_3$ and $(\mu)^\sigma = \mu$, we find that

$$(5.4)\quad \frac{1}{2} \sum_{\chi \in \{\chi_3, \chi_3\mu\}} \sum_{\psi \in \{\psi'_\ell, \psi''_\ell\}} \Delta_c\big(0, {}_f(\chi\psi)\big) = \sum_{\chi \in \{\chi_3, \chi_3\mu\}} \Delta_c\big(0, {}_f(\chi\psi'_\ell)\big),$$

which is in $2^6\mathbf{Z}_2$, by Lemma 2.2.

Finally, recall that our choice of c in Proposition 5.1 forced $\mu(c)$ to be -1. Thus, although $(1 - \chi_3(c)) = (1 - \chi_3\psi_\ell(c)) = 0$, we find that $(1 - \chi_3\mu(c)) = (1 - \chi_3\mu\psi_\ell(c)) = 2$. Hence, we have now shown that

$$\begin{aligned}(5.5)\quad &\frac{1}{2} \sum_{\chi \in \{\chi_1, \chi_1\mu, \chi_3, \chi_3\mu\}} \sum_{\psi \in S_\ell} \Delta_c\big(0, {}_f(\chi\psi)\big)\\ &\equiv \frac{1}{2} \sum_{\chi \in \{\chi_3, \chi_3\mu\}} \sum_{\psi \in \{\mathbf{1}_\ell, \psi_\ell\}} \Delta_c\big(0, {}_f(\chi\psi)\big)\\ &= \big(1 - \chi_3\mu(P_\ell)\big)\big(1 - \chi_3\mu(\overline{P}_\ell)\big)L(0, \chi_3\mu) + L(0, \chi_3\mu\psi_\ell).\end{aligned}$$

Now, $\chi_3\mu$ corresponds to a fourth degree Galois extension of $\mathbf{Q}$ in which only p, q, and ∞ ramify (p totally ramified). This field must be the cyclic extension of $\mathbf{Q}$ corresponding to $\lambda_p\tau_q$. Thus,

$$L(0, \chi_3\mu) = L(0, \lambda_p\tau_q)L(0, \lambda_p^3\tau_q).$$

We apply Corollary 2.3 to conclude that $L(0, \chi_3\mu) \in 2^4\mathbf{Z}_2$. Since 4 divides the value

$$(1 - \chi_3\mu(P_\ell))(1 - \chi_3\mu(\bar{P}_\ell)),$$

we have

$$(1 - \chi_3\mu(P_\ell))(1 - \chi_3\mu(\bar{P}_\ell))L(0, \chi_3\mu) \in 2^6\mathbf{Z}_2.$$

Similarly,

$$L(0, \chi_3\mu\psi_\ell) = L(0, \lambda_p\tau_q\tau_\ell)L(0, \lambda_p^3\tau_q\tau_\ell).$$

Since $\left(\frac{\ell}{q}\right) = -\left(\frac{\ell}{p}\right) = -1$, we may use Corollary 2.3 to see that

$$L(0, \chi_3\mu\psi_\ell) \in 2^5\mathbf{Z}_2[i] \setminus 2^6\mathbf{Z}_2[i],$$

as we have chosen our q such that $\left(\frac{q}{p}\right)_4 = 1$.

Thus, we have proven our proposition.

PROPOSITION 5.3. *For $c \in G_f$ as above,*

$$\frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_f(\chi\psi)) \in 2^6\mathbf{Z}_2[i].$$

Proof. Let $f' = f/P_p = P_q\bar{P}_qP_\ell\bar{P}_\ell$. Now, using $P_p = (\sqrt{p})$, we find $\chi_q(P_p) = -1$ and, for $\psi \in S_\ell$, $\psi(P_p) = 1$. Thus,

$$(5.6)\quad \frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_f(\chi\psi)) = \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_{f'}(\chi\psi)).$$

By the Deligne and Ribet theorem,

$$\sum_{\chi \in \{\chi_q, \chi_q^{-1}, \chi_1, \chi_1\mu\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_{f'}(\chi\psi)) \in 2^6\mathbf{Z}_2[i].$$

Therefore,

$$(5.7) \qquad \frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_f(\chi\psi)) = - \sum_{\chi \in \{\chi_1, \chi_1\mu\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_{f'}(\chi\psi)) \mod 2^6\mathbf{Z}[i].$$

But,

$$\chi_1(P_\ell)\chi_1(\bar{P}_\ell) = \chi_1(P_\ell)\chi_1\mu(P_\ell) = \mu(P_\ell) = -1.$$

Similarly, $\chi_1\mu(P_\ell)\chi_1\mu(\bar{P}_\ell) = -1$. Therefore,

$$L(0, {}_{f'}(\chi_1 \mathbf{1}_\ell)) = L(0, {}_{f'}(\chi_1\mu \mathbf{1}_\ell)) = 0.$$

Now, χ_1 is an odd quadratic character of k, non-Galois over $\mathbf{Q}$. Thus, Lemma 2.3 gives that $\chi_1(P_p\bar{P}_q) = -1$. Since $\chi_1(P_p) = 1$, we find $\chi_1(\bar{P}_q) = -1$. This, and $\psi_\ell(\bar{P}_q) = -1$, give

$$\left(1 - \chi_1\psi_\ell(\bar{P}_q)\right) = \left(1 - \chi_1\mu\psi_\ell(\bar{P}_q)\right) = 0.$$

Hence,

$$L(0, {}_{f'}(\chi_1\psi_\ell)) = L(0, {}_{f'}(\chi_1\mu\psi_\ell)) = 0.$$

By Lemma 2.3, $\chi_1\psi'_\ell(P_p\bar{P}_qP_\ell) = -1$. But, $\chi_1\psi'_\ell(P_p) = 1$. Therefore, $\chi_1\psi'_\ell(\bar{P}_qP_\ell) = -1$ and

$$\left(1 - \chi_1\psi'_\ell(\bar{P}_q)\right)\left(1 - \chi_1\psi'_\ell(P_\ell)\right) = 0.$$

The same holds true for $\chi_1\psi''_\ell(\bar{P}_\ell)$ replacing $\chi_1\psi'_\ell(P_\ell)$, and similarly for $\chi_1\mu\psi'_\ell(P_\ell)$ and $\chi_1\mu\psi''_\ell(\bar{P}_\ell)$. Hence,

$$L(0, {}_{f'}(\chi_1\psi'_\ell)) = L(0, {}_{f'}(\chi_1\psi''_\ell)) = L(0, {}_{f'}(\chi_1\mu\psi'_\ell)) = L(0, {}_{f'}(\chi_1\mu\psi''_\ell)) = 0.$$

Thus,

$$(5.7) \qquad \frac{1}{2} \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in S_\ell} \Delta_c(0, {}_f(\chi\psi)) \equiv - \sum_{\psi \in S_\ell} \Delta_c(0, {}_{f'}(\chi_1\psi)) \mod 2^6\mathbf{Z}_2[i],$$

which is equal to zero. Therefore, we have proved our proposition.

PROPOSITION 5.4. *For $c \in G_f$ as above,*

$$\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}}\Delta_c(0,{}_f(\chi\psi)) \equiv -4L(0,V) \mod 2^6\mathbf{Z}_2[i].$$

Proof. Let $f'' = f/\bar{P}_\ell = P_pP_q\bar{P}_qP_\ell$. Since $\chi_2(c) \in \{\pm i\}$, $\psi'_\ell(c) = \psi''_\ell(c) \in \{\pm 1\}$ and the action of the non-trivial element σ of $\mathrm{Gal}(k/\mathbf{Q})$ takes χ_2 to χ_2^{-1} and ψ'_ℓ to ψ''_ℓ, we have

$$(5.9)\qquad \begin{aligned}&\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}}\Delta_c(0,{}_f(\chi\psi))\\ &\quad= \sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\left(1-\chi\psi'_\ell(\bar{P}_\ell)\right)L(0,{}_{f''}(\chi\psi'_\ell))\end{aligned}$$

Now, let c'' in $G_{f''}$ be such that $\chi_2\psi'_\ell(c'') = \chi_2\psi'_\ell(\bar{P}_\ell)$ (note that this forces $\mu(c'')$ to be -1.) We will further restrict our choice of c'' as we proceed with our proof. Now,

$$\sum_{\chi\in S}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}\Delta_{c''}(0,{}_{f''}(\chi\psi)) \in 2^6\mathbf{Z}_2[i],$$

by the Deligne and Ribet theorem. Therefore,

$$(5.10)\qquad \begin{aligned}&\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}}\Delta_c(0,{}_f(\chi\psi))\\ &\quad= \sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}(0,{}_{f''}(\chi\psi'_\ell))\\ &\quad\equiv -\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}(0,{}_{f''}(\mathbf{1}_\ell))\\ &\qquad -\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}\Delta_{c''}(0,{}_{f''}(\chi\psi))\\ &\qquad -\sum_{\chi\in\{\chi_1,\chi_1\mu,\chi_3,\chi_3\mu\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}\Delta_{c''}(0,{}_{f''}(\chi\psi))\end{aligned}$$

But, the $\psi(P_p)$, $\chi_1(P_p)$ and $\chi_1\mu(P_p)$ are equal to 1. Since χ_3 corresponds to a cyclic order 4 character of $\mathbf{Q}$ of conductor p, we find that $\chi_3(P_q) =$

$\chi_3(P_\ell) = 1$. Also, $\chi_3\psi'_\ell(P_q)\chi_3\psi'_\ell(\bar{P}_q) = \mu(P_q) = -1$. Recall that $L(0, \chi_3\mu) \in 2^4\mathbf{Z}_2$, hence

$\Delta_{c''}(0, {}_{f''}(\chi_3\mu\mathbf{1}_\ell)) \in 2^6\mathbf{Z}_2$. If we now restrict our choice of c'' such that $\chi_3\mu\psi'_\ell(c'') = 1$, then

$$(5.11)\quad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}}\Delta_c\big(0,{}_f(\chi\psi)\big)$$
$$\equiv -\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}\big(0,{}_{f''}(\chi\mathbf{1}_\ell)\big)$$
$$-\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}\Delta_{c''}\big(0,{}_{f''}(\chi\psi)\big)$$
$$\equiv -\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}\big(0,{}_{f''}(\chi\mathbf{1}_\ell)\big)$$
$$-\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}\big(1-\chi\psi(P_p)\big)\Delta_{c''}\big(0,{}_{f'''}(\chi\psi)\big)$$
$$\big(\text{for } f''' = f''/P_p = P_q\bar{P}_qP_\ell\big)$$
$$\equiv -\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}\big(0,{}_{f''}(\chi\mathbf{1}_\ell)\big)$$
$$-\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}2\Delta_{c''}\big(0,{}_{f'''}(\chi\psi)\big)$$

as $\chi_q(P_p) = -1 = -\psi(P_p)$.

Since

$$\sum_{\chi\in\{\chi_q,\chi_q^{-1},\chi_1,\chi_1\mu\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}2\Delta_{c''}\big(0,{}_{f'''}(\chi\psi)\big)\in 2^6\mathbf{Z}_2[i],$$

we have

$$(5.12)\quad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}}\Delta_c\big(0,{}_f(\chi\psi)\big)$$
$$\equiv -\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}\big(0,{}_{f''}(\chi\mathbf{1}_\ell)\big)$$
$$+\sum_{\chi\in\{\chi_1,\chi_1\mu\}}\sum_{\psi\in\{\mathbf{1}_\ell,\psi'_\ell\}}2\Delta_{c''}\big(0,{}_{f'''}(\chi\psi)\big).$$

By setting $\chi_1(c'') = \chi_1(\bar{P}_\ell)[= -\chi_1(P_\ell)]$, one has $(1 - \chi_1(c''))(1 - \chi_1(P_\ell)) = 0$. That is,

$$\Delta_{c''}\left(0, {}_{f'''}(\chi_1 \mathbf{1}_\ell)\right) = 0.$$

Since $\mu(c'') = -1$ and $\chi_1\mu(P_\ell) = -\chi_1(P_\ell)$, we also find $\Delta_{c''}(0, {}_{f'''}(\chi_1\mu\mathbf{1}_\ell)) = 0$. Now,

$$(1 - \chi_1\psi'_\ell(c''))\left(1 - \chi_1\psi'_\ell\left(\bar{P}_q\right)\right) = \left(1 - \chi_1\left(\bar{P}_\ell\right)\psi'_\ell(c'')\right)\left(1 - \chi_1\psi'_\ell\left(\bar{P}_q\right)\right.$$

and Lemma 2.3 gives $\chi_1\psi'_\ell(P_p\bar{P}_q\bar{P}_\ell) = -1$, i.e. that $\chi_1\psi'_\ell(\bar{P}_q\bar{P}_\ell) = -1$. Let us choose c'' such that $\psi'_\ell(c'') = \psi'_\ell(\bar{P}_\ell)$. Hence, $\Delta_{c''}(0, {}_{f'''}(\chi_1\psi'_\ell)) = 0$. We also find $\Delta_{c''}(0, {}_{f'''}(\chi_1\mu\psi'_\ell)) = 0$. Therefore,

$$\begin{aligned}(5.13)\qquad \frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}} &\Delta_c\left(0, {}_f(\chi\psi)\right)\\ &\equiv -\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\Delta_{c''}\left(0, {}_{f''}(\chi\mathbf{1}_\ell)\right)\\ &= -\left[(1-\chi_2(c''))(1-\chi_2(P_\ell))\right.\\ &\qquad\left.+(1-\chi_2^{-1}(c''))(1-\chi_2^{-1}(P_\ell))\right]L(0,\chi_2).\end{aligned}$$

We have chosen c'' such that $\chi_2\psi'_\ell(c'') = \chi_2\psi'_\ell(\bar{P}_\ell)$ and $\psi'_\ell(c'') = \psi'_\ell(\bar{P}_\ell)$. Thus,

$$\chi_2(c'') = \chi_2\left(\bar{P}_\ell\right) = \chi_2^{-1}(P_\ell).$$

Therefore,

$$\frac{1}{2}\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\psi\in\{\psi'_\ell,\psi''_\ell\}}\Delta_c\left(0, {}_f(\chi\psi)\right) \equiv -4L(0,\chi_2) = -4L(0,V).$$

Combining Propositions 5.1–5.4 and using that $L(0,V)$ is exactly divisible by 2^3, we find that

$$L(0,V[\ell]) \equiv 6L(0,V) \mod 2^6\mathbf{Z}_2.$$

By example 3.2, both of the above L-values are rational integers, thus we have proved our theorem.

6. Governing fields

Let F be a biquadratic extension of $\mathbf{Q}$ with odd class number. Fix a complex quaternion extension N of $\mathbf{Q}$ containing F. Let S be the set of primes of N ramified over $\mathbf{Q}$. Let S' be the corresponding set of primes of F, and S'' that of $\mathbf{Q}$.

Let A be the set of all rational primes with a given unramified splitting configuration to F and of a given residue modulo $4\mathbf{Z}$. For ℓ in A, consider $N[\ell]$, a complex quaternion extension of $\mathbf{Q}$ containing F and ramified at $S[\ell]$, the set of primes dividing ℓ and S''. Let t be the cardinality of $S[\ell]$ and let T be $2+[t/2]$, where $[x]$ denotes the integer part of x. Let $L(s, V[\ell])$ be the Artin L-function of the unique irreducible two-dimensional representation of the Galois group of $N[\ell]$ over $\mathbf{Q}$.

Let K be the maximal abelian unramified extension of F to which all of the primes of S' split. Let K' be the fixed field of the maximal subgroup of $Gal(K/F)$ of order powers of primes congruent to 1 or 7 mod $8\mathbf{Z}$. Let H_S be the field fixed by the unique subgroup of $Gal(K'/\mathbf{Q})$ of order 4.

Let $f_S(\ell)$ be the class of the $S[\ell]$-Class group of $N[\ell]$, $Cl_{S[\ell]}N[\ell]$, in $Cl(\mathbf{Z}[H_8])$, the finite torsion subgroup of the Grothendieck group of finitely generated $\mathbf{Z}[H_8]$-modules of finite projective dimension.

PROPOSITION 6.1. *If*
(i) $L(0, N[\ell])/2^T$ *is odd and*
(ii) $L(0, N[\ell])/2^T$ mod $4\mathbf{Z}$ *is a constant function of* ℓ,
then H_S *is a minimal governing field for* $f_S(\ell)$.

Proof. Let χ_+ (respectively χ_-) be the non-trivial even (resp. odd) quadratic Dirichlet character of conductor 8 (resp. 4). Let $W_{N[\ell]/\mathbf{Q}}$ be the Artin root number of the two-dimensional irreducible representation $V[\ell]$ of $Gal(N[\ell]/\mathbf{Q})$. Furthermore, let $Cl_{S[\ell]}F$ be the $S'[\ell]$-Class group of F. Since $Cl(\mathbf{Z}[H_8])$ is a group of order 2, we identify it in the natural manner with $\{1, -1\}$. From our assumption (i), Chinburg [Ch2; Proposition 4.3.7] gives that the image of the class of $Cl_{S[\ell]}N[\ell]$ in $Cl(\mathbf{Z}[H_8])$ is equal to

$$\chi_+(Cl_{S[\ell]}F)\chi_-(L(0, V[\ell])/2^T)W_{N[\ell]/\mathbf{Q}}.$$

From our assumption that all ℓ in A have the same residue modulo $4\mathbf{Z}$, results of Fröhlich [F1] give that $W_{N[\ell]/\mathbf{Q}}$ is constant.

From our assumption (ii), $\chi_-(L(0, V[\ell])/2^T)$ is constant.

Classfield theory gives that the image of the Artin map for K over F of the primes of F above ℓ determines $Cl_{S[\ell]}F$. A restriction to K' over F determines $\chi_+(Cl_{S[\ell]}F)$. Standard density results show that K' is minimal for this property. However, for all ℓ in A, the splitting configuration of ℓ to F is known. Since K' is the composition of F and H_S, H_S is indeed a minimal governing field for $f_S(\ell)$.

It is now clear that Corollary I and Corollary II follow from the above results.

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