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AMENABLE HYPERGROUPS

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1. Introduction

The purpose of this paper is to initiate a systematic study of amenable hypergroups. The theory of hypergroups was initiated by Dunkl [13], Jewett [28] and Spector [49] and has received a good deal of attention from harmonic analysts. Hypergroups naturally arise as double coset spaces of locally compact groups by compact subgroups. In [42], Pym also considers convolution structures which are close to hypergroups. A fairly complete history is given in Ross' survey article [45].

Throughout, K will denote a hypergroup with a left Haar measure λ. It is still unknown if an arbitrary hypergroup admits a left Haar measure, but all the known examples such as commutative hypergroups [50] and central hypergroups [24] do.

Let $L_\infty(K)$ be the Banach space of all bounded Borel measurable functions on K with the essential supremum norm. A left invariant mean on $L_\infty(K)$ is a positive linear functional of norm one, which is invariant under left translations by elements in K. K is said to be amenable if there is a left invariant mean on $L_\infty(K)$.

Section 2 consists of notations used throughout this paper.

In Section 3, we give examples and discuss stability properties of amenable hypergroups. In contrast to the result of Granirer [21] and Rudin [47] for the group case, we exhibit a class of commutative hypergroups K for which every invariant mean on $L_\infty(K)$ is topologically invariant.

In Section 4, Reiter's condition (P_1) is shown to characterize amenability of hypergroups. It is also shown that, if a hypergroup satisfies (P_2), then it has property (P_1), and that the converse is not true in general. This is again in contrast to the group case.

In [33], Lau introduced and studied a class of Banach algebras which include $L_1(K)$. He called such algebras F-algebras. He extended several important characterizations of amenable locally compact groups to left

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amenable F-algebras. The F-algebra $L_1(K)$ is amenable if and only if K is amenable.

Let G be a non compact amenable locally compact group, and let $\mathrm{TLIM}(L_\infty(G))$ be the set of all topological left invariant means on $L_\infty(G)$. Lau and Paterson proved in [36, Theorem 1] that $|\mathrm{TLIM}(L_\infty(G))| = 2^{2^d}$, where d is the smallest cardinality of a cover of G by compact sets. Later, Yang improved their result in [57]. In fact, he showed in [57, Corollary 3.4] that

$$|\mathrm{TLIM}(L_\infty(G))| = |\mathrm{TIM}(L_\infty(G))| = |\mathrm{TIIM}(L_\infty(G))| = 2^{2^d}.$$

Earlier references on the subject include [8], [18], [5], [6], [7], [20] and [35]. Inspired by these, we prove in Section 5 that

$$|\mathrm{TIM}(L_\infty(K))| = |\mathrm{TIIM}(L_\infty(K))| = 2^{2^d},$$

for an arbitrary amenable noncompact hypergroup, where d is defined exactly as before. We also show that, if the maximal subgroup $G(K)$ of K is open, then $|\mathrm{TLIM}(L_\infty(K))| = 2^{2^d}$. Finally, we give some applications of these theorems.

2. Preliminaries

Throughout this paper, K will denote a hypergroup (Same as convo in Jewett [28]) with a fixed left Haar measure λ. Unless otherwise specified, our notation will follow that of [28]. The following notations are different from those in [28]:

δ_x	The point mass at $x \in K$
1_A	The characteristic function of the non empty set $A \subseteq K$
$\mathrm{cl}\,A$ or $\bar{A}$	The closure of the set $A \subseteq K$
$C(K)$	The bounded continuous complex valued functions on K
$\Vert f\Vert_\infty$	$\mathrm{Sup}\,\vert f(x)\vert$

The involution on K is denoted by $x \to \check{x}$.

If f is a Borel function on K and $x, y \in K$, the left translation ${}_xf$ or L_xf and the right translation f_y or R_yf are defined by

$$L_xf(y) = {}_xf(y) = f_y(x) = R_y(x) = \int_K f d\delta_x * \delta_y = f(x * y),$$

if the integral exists. The functions $\check{f}, \tilde{f}$ are given by $\check{f}(x) = f(\check{x})$, $\tilde{f}(x) = \overline{f(\check{x})}$, respectively. The integral $\int \dots d\lambda(x)$ is often denoted by $\int \dots dx$. Let $(L_p(K), \|\cdot\|_p)$, $1 \le p \le \infty$, denote the usual Banach spaces of Borel functions [28, 6.2]. For $f \in L_1(K)$, we write

$$f^{\#}(x) = \overline{f(\check{x})}\,\Delta(\check{x}),$$

where Δ is the modular function on K. Then $\|f^{\#}\|_1 = \|f\|_1$. If $f \in L_p(K)$, $1 \le p \le \infty$, $x \in K$, then $\|_x f\|_p \le \|f\|_p$, and this is in general not an isometry [28, 3.3]. The mapping $x \to {}_x f$ is continuous from K to $(L_p(K), \|\cdot\|_p)$, $1 \le p < \infty$ [28, 2.2B and 5.4H]. For $f \in L_p(K)$, $x \in K$, write

$$f \odot \delta_x = f_{\check{x}}\,\Delta(\check{x})$$

(Note that $f \odot \delta_x$ is the same as $f * \delta_x$ in [25, §20]). Then it is easy to see that, for $f \in L_p(K)$, $1 \le p \le \infty$, $x \in K$, f_x, $f \odot \delta_x \in L_p(K)$ with $\|f_x\| \le \|f_x\|_p\, \Delta(\check{x})^{1/p}$ and $\|f \odot \delta_x\| \le \|f\|_p$. Also, if $1 \le p < \infty$, the mappings $x \to f_x$ and $x \to f \odot \delta_x$ from K to $L_p(K)$ are continuous (see [25, 20.4]).

The proof of the next lemma is similar to the group case (see [25, 20.15] and [28, §5]).

LEMMA 2.1. *Let $\mathscr{U}$ be the family of all neighbourhoods of e and regard $\mathscr{U}$ as a directed set in the usual way: $U \ge V$ if $U \subseteq V$. For each $U \in \mathscr{U}$, choose a function $\phi_U \in C_c^+(K)$ such that $\int_K \phi_U(x)\,dx = 1$ and ϕ_U vanishes outside U. Then $\{\phi_U\}_{U \in \mathscr{U}}$ is a bounded approximate identity for $L_1(K)$.*

Let $G(K) = \{x \in K\colon\ \delta_x * \delta_{\check{x}} = \delta_{\check{x}} * \delta_x = \delta_e\}$. Then $G(K)$ is a (closed) subhypergroup of K and a locally compact group [28, 10.4C]. It is called the *maximal subgroup* of K. For each $x \in K$ and $y \in G(K)$, there exists a unique $z \in K$ such that $\delta_x * \delta_y = \delta_z$ [28, 10.4B]. We write $z = xy$.

Let H be a compact subhypergroup of K. Then the space $K//H$ of double cosets of H in K is a hypergroup under the convolution defined by

$$\int_{K//H} f\, d\delta_{HxH} * \delta_{HyH} = \int_H f \circ \pi(x * t * y)\, dt, \qquad f \in C_c(K//H),\ x, y \in K,$$

where π is the natural projection of K onto $K//H$ [28, §14].

We next recall the definition of hypergroup joins which we use very often in this paper. Let H be a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$, where e is the identity of both hypergroups. Let $K = H \cup J$ have the unique topology for which both H and J are closed subspaces of K. Let σ be the normalized Haar measure on H. Define the operation $\cdot$ on K

as follows:

(i) If $s, t \in H$, then $\delta_s \cdot \delta_t = \delta_s * \delta_t$;
(ii) If $a, b \in J$, $a \neq b$, then $\delta_a \cdot \delta_b = \delta_a * \delta_b$;
(iii) If $s \in H$, $a \in J(a \neq e)$, then $\delta_s \cdot \delta_a = \delta_a \cdot \delta_s = \delta_a$;
(iv) If $a \in J$, $a \neq e$, and $\delta_{\check{a}} * \delta_a = \sum_{b \in J} c_b \delta_b$, the c_b's are non-negative, only finitely many are non-zero and $\sum_{b \in J} c_b = 1$, then

$$\delta_{\check{a}} \cdot \delta_a = c_e \sigma + \sum_{b \in J \setminus \{e\}} c_b \delta_b.$$

We call the hypergroup K the *join* of H and J, and write $K = H \vee J$. Observe that H is a subhypergroup of K, but J is not a subhypergroup unless J or H is equal to $\{e\}$. The hypergroup $K = H \vee J$ always has a left Haar measure, as shown by Vrem. In fact,

$$\sigma + \sum_{x \in J \setminus \{e\}} [x]\delta_x, \quad \text{where } [x] = \frac{1}{\delta_{\check{x}} * \delta_x(\{e\})},$$

is a left Haar measure on K [53, Proposition 1.1]. He has also showed in [53, Proposition 1.3] that $K//H \cong J$ as hypergroups.

Let K be a hypergroup, and write

$$UC_r(K) = \{f \in C(K) \colon x \to {}_x f \text{ is continuous from } K \text{ to } (C(K), \|\cdot\|_\infty)\},$$
$$UC_l(K) = \{f \in C(K) \colon x \to f_x \text{ is continuous from } K \text{ to } (C(K), \|\cdot\|_\infty)\},$$

and

$$UC(K) = UC_r(K) \cap UC_l(K).$$

Functions in $UC_r(K)$ $[UC_l(K)]$ are called *bounded right* [*left*] *uniformly continuous*, and functions in $UC(K)$ are said to be *uniformly continuous*.

A subset $X \subseteq L_\infty(K)$ is called *left* [*right*] *translation invariant* if ${}_x f \in X$ $[f_x \in X]$ for all $f \in X$, $x \in K$. Both $C(K)$ and $L_\infty(K)$ are (two-sided) translation invariant [28, 3.1B and 6.2B]. Note that $C(K)$ is a norm closed subspace of $L_\infty(K)$ in a natural way.

LEMMA 2.2. *Each of the spaces $UC(K), UC_r(K), UC_l(K)$ is a norm closed, conjugate closed, translation invariant subspace of $C(K)$ containing the constants and the continuous functions vanishing at infinity. Furthermore,*
(i) $UC_r(K) = L_1(K) * UC_r(K) = L_1(K) * L_\infty(K)$,
(ii) $UC_l(K) = UC_l(K) * L_1(K)^v = L_\infty(K) * L_1(K)^v$
(iii) $UC_r(K) * L_1(K) \subseteq UC_r(K)$ and $L_1(K) * UC_l(K) \subseteq UC_l(K)$;
(iv) $UC(K) = L_1(K) * UC(K) = UC(K) * L_1(K)^v$.

Proof. Let $\phi \in L_1(K)$, $f \in L_\infty(K)$. Then

$$|\phi * f(x) - \phi * f(y)| = \left| \int_K [{}_x\phi(u) - {}_y\phi(u)] f(\check{u})\, du \right|$$
$$\leq \|f\|_\infty \|{}_x\phi - {}_y\phi\|_1.$$

Since $x \to {}_x\phi$ is continuous from K to $L_1(K)$, $\phi * f$ is continuous. Now,

$$\|\phi * f\|_\infty \leq \|\phi\|_1 \|f\|_\infty$$

and

$$\|\delta_x * (\phi * f) - \delta_y * (\phi * f)\|_\infty \leq \|f\|_\infty \|\delta_x * \phi - \delta_y * \phi\|_1.$$

Thus, $\phi * f \in UC_r(K)$. Then $UC_r(K)$ becomes a Banach left $L_1(K)$-module. Let $\varepsilon > 0$ and $f \in UC_r(K)$ be given. Choose a neighbourhood V of e such that $\|\delta_x * f - f\|_\infty < \varepsilon$ for all $x \in V$. Let ϕ_V be a non-negative function in $L_1(K)$ such that $\|\phi_V\|_1 = 1$ and ϕ_V vanishes outside V. Then,

$$\|\phi_V * f - f\|_\infty \leq \varepsilon.$$

Hence, $L_1(K) * UC_r(K)$ is norm dense in $UC_r(K)$. Since $L_1(K)$ has a bounded approximate identity, by Cohen's factorization theorem [26, 32.22], we have

$$L_1(K) * UC_r(K) = UC_r(K).$$

This proves (i). If $f \in UC_r(K)$, write $f = \phi * h$ where $\phi \in L_1(K)$, $h \in UC_r(K)$. Then, for $x \in K$, $\delta_x * f = (\delta_x * \phi) * h \in UC_r(K)$. So, $UC_r(K)$ is left translation invariant, and it is easily seen to be right translation invariant. To see (ii), note that, if $f \in C(K)$, then $f \in UC_r(K)$ if and only if $\check{f} \in UC_l(K)$. (iii) and (iv) are similar or easy to prove. Finally, by [28, 2.2B and 4.2F], $C_0(K) \subseteq UC(K)$.

Remarks 2.3. (a) Let $f \in C(K)$ be such that $x \to {}_xf$ is continuous from K to $(C(K), \|\cdot\|_\infty)$ at the identity $x = e$. Then $f \in UC_r(K)$. Indeed, if $\{\phi_U\}_{U \in \mathcal{U}}$ is the bounded approximate identity for $L_1(K)$, as in 2.1, then $\{\phi_U * f\}_{U \in \mathcal{U}}$ converges to f in the $\|\cdot\|_\infty$-norm.

(b) If the maximal subgroup $G(K)$ is open in K, then $UC_r(K)$ is an algebra. To see this, let, $f, g \in UC_r(K)$, $x \in G(K)$, $y \in K$. Then,

$${}_x(fg)(y) - (fg)(y) = {}_xf(y)[{}_xg(y) - g(y)] + g(y)[{}_xf(y) - f(y)],$$

since $\delta_x * \delta_y = \delta_{xy}$. Thus, $x \to {}_x(fg)$ is continuous at e, and hence by (a), $fg \in UC_r(K)$.

(c) If K is compact or discrete, then $C(K) = UC(K)$.

(d) If the maximal subgroup $G = G(K)$ of a hypergroup K is open, non-discrete and non-compact, then $UC_r(K) \neq C(K)$. In fact, let $f \in C(G)$, $f \notin UC_r(G)$ [39, Problem 1.3]. Let $\tilde{f}$ be the function on K given by $\tilde{f} = f$ on G and zero otherwise. Then $\tilde{f} \in C(K)$, but $\tilde{f} \notin UC_r(K)$.

(e) Ross has shown in [52, Theorem A.6] that, if K is non-discrete, then $L_\infty(K) \neq C(K)$.

The next result is in contrast to the group case [39, Problem 1.3].

Proposition 2.4. *Let $K = H \vee J$, where H is a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$. Then, $C(K) = UC(K)$.*

Proof. Let $f \in C(K)$ and write $g = f|_H$. Then $g \in C(H) = UC(H)$ because H is compact. If $x \in H$, then

$$ {}_xf(y) - f(y) = \begin{cases} {}_xg(y) - g(y), & y \in H \\ 0, & y \in J \setminus \{e\}. \end{cases} $$

Since H is open in K, the mapping $x \to {}_xf$ (and similarly $x \to f_x$) is continuous at e from K to $(C(K), \|\cdot\|_\infty)$. By 2.3(a), $f \in UC(K)$.

3. Amenable hypergroups: Examples and stability properties

In this section, we give some important examples and discuss stability properties of amenable hypergroups.

Let K be a hypergroup with a left Haar measure λ, and let X be one of the spaces $UC(K)$, $UC_r(K)$, $C(K)$ or $L_\infty(K)$. A linear functional m on X is called a *mean* if:

(i) $m(\bar{f}) = \overline{m(f)}$ for all $f \in X$;

(ii) $f \geq 0$ implies $m(f) \geq 0$ ($f \geq 0$ loc. λa · e implies $m(f) \geq 0$) and $m(1) = 1$.

It is easy to see that a linear functional m on X is a mean if and only if $m(1) = \|m\| = 1$ and thus the set $\Sigma(X)$ of all means on X is a non-empty weak* compact convex set in X^* (see [40, p. 23–27]). A mean m on X is called a *left invariant mean* [LIM] if $m({}_xf) = m(f)$ for all $f \in X$, $x \in K$. A hypergroup K is called *amenable* if there is a LIM on $C(K)$. A *right invariant mean* [RIM] on X is a mean such that $m(f_x) = m(f)$ for all $x \in K$, $f \in X$. Let

$$ P(K) = \{\phi \in L_1(K) : \phi \geq 0, \|\phi\|_1 = 1\} $$

and

$$P_c(K) = P(K) \cap C_c(K).$$

A mean m on X is said to be a *topological left* (*right*) *invariant mean* [TLIM] ([TRIM]) if

$$m(\phi * f) = m(f)\ \big(m(f * \check{\phi}) = m(f)\big) \quad \text{for all } \phi \in P(K),\ f \in X.$$

A mean m on X $(= UC(K), C(K), L_\infty(K))$ is called *inversion invariant* if $m(\check{f}) = m(f)$ for all $f \in X$. Note that, if an inversion invariant mean is one sided invariant, then it is automatically two sided invariant. We denote the set of all [topological] left invariant means on X by LIM(X) [TLIM(X)]. The sets IM(X), TIM(X), IIM(X) and TIIM(X) are similarly defined. For example, TIIM(X) is the set of all topological invariant and inversion invariant means on $X(= UC(K), C(K)$ or $L_\infty(K))$.

LEMMA 3.1. (i) *Every TLIM on X is a LIM;*
(ii) *If $X = UC_r(K)$ or $UC(K)$, then every LIM on X is also a TLIM, and every RIM on $UC(K)$ is a TRIM.*

Proof. (i) Let m be a TLIM on X. Since the modular function Δ is constant on $\{x\} * \{y\}$ with value $\Delta(x)\Delta(y)$ for all $x, y \in K$, it follows that $\phi *_x f = (\phi \odot \delta_{\check{x}}) * f$ for $f \in X$, $\phi \in P(K)$. Also $\phi \odot \delta_{\check{x}} \in P(K)$. Hence,

$$m({}_x f) = m(\phi *_x f) = m((\phi \odot \delta_{\check{x}}) * f) = m(f).$$

(ii) Let m be a LIM on $X(= UC_r(K)$ or $UC(K))$ and $\phi \in P_c(K)$. Since the mapping $x \to \delta_x * f (f \in X)$ is continuous from K to $(C(K), \|\cdot\|_\infty)$ and the point evaluation functionals in X^* separate points of X, we have

$$\phi * f = \int_K (\delta_x * f)\phi(x)\, dx.$$

Thus,

$$\langle m, \phi * f\rangle = \langle m, \int_K (\delta_x * f)\phi(x)\, dx\rangle$$

$$= \int_K \langle m, \delta_x * f\rangle \phi(x)\, dx = \langle m, f\rangle.$$

Hence, $m(\phi * f) = m(f)$ for all $\phi \in P(K)$, $f \in X$, by the density of $P_c(K)$ in $P(K)$. The rest of the proof of (ii) is similar. □

THEOREM 3.2. *Let X be one of the spaces $UC(K)$, $UC_r(K)$, $C(K)$ or $L_\infty(K)$. Then K is amenable if and only if* $\mathrm{LIM}(X) \neq \phi[\mathrm{TLIM}(X) \neq \phi]$. *In this case,* $\mathrm{IM}(X) \neq \phi$ *[and thus* $\mathrm{TIM}(X) \neq \phi$]. *Also,* $\mathrm{TIIM}(X) \neq \phi$ *for $X = UC(K)$, $C(K)$ or $L_\infty(K)$.*

Proof. If K is amenable, let m be a LIM on $UC(K)$ and n a RIM on $C(K)$. Define

$$F(x) = \langle m, f_x \rangle \quad \text{for } f \in UC(K),\ x \in K.$$

Then $F \in C(K)$.

Next, put

$$\langle m_1, f \rangle = \langle n, F \rangle, \quad f \in UC(K).$$

Then m_1 is a two-sided invariant mean on $UC(K)$. Indeed, since

$$\langle m, ({}_yf)_x \rangle = \langle m, {}_y(f_x) \rangle = \langle m, f_x \rangle = F(x),$$

and

$$\langle m, (f_y)_x \rangle = \int_K \langle m, f_u \rangle \, d\delta_x * \delta_y(u) = \int_K F(u) \, d\delta_x * \delta_y(u) = F_y(x),$$

we have

$$\langle m_1, {}_yf \rangle = \langle n, F \rangle = \langle m_1, f \rangle$$

and

$$\langle m_1, f_y \rangle = \langle n, F_y \rangle = \langle n, F \rangle = \langle m_1, f \rangle.$$

Hence, m is a TIM on $UC(K)$ by the previous lemma. Let E be a compact symmetric neighbourhood of e, and put

$$\phi_0 = \frac{1}{\lambda(E)} 1_E \in P(K).$$

Then $\phi_0 * f * \phi_0 \in UC(K)$ for all $f \in L_\infty(K)$. Write

$$M(f) = m_1(\phi_0 * f * \phi_0), \qquad f \in L_\infty(K).$$

Then M is a TIM on $L_\infty(K)$ (see [40, §4]). Finally, note that if M is a TIM on X $(= UC(K), C(K), L_\infty(K))$ then $\frac{1}{2}(M + \check{M})$ is a TIM on X. □

Example 3.3. (a) Commutative hypergroups are amenable. This is a consequence of the Markov-Kakutani fixed point theorem and shown in [33, p. 168].

(b) Compact hypergroups are amenable: The normalized Haar measure is a unique LIM on $C(K)$. It is the unique TLIM on $L_\infty(K)$, and it is also inversion invariant.

Example 3.4. Let K be a hypergroup such that $\{x\}*\{y\}$ is finite for all $x, y \in K$. Write K_d for K when it is equipped with the discrete topology. In this case, the discrete measures $\sum_{i=1}^{\infty}\alpha_i\delta_{x_i}$, $x_i \in K$, $\{\alpha_i\}$ a sequence of complex numbers such that $\sum_{i=1}^{\infty}|\alpha_i| < \infty$ form a closed self adjoint subalgebra of $M(K)$. Hence, the convolution in $M(K)$ naturally induces a hypergroup structure on K_d. Such hypergroups include double coset spaces of locally compact groups by finite subgroups. We say that K is *amenable as a discrete hypergroup* if K_d is amenable. In this case, K is clearly amenable. Also every LIM on $C(K_d)$ is a TLIM.

It is Rickert [44] who initially proved that a closed subgroup of an amenable locally compact group is amenable. There are many other proofs available in the literature now. (See [22], [27] and [43]). We show below that every (closed) subgroup of an amenable hypergroup is amenable. The proof here applies Reiter's methods [43, Ch. 8, 5.5(i)].

Let H be a subgroup of the hypergroup K. Let F be a non-negative continuous function on K such that:

(i) For each $x \in X$, there exists $t \in h$ such that $F(xt) > 0$;

(ii) If $W \subseteq K$ is compact, then F coincides on WH with some function $\psi \in C_c^+(K)$ [24, Lemma 1.2].

Write $F_1(x) = \int_H F(xt)\,dt$ $(x \in K)$. Then the integral exists and is positive. Also F_1 is continuous. For, let W be a compact neighbourhood of $x \in K$ and ψ as in (ii) above. Then, for $y \in W$, we have

$$|F_1(x) - F_1(y)| = \left|\int_{H\cap \check{W}*\operatorname{spt}\psi} \psi(xt)\,dt - \int_{H\cap \check{W}*\operatorname{spt}\psi} \psi(yt)\,dt\right|.$$

If $\varepsilon > 0$ is given, then by [28, 2.2B and 4.2F], we can find a neighborhood V of x contained in W such that

$$\|{}_x\psi - {}_y\psi\|_\infty\, \sigma\big(H \cap \check{W} * \operatorname{spt}\psi\big) < \varepsilon \quad \text{for } y \in V,$$

where σ is a fixed left Haar measure on H. This shows that F_1 is continuous.

Next put

$$\beta(x) = \frac{F(x)}{F_1(x)}, \qquad x \in K.$$

Then:

(i) $\int_H \beta(xt)\,dt = 1$;

(ii) If $W \subseteq K$ is compact, then β coincides on WH with some $\psi \in C_c^+(K)$.

β is called a *Bruhat function for H* [43, p. 163].

PROPOSITION 3.5. *Every subgroup H of an amenable hypergroup K is amenable. In particular, the maximal subgroup $G(K)$ is amenable.*

Proof. Let β be a Bruhat function for H. For $\phi \in C(H)$, put

$$f_\phi(x) = \int_H \beta(\check{x}t)\phi(t)\,dt, \qquad x \in K.$$

Then f_ϕ is continuous (this can be proved as above) and $\|f_\phi\|_\infty \le \|\phi\|_\infty$. It is easy to check that $({}_hf)_\phi = {}_h(f_\phi)$ for $h \in H$. Let m be a LIM on $C(K)$. Define

$$\langle m^1, \phi\rangle = \langle m, f_\phi\rangle, \qquad \phi \in C(H).$$

Them m^1 is a LIM on $C(H)$. □

A subgroup H of K is called *normal* if $xH = Hx$ for all $x \in K$. Let H be a normal subgroup of K, and let K/H be the set of all cosets xH, $x \in K$, equipped with the quotient topology with respect to the natural projection $p(x) = xH$. Then K/H becomes a hypergroup under the convolution

$$\int_{K/H} f\,d\delta_{xH} * \delta_{yH} = \int_K f \circ p\,d\delta_x * \delta_y, \qquad x, y \in K, \quad f \in C_c(K/H).$$

PROPOSITION 3.6. *K is amenable if and only if both H and K/H are amenable.*

Proof. Let m be a LIM on $C(K)$, and write

$$\langle M, f\rangle = \langle m, f \circ p\rangle, \qquad f \in C(K/H).$$

We have

$$\begin{aligned} {}_x(f \circ p)(y) &= \int_K f \circ p\,d\delta_x * \delta_y = \int_{K/H} f\,d\delta_{xH} * \delta_{yH} \\ &= {}_{xH}f \circ p(y) \quad \text{for all } x, y \in K, f \in C(K/H) \quad \text{(See [28, §2])}. \end{aligned}$$

Hence, M is a LIM on $C(K/H)$, and so K/H is amenable. Conversely, let m_1 be a LIM on $C(H)$ and m_2 a LIM on $C(K/H)$. For $f \in UC_r(K)$, write

$$f^1(x) = \langle m_1, {}_xf|_H \rangle \qquad (x \in K).$$

Then f^1 is bounded, continuous and constant on the cosets of H in K, and hence we can write $f^1 = F \circ p$, $F \in C(K/H)$. Put $\langle m, f \rangle = \langle m_2, F \rangle$. We have

$${}_{xH}F(yH) = f^1(x * y) = \int_K \langle m_1, {}_uf|_H \rangle d\delta_x * \delta_y(u) = \langle m_1, {}_y({}^{x}f)|_H \rangle,$$

since $u \to {}_uf|_H$ is continuous from K to $(C(H), \|\cdot\|_\infty)$ and the point evaluation functionals in $C(H)^*$ separate points of $C(H)$. That is, ${}_{xH}F \circ p = ({}_xf)^1$. Hence, m is a LIM on $UC_r(K)$. □

Let K be a hypergroup, and let

$$Z(K) = \{x \in K \colon \delta_y * \delta_x = \delta_x * \delta_y \text{ for each } y \in K\}.$$

K is called a *central hypergroup or a Z-hypergroup* if $K/_{Z(K) \cap G(K)}$ is compact [24]. Central hypergroups admit left Haar measures and are unimodular [24, p. 93].

COROLLARY 3.7. *Central hypergroups are amenable.*

Let J, L be hypergroups with left Haar measures. Then it is easy to see that the hypergroup $J \times L$ has a left Haar measure. The next result is a consequence of 3.6 if either J or L is a group.

PROPOSITION 3.8. *$J \times L$ is amenable if and only if both J and L are amenable.*

Proof. Let m_1 be a LIM on $UC_r(L)$ and m_2 a LIM on $C(J)$. Write

$$(f : x)(y) = f(x, y) \quad \text{for } f \in UC_r(J \times L),\ x \in J,\ y \in L.$$

Then $(f : x) \in UC_r(L)$ because $\|{}_y(f : x) - {}_{y_0}(f : x)\|_\infty \leq \|{}_{(e,y)}f - {}_{(e,y_0)}f\|_\infty$.
Let $F(x) = \langle m_1, (f : x) \rangle$ $(x \in J)$; since

$$\|(f : x) - (f : x_0)\|_\infty \leq \|{}_{(x,e)}f - {}_{(x_0,e)}f\|_\infty, \qquad F \in C(J).$$

Next, put $\langle m, f\rangle = \langle m_2, F\rangle$. We have

$$\begin{aligned}({}_{(a,b)}f : x)(y) &= {}_{(a,b)}f(x,y)\\ &= \int_J\int_L f(u,v)\, d\delta_a * \delta_x(u)\, d\delta_b * \delta_y(v)\\ &= \int_J {}_b(f : u)(y)\, d\delta_a * \delta_x(u).\end{aligned}$$

Hence, $({}_{(a,b)}f : x) = \int_J {}_b(f : u)\, d\delta_a * \delta_x(u)$, because the mapping $u \to {}_b(f : u)$ is continuous from J into $(C(L), \|\cdot\|_\infty)$ and the point evaluation functionals in $C(L)^*$ separate points of $C(L)$. Thus

$$\langle m_1, ({}_{(a,b)}f : x)\rangle = \int_J \langle m_1, {}_b(f : u)\rangle\, d\delta_a * \delta_x(u) = F_a(x).$$

So, m is a LIM on $UC_r(J \times L)$ and hence $J \times L$ is amenable.

The converse is easy. □

PROPOSITION 3.9. *If the hypergroup K is the directed union of a system of amenable subhypergroups, then K is amenable.*

Proof. See [40, Proposition 13.6]. □

PROPOSITION 3.10. *Let H be a compact subhypergroup of K. If K is amenable, so is $K//H$. If $\delta_x * \sigma = \sigma * \delta_x$ for each $x \in K$, where σ is the normalized Haar measure of H, then the converse is also true* (*cf.* [3, 2.2] *and* [30, *Remark* 2.2]).

Proof. Let m be a TLIM on $C(K)$. For $f \in C(K//H)$, write $\langle M, f\rangle = \langle m, f \circ \pi\rangle$, where π is the projection of K onto $K//H$. Let $\phi \in P_c(K//H)$. Then an easy computation shows that

$$(\phi * f) \circ \pi = \sigma * (\phi \circ \pi) * (f \circ \pi) \quad (\text{See } [28, 14.2G]).$$

Thus M is a TLIM on $C(K//H)$ because $\sigma * (\phi \circ \pi) \in P(K)$. To prove the converse, let

$$f'(x) = \int_H f(x * t)\, d\sigma(t) = f * \sigma(x), \qquad f \in C(K).$$

Then f' is continuous, bounded and constant on the cosets. Indeed, if $f \geq 0$,

$$f'(z_0) = \sup_{z \in \{x\} * H} f'(z), \quad z_0 \in \{x\} * H;$$

then since $f'(z_0 * t) = f'(z_0)$ for $t \in H$, we have $f'(u) = f'(z_0)$ for all $u \in \{z_0\} * \{t\}$. Thus, f' is constant on $\{x\} * H = \operatorname{spt} \delta_x * \sigma = \operatorname{Spt} \sigma * \delta_x = H * \{x\}$, $x \in K$. Next observe that, if $f' = F \circ \pi$, then

$$\begin{aligned} F(xH * yH) &= \int_K f' \, d\delta_x * \sigma * \delta_y \\ &= \int_K f' \, d\delta_x * \delta_y * \sigma \\ &= \int_K \int_K f'(u * t) \, d\delta_x * \delta_y(u) \, d\sigma(t) \\ &= \int_K \int_K f'(u) \, d\delta_x * \delta_y(u) \, d\sigma(t) [f'(u * t) = f'(u) \text{ for } t \in H]. \end{aligned}$$

Thus,

$$({}_x f)'(y) = {}_x F \circ \pi(y).$$

Finally, if m is a LIM on $C(K//H)$, put

$$\langle M, f \rangle = \langle m, F \rangle, f \in C(K).$$

Then, M is a LIM on $C(K)$. □

Let G be a locally compact group and let B denote a subgroup of the topological automorphism group Aut G. We say that G is an $[FIA]_B$-group provided the closure B^- of B in Aut G is compact [37]. Let G be an $[FIA]_B$-group. Then the space G_B of B-orbits $[x](x \in G)$, forms a hypergroup under the convolution defined by

$$\begin{aligned} \int_{G_B} f \, d\delta_{[x]} * \delta_{[y]} &= \int_{B^-} f \circ \pi(\beta(x) y) \, d\beta \\ &= \int_{B^-} (f \circ \pi)(x \beta(y)) \, d\beta, \qquad f \in C_c(G_B) \text{ [28, 8.3]}. \end{aligned}$$

Corollary 3.11. *Let G be an amenable $[FIA]_B$-group. Then the hypergroup G_B is amenable.*

Proof. Let G' be the semidirect product of G and B^-, and let $H' = \{e\} \times B^-$. Then $G'//H' \cong G_B$ as hypergroups [28, 8.3B]. So, G_B is amenable by 3.10. □

COROLLARY 3.12. *Let $K = H \vee J$, where H is a compact hypergroup and J is a discrete hypergroup with $H \cap J = \{e\}$. Then, K is amenable if and only if J is amenable.*

Proof. H is a compact subhypergroup of K, $\sigma * \delta_x = \delta_x * \sigma$ for every $x \in K$, and $K//H \cong J$ [53, Propositions 1.2 and 1.3]. Now, the result follows from 3.10. □

Following [3], we say that a compact subhypergroup H of K is *supernormal* in K if $\{\check{x}\} * H * \{x\} \subseteq H$ for each $x \in K$ (observe that the compactness is not needed for this definition). If H is supernormal in K, then $\delta_x * \sigma = \sigma * \delta_x$ for $x \in K$ [3, Lemma 2.2.1]. The converse is not true in general. In fact, $\{e\}$ is supernormal in K if and only if K is a group. Let H be supernormal in K. Then $K//H(= K/H)$ becomes a locally compact group under the multiplication:

$$\int_{K/H} f \, d\delta_{xH} * \delta_{yH} = \int_K f \circ \pi \, d\delta_x * \delta_y$$

$$= \int_H f \circ p(x * t * y)\, dt, \quad x, y \in K, f \in C_c(K/H)$$

(See [54, Theorem 2.1])

Hence, it follows from the proof of 3.10 that, if an arbitrary hypergroup K has a supernormal subhypergroup H, then K admits a left Haar measure such that

$$\int_K f(x)\, dx = \int_{K/H} \int_H f(x * t)\, dt\, dxH, \qquad f \in C_c(K).$$

One should note that if H is a compact hypergroup and J any discrete group with $H \cap J = \{e\}$, then H is a supernormal subhypergroup of $K = H \vee J$. The next result follows immediately from 3.10.

COROLLARY 3.13. *If K admits a supernormal subhypergroup H, then K is amenable if and only if K/H is amenable.*

Examples 3.14. (a) Let $SL(2, \mathbf{C})$ be the locally compact group (with the usual topology) of all 2×2 complex matrices with determinant 1, and $SU(2)$ the compact subgroup of unitary matrices in $SL(2, \mathbf{C})$. Then, $SL(2, \mathbf{C})$ is non-amenable [40, Corollary 14.6], but the hypergroup $SL(2,\mathbf{C})//SU(2)$ is commutative [28, 15.5] and hence amenable.

(b) Let H be a compact group and G (a discrete) free group on two generators with $H \cap G = \{e\}$. Since G is non-amenable [40, Proposition 14.1], the hypergroup $K = H \vee G$ is non-amenable by 3.12. But, the maximal subgroup of K is H which is compact (and hence amenable).

Granirer [21] and Rudin [47] established independently that if G is a non-discrete locally compact group which is amenable as a discrete group, then $\mathrm{LIM}(L_\infty(G)) \setminus \mathrm{TLIM}(L_\infty(G)) \neq \phi$. The next result is in contrast to theirs.

THEOREM 3.15. *Let H be a compact hypergroup, J a discrete hypergroup with $|J| \geq 2$, $H \cap J = \{e\}$. Let $K = H \vee J$ be amenable. Then every LIM M on $L_\infty(K)$ satisfies the equation*

$$\langle M, f \rangle = \langle M, f|_{J^*} \rangle + \langle M, 1_H \rangle \int_H f\,d\sigma,\ f \in L_\infty(K),$$

where $J^* = J \setminus \{e\}$, and σ is the normalized Haar measure on H. In particular, every LIM on $L_\infty(K)$ is a TLIM.

Proof. Let $x \in J^*$. Then, for $f \in L_\infty(K)$, we have

$$_xf(y) = f(x \cdot y) = \begin{cases} f(x), & y \in H \\ f(x * y), & y \in J^*,\ y \neq \check{x} \\ c_e \int_H f\,d\sigma + \Sigma_{b \in J^*} c_b f(b), & y = \check{x} \end{cases}$$

where $\delta_x * \delta_{\check{x}} = \Sigma_{b \in J} c_b \delta_b$, $c_b \geq 0$, only finitely many nonzero, $\Sigma_{b \in J} c_b = 1$. (Recall that $\cdot$ is the convolution in K, and that the points J^* are isolated in K). Now,

$$_x(f|_{J^*})(y) = \begin{cases} f(x), & y \in H \\ f(x * y), & y \in J^*,\quad y \neq \check{x} \\ \Sigma_{b \in J^*} c_b f(b), & y = \check{x} \end{cases}$$

and

$$_x(f|_H)(y) = \begin{cases} 0, & y \neq \check{x} \\ c_e \int_H f\,d\sigma, & y = \check{x}. \end{cases}$$

This implies that ${}_xf = {}_x(f|_{J^*}) + {}_x(1_H)\int_H f\,d\sigma$. Hence,

$$\begin{aligned}\langle M, f\rangle &= \langle M, {}_x(f|_{J^*})\rangle + \langle M, {}_x(1_H)\rangle \int_H f\,d\sigma \\ &= \langle M, f|_{J^*}\rangle + \langle M, 1_H\rangle \int_H f\,d\sigma \end{aligned} \tag{1}$$

Let $\phi \in P(K)$. Since

$$\sum_{y\in J^*} \phi(y)[y] = \int_{J^*} \phi(y)\,dy < \infty$$

and

$$\sum_{y\in J^*} \phi(y)[y]\delta_y * f(z) = (\phi|_{J^*}) * f(z),$$

we have

$$\langle M, (\phi|_{J^*}) * f\rangle = \left(\int_{J^*} \phi(y)\,dy\right)\langle M, f\rangle \tag{2}$$

Next,

$$(\phi|_H) * f(z) = \begin{cases} \int_H \phi(y) f(\check{y} * z)\,dy, & z \in H \\ \left(\int_H \phi(y)\,dy\right) f(z), & z \in J^* \end{cases}$$

Hence,

$$\begin{aligned}\langle M, (\phi|_H) * f\rangle &= \langle M, (\phi|_H) * f|_{J^*}\rangle + \langle M, 1_H\rangle \int_H (\phi|_H) * f\,d\sigma \text{ (by (1))} \\ &= \left(\int_H \phi(y)\,dy\right)\langle M, f|_{J^*}\rangle + \langle M, 1_H\rangle\left(\int_H \phi(y)\,dy\right)\left(\int_H f\,d\sigma\right) \end{aligned} \tag{3}$$

By (2) and (3), we get

$$\begin{aligned}\langle M, \phi * f\rangle &= \langle M, 1_H\rangle\left(\int_H \phi(y)\,dy\right)\left(\int_H f\,d\sigma\right) + \langle M, f\rangle \\ &\quad - \left(\int_H \phi(y)\,dy\right)\langle M, f|_H\rangle \\ &= \langle M, f\rangle, \text{ by } (1).\end{aligned}$$

Thus, M is a TLIM.

Example 3.16. Let $\mathbf{I}_+$ be the nonnegative integers and $\mathbf{I}_+ \cup \{\infty\}$ its one point compactification. Let $0 < a \le \frac{1}{2}$. Let δ_∞ be the identity element, and define

$$\delta_m * \delta_n = \delta_{\min(m,n)}, \quad m, n \in \mathbf{I}^+, m \neq n,$$

$$\delta_n * \delta_n(\{t\}) = \begin{cases} 0, & t < n \\ \dfrac{1-2a}{1-a}, & t = n \\ a^k, & t = n + k > n, \end{cases}$$

and $\check{n} = n$ for all n.

The compact hypergroup obtained this way is denoted by H_a. This class of hypergroups is studied by Dunkl and Ramirez [14]. The normalized Haar measure on H_a is given by

$$\lambda(\{k\}) = \begin{cases} (1-a)a^k, & k \neq \infty \\ 0, & k = \infty. \end{cases}$$

Consider the subhypergroup $H = \{1, 2, \ldots, \infty\}$ of the hypergroup H_a and the hypergroup $J_0 = \{0, \infty\}$, the convolution on J_0 being given by

$$\delta_0 * \delta_0 = \frac{a}{1-a}\delta_\infty + \frac{1-2a}{1-a}\delta_0.$$

Then $H_a = H \vee J_0$ [53, Example 4.5], and hence by the previous theorem Haar measure is the unique LIM on $L_\infty(H_a)$. This is also easy to see without referring to 3.14: Let m be a LIM on $L_\infty(H_a)$, $f \in L_\infty(H_a)$, $\phi \in P(H_a)$. Then $\|\phi_n - \phi\|$, converges to zero, where $\phi_n = \phi$ on $\{0, 1, \ldots, n\}$ and zero otherwise. Hence,

$$m(\phi * f) = \lim_n m\left(\sum_{k=0}^{n} \phi(k)(1-a)a^k\right) m(f) = m(f).$$

Thus m is a TLIM on $L_\infty(H_a)$, and hence $m = \lambda$. $\square$

We close this section with the following remark. Let K be a hypergroup with a left Haar measure. Suppose that $\{x\}*\{y\}$ is finite for all $x, y \in K$. Let K have a nondiscrete normal subgroup of finite index. If H_d is amenable, then

$$\mathrm{LIM}(L_\infty(K)) \setminus \mathrm{TLIM}(L_\infty(K)) \neq \varnothing.$$

In fact, let $m \in \mathrm{LIM}(L_\infty(H)) \setminus \mathrm{TLIM}(L_\infty(H))$. Let ν be the normalized Haar measure on K/H. We take the restriction of λ to H to be the left Haar measure on H (H is open in K). For $x \in K$, $f \in L_\infty(K)$, put $f^1(x) = \langle m, {}_x f|_H\rangle$. Then f^1 is bounded, continuous and constant on the cosets of H in K (see [21, p. 619–620]). Next, write $\langle M, f\rangle = \langle \nu, F\rangle$, where $f^1 = F \circ \pi$. Then

$${}_{xH}F \circ \pi(y) = F(xH * yH) = f^1(x * y) = \langle m, {}_y({}_x f)\rangle = ({}_x f)^1(y)$$

for all $x, y \in K$, since $\{x\}*\{y\}$ is finite. Hence, M is a LIM on $L_\infty(K)$. For $f \in L_\infty(H)$, let $\tilde{f}$ be the function on $L_\infty(K)$ given by $\tilde{f} = f$ on H, and zero otherwise. Then

$$(\tilde{f})^1(x) = \begin{cases} \langle m, f\rangle, & x \in H \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle M, \tilde{f}\rangle = \nu(\{H\})\langle m, f\rangle$. Since m is not a TLIM, it follows that M is not a TLIM.

4. Reiter's conditions

Let K be a hypergroup with a left Haar measure λ. We say that K satisfies (P_r) $[(\mathrm{P}_r^*)]$, $r = 1$ or 2, if whenever $\varepsilon > 0$ and a compact [finite] set $E \subseteq K$ are given, then there exists a $\phi \in L_r(K)$, $\phi \geq 0$, $\|\phi\|_r = 1$ such that

$$\|\delta_x * \phi - \phi\|_r < \varepsilon \quad \text{for every } x \in E.$$

We say that K satisfies Reiter's condition if it has property (P_1). The proof of the next result is adapted from Hulanicki [27, §4].

THEOREM 4.1. *K is amenable if and only if it has property* (P_1) $[(\mathrm{P}_1^*)]$.

Proof. If K is amenable, let $\varepsilon > 0$ and $E \subseteq K$ compact be given. Fix $\beta \in P(K)$. Choose $x_1, \dots, x_n \in E$ and open neighborhoods V_i of x_i, $1 \leq i \leq$

n such that

$$E \subseteq \bigcup_{i=1}^{n} V_i \quad \text{and} \quad \|\delta_y * \beta - \delta_{x_i} * \beta\|_1 < \varepsilon \text{ for each } y \in V_i.$$

Next, we can find $\psi \in P(K)$ such that $\|\psi * \beta - \beta\|_1 < \varepsilon$.

Now, there exists a net $\{\phi_\alpha\} \subseteq P(K)$ such that $\|\phi * \phi_\alpha - \phi_\alpha\|_1$ converges to zero for all $\phi \in P(K)$ (see [33, Theorem 4.6] or [22, §2.4]). Choose $\phi_0 = \phi_{\alpha_0} \in P(K)$ such that

$$\|\beta * \phi_0 - \phi_0\|_1 < \varepsilon,$$

and

$$\|(\delta_{x_k} * \psi) * \phi_0 - \phi_0\| < \varepsilon, \qquad 1 \leq k \leq n, (\delta_{x_k} * \psi \in P(K)).$$

Put $\phi = \beta * \phi_0 \in P(K)$. Then,

$$\|\psi * \phi - \phi\|_1 = \|(\psi * \beta) * \phi_0 - \beta * \phi_0\|_1 \leq \|\psi * \beta - \beta\|_1 < \varepsilon.$$

This implies that

$$\|\delta_{x_k} * (\psi * \phi) - \delta_{x_k} * \phi\|_1 < \varepsilon, \qquad 1 \leq k \leq n.$$

Let $z \in V_k$ for some $1 \leq k \leq n$. Then

$$\begin{aligned}
\|\delta_z * \phi - \phi\|_1 &\leq \|\delta_z * \phi - \delta_{x_k} * \phi\|_1 + \|\delta_{x_k} * \phi - \phi\|_1 \\
&= \|(\delta_z * \beta) * \phi_0 - (\delta_{x_k} * \beta) * \phi_0\|_1 + \|\delta_{x_k} * \phi - \phi\|_1 \\
&\leq \|\delta_z * \beta - \delta_{x_k} * \beta\|_1 + \|\delta_{x_k} * \phi - \phi\|_1 \\
&< \varepsilon + \|\delta_{x_k} * \phi - \delta_{x_k} * (\psi * \phi)\|_1 + \|\delta_{x_k} * (\psi * \phi) - \phi\|_1 \\
&< 2\varepsilon + \|\delta_{x_k} * (\psi * \phi) - \phi\|_1 \\
&= 2\varepsilon + (\delta_{x_k} * \psi) * \beta * \phi_0 - \beta * \phi_0\|_1 \\
&\leq 2\varepsilon + \|(\delta_{x_k} * \psi) * \beta * \phi_0 - (\delta_{x_k} * \psi) * \phi_0\|_1 \\
&\qquad + \|(\delta_{x_k} * \psi) * \phi_0 - \phi_0\|_1 + \|\beta * \phi_0 - \phi_0\|_1 \\
&< 2\varepsilon + 2\|\beta * \phi_0 - \phi_0\|_1 + \|(\delta_{x_k} * \psi) * \phi_0 - \phi_0\|_1 < 5\varepsilon.
\end{aligned}$$

If K satisfies (P_1^*), then it is easy to see that there is a net $\{\phi_\alpha\} \subseteq P(K)$ such that $\|\delta_x * \phi_\alpha - \phi_\alpha\|_1$ converges to zero for all $x \in K$. Hence, there is a LIM on $L_\infty(K)$. □

COROLLARY 4.2. *Let $M^1(K)$ be the set of all probability measures on K. Then K is amenable if and only if there is a net $\{\phi_\alpha\} \subseteq P(K)$ such that $\|\mu * \phi_\alpha - \phi_\alpha\|_1$ converges to zero for all $\mu \in M^1(K)$.*

Proof. See [39, p. 127] or [56, Lemma 5.1] □

The proof of the next theorem is slightly more delicate than the group case because the relation ${}_x(fg) = {}_xf\,{}_xg$ does not hold, in general, for hypergroups.

THEOREM 4.3. *If K satisfies* (P_2), *then it has property* (P_1). *Conversely, if ϕ in Reiter's condition* (P_1) *can be chosen of the form $1/\lambda(A)\, 1_A$, where A is a Borel set in K with $0 < \lambda(A) < \infty$, then K has property* (P_2).

Proof. If K satisfies (P_2), let $\varepsilon > 0$ and compact $E \subseteq K$ be given. Let $\phi \in L_2(K)$, $\phi \geq 0$, $\|\phi\|_2 = 1$ be such that $\|\delta_x * \phi - \phi\|_2 < \varepsilon$ for all $x \in E$, and put $\psi = \phi^2 \in P(K)$. Following [4, p. 319], write

$$\begin{aligned}\delta_x * \psi(y) - \psi(y) &= \int_K [\psi(z) - \psi(y)]\, d\delta_{\check{x}} * \delta_y(z) \\ &= \int_K [\phi(z) - \phi(y)]^2\, d\delta_{\check{x}} * \delta_y(z) \\ &\quad + 2\big[\phi(y)\delta_x * \phi(y) - \phi^2(y)\big] = G_1(y) + G_2(y).\end{aligned}$$

We have

$$\int_K |G_2(y)|\, dy \leq 2\|\phi\|_2 \|\delta_x * \phi - \phi\|_2 < 2\varepsilon \quad \text{for all } x \in E,$$

$$\begin{aligned}G_1(y) &= \int_K \big[\phi^2(z) - 2\phi(z)\phi(y) + \phi^2(y)\big]\, d\delta_{\check{x}} * \delta_y(z) \\ &= \delta_x * \phi^2(y) - 2\delta_x * \phi(y)\phi(y) + \phi^2(y) \\ &= [\delta_x * \phi(y) - \phi(y)]^2 + \delta_x * \phi^2(y) - (\delta_x * \phi)^2(y).\end{aligned}$$

So,

$$\begin{aligned}\int_K G_1(y)\, dy &= \|\delta_x * \phi - \phi\|_2^2 + \|\phi\|_2^2 - \|(\delta_x * \phi)\|_2^2 \leq 2\|\delta_x * \phi - \phi\|_2 \\ &\quad + 2[\|\phi\|_2 - \|\delta_x * \phi\|_2] \leq 4\|\delta_x * \phi - \phi\|_2 < 4\varepsilon\end{aligned}$$

for $x \in E$. Hence,

$$\|\delta_x * \psi - \psi\|_1 < 6\varepsilon \quad \text{for all } x \in E.$$

Conversely, if $\phi = 1/\lambda(A)\, 1_A \in P(K)$ satisfies $\|\delta_x * \phi - \phi\|_1 < \varepsilon^2$ for all $x \in E$, let

$$\psi = \phi^{1/2} = \frac{1}{\lambda(A)^{1/2}} 1_A.$$

Then,

$$\|\delta_x * \psi - \psi\|_2 = \frac{1}{\lambda(A)} \int_K |\delta_x * 1_A(y) - 1_A(y)|^2 \, dy$$
$$\leq \frac{2}{\lambda(A)} \int_K |\delta_x * 1_A(y) - 1_A(y)| \, dy = 2\|\delta_x * \phi - \phi\|_1.$$

Hence, $\|\delta_x * \psi - \psi\|_2 < \varepsilon$ for all $x \in E$. □

Let $\mu \to T_\mu$ be the left regular representation of K on $L_2(K)$, given by

$$T_\mu f = \mu * f, \quad f \in L_2(K), \mu \in M(K).$$

LEMMA 4.4. *The following two statements are equivalent.*
(i) *K satisfies* (P_2).
(ii) *K satisfies* (F): *There is a net $\{f_\alpha\} \subseteq L_2(K)$, $\|f_\alpha\|_2 = 1$, such that $f_\alpha * f_\alpha^{\sim}$ converges to* 1 *uniformly on compact subsets of K.*
In this case, we have:
(G) $|\int_K d\mu| \leq \|T_\mu\|$ *for all* $\mu \in M(K)$;
(D_2) $\|T_\mu\| = \|\mu\|$ *for all* $\mu \in M^+(K)$.

Proof. (ii) ⇒ (i) If $\varepsilon > 0$ and compact $E \subseteq K$ are given, choose $f \in L_2(K)$, $\|f\|_2 = 1$ such that $|1 - f * f^{\sim}(x)| < \varepsilon$ for all $x \in E$, and let $\phi = |f|$. Then,

$$0 \leq |f * f^{\sim}(x)| \leq \phi * \phi^{\sim}(x)$$

and

$$0 \leq 1 - \phi * \phi^{\sim}(x) \leq 1 - |f * f^{\sim}(x)| \leq |1 - f * f^{\sim}(x)| < \varepsilon, \qquad x \in E.$$

This shows that, $\|\delta_x * \phi - \phi\|_2 < \sqrt{2\varepsilon}$, all $x \in E$ (see [27, p. 100–101]).
(i) ⇒ (ii) Easy.
(G) If $f_\alpha * f_\alpha^{\sim}$ converges to 1 uniformly on compact subsets of K, then

$$\int_K d\mu = \lim_\alpha \int_K f_\alpha * f_\alpha^{\sim} \, d\mu = \lim_\alpha \int_K \int_K f_\alpha(y) \overline{f_\alpha(\check{x} * y)} \, dy \, d\mu(x)$$
$$= \lim_\alpha \int_K T_\mu \overline{f_\alpha}(y) \, dy.$$

This implies that $|\int_K d\mu| \leq \|T_\mu\|$ for each $\mu \in M(K)$.

(D_2) follows from (G). □

The next result can be found in [16, p. 61, Theorem 1.4] for a second countable commutative hypergroup. For the sake of completeness, we give a proof.

LEMMA 4.5. *Let K be a commutative hypergroup with the Plancherel measure π on the dual $\hat{K}$. Then* $\operatorname{spt}\pi$ *contains the trivial character* 1 *if and only if K satisfies* (F) *or equivalently* (P_2).

Proof. If $1 \in \operatorname{spt}\pi$, then by considering the inverse Fourier transform one can easily find a net $\{f_\alpha\} \subseteq L_2(K)$, $\|f_\alpha\|_2 = 1$ such that $f_\alpha * f_\alpha^{\sim}$ converges to 1 uniformly on compact subsets of K (see [28, p. 87, Proof B]). Converse follows from 4.4 (a) and [28, 7.3I].

Example 4.6. Let K be the hypergroup given in [28, 9.5]. This is known as Naimark's example. Then $1 \notin \operatorname{spt}\pi$ and hence K does not satisfy (P_2). But K satisfies (P_1) because it is commutative. An example of a commutative discrete hypergroup which does not satisfy (P_2) can be found in [32, Example 2f].

THEOREM 4.7. *If a hypergroup K has a supernormal subhypergroup H, then K satisfies* (P_2) *if* (*and only if*) *K is amenable.*

Proof. If K is amenable, then $K//H$ is an amenable locally compact group by 3.10 and [54, Theorem 2.1]. If σ is the normalized Haar measure of H, then $\delta_x * \sigma = \sigma * \delta_x$ for each $x \in K$ [3, Lemma 2.21]. Let $f, g \in C_c(K//H)$. Then it is easy to verify that $(f * g\sim)\circ\pi = (f\circ\pi)*(g\circ\pi)^{\sim}$. Since $K//H$ satisfies (F), it follows that K has property (F).

Remarks 4.7. Some important characterizations of amenable locally compact groups are extended to hypergroups in [48]. In particular, analogs of Day-Rickert fixed point theorem [22, §3.3] and Reiter-Glicksberg property [43, Chapter 8, §6] are obtained for hypergroups. (See also [15] and [23]) The statements and proofs are similar to the group case, and therefore the details are omitted to save space.

In [33], Lau introduced and studied a class of Banach algebras which include $L_1(K)$. He called such algebras F-algebras. Using the theory of von Neumann algebras he extended several fundamental characterizations of amenable locally compact groups to F-algebras which admit topological left invariant means. The F-algebra $L_1(K)$ has a topological left invariant mean if and only if K is amenable. The interested reader should note that

F-algebras are called Lau algebras in [41]. Properties related to amenability are also considered in [16, Chapter IV] for commutative hypergroups.

Our last result of this section is also in contrast to the group case. Let E be a two-sided Banach $L_1(K)$-module. Then E^* is also a two-sided Banach $L_1(K)$-module. By a derivation D of $L_1(K)$ into E^*, we mean a linear map

$$D\colon L_1(K) \to E^*$$

such that

$$D(\phi * \psi) = D(\phi) \cdot \psi + \phi \cdot D(\phi) \quad \text{for all } \phi, \psi \in L_1(K).$$

If $f \in E^*$, then the map $\delta_f\colon L_1(K) \to E$, given by $\delta_f(\phi) = \phi \cdot f - f \cdot \phi$ is a bounded derivation, called an inner derivation. Following Johnson [29], we say that the Banach algebra $L_1(K)$ is amenable if every bounded derivation of $L_1(K)$ into E^* is an inner derivation. The next result follows from [33, Theorem 4.1].

PROPOSITION 4.9. *If the Banach algebra $L_1(K)$ is amenable, then K is amenable.*

The converse of the above result is not true in general. The author is thankful to Dr. Brian Forrest for suggesting the following:

Example 4.10. This is the same as [4, Example 4.5]. Let $G = \mathbf{R}^n$ and let B be the group of rotations in G. Consider the hypergroup $K = G_B$ (see the remarks prior to 3.11). As a set, K is identified with $\mathbf{R}^+ = [0, \infty)$. The hypergroup $\hat{K}$ is isomorphic with K and so $L_1(K)$ and $A(K)$ are isometrically isomorphic, where $A(K)$ is the pointwise algebra of Fourier transforms on K. The functions in $A(K)$ are continuously differentiable in $(0, \infty)$. Let δ be the derivative evaluated at ρ: $\delta(f) = f'(\rho)$ for $f \in A(K)$. For $n \geq 3$, δ is continuous in the topology of $A(K)$ (See [43, Chapter 2, 6.3(4)]). Accordingly, δ is a point derivation at ρ (See [4, §4] and [1, page 360]). This shows that $L_1(K)$ is not weakly amenable and hence not amenable. (The commutative Banach algebra $L_1(K)$ is weakly amenable [1] if every bounded derivation of $L_1(K)$ into a commutative Banach module is necessarily zero.) Since K is commutative, it is amenable. One should note that $L_1(K)$ is isometrically *-isomorphic to the closed self adjoint subalgebra of $L_1(\mathbf{R}^n)$ consisting of radial functions. Thus, the amenable Banach algebra $L_1(\mathbf{R}^n)$ has a closed self adjoint subalgebra with a bounded approximate identity, which is not even weakly amenable. □

5. On the size of the set of topological invariant means on $L_\infty(K)$

In this section, we obtain the exact cardinality of the set of topological invariant means on $L_\infty(K)$. Throughout this section, K will denote a non-compact amenable hypergroup. Let d be the smallest cardinality of a cover of K by compact sets.

LEMMA 5.1. *Let A be a closed set in K that can be written as the union of less than d compact subsets of K. Then $m(A) = 0$ for all LIM's m on $L_\infty(K)$, where $m(A) = m(1_A)$.*

Proof. For $x \in K$, we have $\{x\} * A \cap A \neq \phi$ if and only if $x \in A * \check{A}$ [28, 4.1B]. Since $A * \check{A}$ is the union of less than d compact subsets of K (See [28, 3.2B]), there exists an $x \in K$ such that $\{x\} * A \cap A = \phi$. By induction, we can find a sequence $\{x_n\}_{n=1}^\infty \subseteq K$ such that $\{x_i\} * A \cap \{x_j\} * A = \phi$ $(i \neq j)$.

Now, $\{\check{x}\} * \{y\} \cap A \neq \phi$ if and only if $y \in \{x\} * A$, for all $x, y \in K$. Also, $\delta_{\check{x}} * \delta_y$ is a probability measure. Hence, $\delta_x * 1_A$ vanishes outside $\{x\} * A$, and is less than or equal to one on $\{x\} * A$. That is, $\delta_x * 1_A \leq 1_{\{x\} * A}$ for $x \in K$. This implies that $m(A) \leq 1/n$ for each positive integer n, and hence $m(A) = 0$.

Let $\mathscr{V}$ be a cover of K by compact sets with $|\mathscr{V}| = d$, where $|\mathscr{V}|$ is the cardinality of $\mathscr{V}$. Let $\Omega = \Omega(\mathscr{V})$ be the set of all finite subsets of $\mathscr{V}$ and consider Ω as a directed set in the usual way: $\lambda \geq \lambda^1$ if $\lambda \supseteq \lambda^1$. Fix a TLIM m_0 on $L_\infty(K)$. Let U be a compact symmetric neighbourhood of e and $\{f_k\}_{k=1}^\infty$ a countable set in $L_\infty(K)$.

The next lemma and its proof are inspired by recent work of Yang [57, Theorem 3.3].

LEMMA 5.2. *There exists a net $\{\psi_\lambda\}_{\lambda \in \Omega} \subseteq P_c(K)$ such that:*

(i) *If $\lambda \neq \lambda^1$, then $U * \operatorname{spt} \psi_\lambda \cap U * \operatorname{spt} \psi_{\lambda^1} = \phi$;*

(ii) *$\|\phi * \phi_\lambda - \phi_\lambda\|_1$ (and $\|\phi_\lambda * \phi - \phi_\lambda\|_1$ if m_0 is a TIM) converges to zero for every $\phi \in P(K)$;*

(iii) *If m_0 is inversion invariant, so is each ψ_λ;*

(iv) *If m is any weak* cluster point of $\{\psi_k\}$ in $L_\infty(K)^*$, then $m(f_n) = m_0(f_n)$, $n = 1, 2, \ldots$. If K is σ-compact, then we can find a sequence $\{\psi_n\} \subseteq P_c(K)$ satisfying* (i), (ii), (iii) *and* (iv).

Proof. We assume that m_0 is inversion invariant (the other cases are even easier) and for convenience that $\|f_n\|_\infty \leq 1$ for all n. Let $\{\phi_\alpha\}$ be a net in $P_c(K)$ converging to m_0 in the weak* topology with $\phi_\alpha^\# = \phi_\alpha$ for all α. By [20, page 17–18], we can assume that

$$\lim_\alpha \|\phi * \phi_\alpha - \phi_\alpha\|_1 = 0 = \lim_\alpha \|\phi_\alpha * \phi - \phi_\alpha\|_1 \quad \text{for all } \phi \in P(K),$$

and

$$w^* - \lim_{\alpha} \phi_\alpha = m_0.$$

Now, we'll order the set Ω by $\{\lambda_\alpha\}_{1 \le \alpha < d}$, and let $\alpha < d$ be an ordinal. Suppose that for each $\beta < \alpha$, we have constructed a mean $\psi_{\lambda_\beta} \in P_c(K)$ satisfying:

(a) If $\beta < \gamma < \alpha$, then $U * \operatorname{spt} \psi_{\lambda_\beta} \cap U * \operatorname{spt} \psi_{\lambda_\gamma} = \phi$;
(b) If $\beta < \alpha$, then

$$\|\delta_s * \psi_{\lambda_\beta} - \psi_{\lambda_\beta}\|_1 < \frac{1}{|\lambda_\beta|} \quad \text{and} \quad \|\psi_{\lambda_\beta} \odot \delta_s - \psi_{\lambda_\beta}\|_1 < \frac{1}{|\lambda_\beta|}$$

for each $s \in U\lambda_\beta$;
(c) If $\beta < \alpha$, then ψ_{λ_β} is inversion invariant;
(d) If $\beta < \alpha$, then

$$|\psi_{\lambda_\beta}(f_j) - m_0(f_j)| < \frac{1}{|\lambda_\beta|}$$

for $1 \le j \le |\lambda_\beta|$.

Write

$$A_\alpha = \bigcup_{\beta < \alpha} \operatorname{spt} \psi_{\lambda_\beta}, \qquad A_1 = \phi.$$

For $s \in K$, $U * \{s\} \cap \operatorname{spt} \psi_\beta = \phi$ $(\beta < \alpha)$ if and only if $s \in U * \operatorname{spt} \psi_\beta$. Hence, the neighbourhood $U * \{s\}$ of s meets at most one element of the family $\{\operatorname{spt} \psi_\beta\}_{1 \le \beta < d}$. Thus A_α and hence $U^3 * A_\alpha * U^3$ is closed in K [28, 4.1E]. Since the latter set is the union of less than d compact sets, by 5.1, $m_0(U^3 * A_\alpha * U^3) = 0$. Fix $\phi \in P_c(K)$ with $\operatorname{spt} \phi \subseteq U$, $\phi = \phi^\#$. Let $0 < \varepsilon < 1$ be given. Choose $s_1, \ldots, s_n$ in $\bigcup \lambda_\alpha$ and neighbourhoods V_i of s_i such that

$$\bigcup \lambda_\alpha \subseteq \bigcup_{i=1}^{n} V_i, \ \|\delta_s * \phi - \delta_{s_i} * \phi\|_1 < \varepsilon$$

and

$$\|\phi \odot \delta_s - \phi \odot \delta_{s_i}\|_1 < \varepsilon \quad \text{for all } s \in V_i,\ 1 \le i \le n.$$

Next, find $\psi \in P_c(K)$ such that $\|\phi * \psi - \phi\|_1 < \varepsilon$, $\|\psi * \phi - \phi\|_1 < \varepsilon$.

Finally, choose $\phi_0 = \phi_{\alpha_0}$ such that

$$\|\phi * \phi_0 - \phi_0\|_1 < \varepsilon,\ \|\phi_0 * \phi - \phi_0\|_1 < \varepsilon,$$
$$\|(\delta_{s_i} * \psi) * \phi_0 - \phi_0\|_1 < \varepsilon,\ \|\phi_0 * (\psi \odot \delta_{s_i}) - \phi_0\|_1 < \varepsilon \quad \text{for } 1 \le i \le n,$$
$$|\phi_0(f_j) - m_0(f_j)| < \varepsilon, \qquad 1 \le j \le |\lambda_\alpha|$$

and

$$\phi_0(U^3 * A_\alpha * U^3) < \varepsilon.$$

Let B_α be any symmetric compact set contained in $K \setminus (U^3 * A_\alpha * U^3)$ with $\phi_0(B_\alpha) > 1 - \varepsilon$ (A_α is symmetric). Define $\phi_0^1 \in P(K)$ by

$$\langle \phi_0^1, f \rangle = \frac{1}{\phi_0(B_\alpha)} \langle \phi_0, f 1_{B_\alpha} \rangle, \qquad f \in L_\infty(K).$$

Then

$$\|\phi_0^1 - \phi_0\|_1 < \varepsilon^1, \qquad \varepsilon < \varepsilon^1 < \frac{2\varepsilon}{1-\varepsilon}.$$

Let $\psi_{\lambda_\alpha} = \phi * \phi_0^1 * \phi$. Then $\psi_{\lambda_\alpha} \in P_c(K)$, and it is easy to see that

$$\|\delta_s * \psi_{\lambda_\alpha} - \psi_{\lambda_\alpha}\|_1 < 7\varepsilon^1, \qquad \|\psi_{\lambda_\alpha} \odot \delta_s - \psi_{\lambda_\alpha}\|_1 < 7\varepsilon^1$$

for each $s \in \bigcup_{i=1}^n V_i$ and

$$|\psi_{\lambda_\alpha}(f_j) - m_0(f_j)| < 4\varepsilon^1, \qquad 1 \le j \le |\lambda_\alpha|$$

(See the proof of 4.1). It is also not hard to show, by repeated applications of [28, 4.1B], that $U * A_\alpha \cap U * \operatorname{spt} \psi_{\lambda_\alpha} = \phi$. Since B_α is symmetric, ϕ_0^1 and hence ψ_{λ_α} is inversion invariant. So, ψ_{λ_α} satisfies (a), (b), (c) and (d). Thus, by transfinite induction, we have a net $\{\psi_\lambda\}_{\lambda \in \Omega} \subseteq P_c(K)$ such that each ψ_λ satisfies (a), (b), (c) and (d). It is now easy to verify that the net $\{\psi_\lambda\}_{\lambda \in \Omega}$ satisfies all the properties of the lemma. By easy modifications of the above arguments, we have the last statement (See [7, §V]). □

Let Ω be a directed set, and let $l_\infty(\Omega)$ be the Banach space of all bounded real valued functions on Ω, with the supremum norm. Write

$$\Phi = \left\{\phi \in l_\infty(\Omega)^* : \phi(x) \le \lim_{\lambda \in \Omega} \sup x(\lambda) \text{ for all } x \in l_\infty(\Omega)\right\}.$$

Then Φ is the set of all $\phi \in l_\infty(\Omega)^*$ such that $\|\phi\| = 1$ and $\phi(x) = \lim_\lambda x(\lambda)$ whenever the limit exists. Let A be an infinite set and $\Omega = \Omega(A)$ be the set of all finite subsets of A directed by inclusion. Then it is shown in [57, Lemma 2.1] that $|\Phi| = 2^{2^{|A|}}$.

Let $L_\infty^r(K)$ be the Banach space of all bounded Borel measurable real valued functions on K with the essential supremum norm. Let $\{\phi_\lambda\}_{\lambda\in\omega}$ be a net in $P_c(K)$. Suppose that for each $s \in K$, there is a neighbourhood U of s which meets at most one element of the family $\{\operatorname{spt}\phi_\lambda\}_{\lambda\in\Omega}$. Let Ψ be the weak* closed convex hull of the set of all weak* cluster points of $\{\phi_\lambda\}_{\lambda\in\Omega}$ in $L_\infty^r(K)^*$. Then Ψ is a non empty weak* compact convex subset of the set $\Sigma(L_\infty^r(K))$ of all means on $L_\infty^r(K)$. Let Φ be defined, as before, for the directed set Ω. The proof of the next result is exactly as in [57, Lemma 3.1]. Note that the group structure and the topological invariance of the net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ are not used here.

LEMMA 5.3. *There exists a linear isometry of $l_\infty(\Omega)^*$ into $L_\infty^r(K)^*$ which maps Φ weak* homeomorphically onto Ψ.*

The next theorem is due to Chou [5, Theorem 5.3] for a σ-compact and noncompact amenable locally compact group. Granirer, assuming the continuum hypothesis, gives a different proof of this result in [20, p. 61]. It is due to Yang [57, Corollary 3.4] for an arbitrary locally compact group (See also [36, Theorem 1]).

THEOREM 5.4. *Let K be a non compact amenable hypergroup, m_0 a TIIM on $L_\infty(K)$ and $\{f_k\}_{k=1}^\infty \subseteq L_\infty(K)$. Then the cardinality of the set*

$$A = \{m \in \operatorname{TIIM}(L_\infty(K)) \colon m_0(f_n) = m(f_n), n = 1, 2, \dots\}$$

is at least 2^{2^d}. In particular, $|\operatorname{TIIM}(L_\infty(I))| \geq 2^{2^d}$.

Proof. Follows easily from Lemmas 5.2, 5.3 and [57, Lemma 2.1].

The next theorem is essentially due to Lau and Paterson [36, Theorem 1] for the case when K is a group.

THEOREM 5.5. *Let K be a non compact amenable hypergroup. Then*

$$|\operatorname{TIIM}(L_\infty(K))| = |\operatorname{TIM}(L_\infty(K))| = 2^{2^d}.$$

If the maximal subgroup $G(K)$ is open, then $|\operatorname{TLIM}(L_\infty(K))| = 2^{2^d}$.

Proof. To prove the first statement, by 5.4, we only need to show that

$$|\operatorname{TIM}(L_\infty(K)| \leq 2^{2^d}.$$

Let H be a compact subhypergroup of K such that $K//H$ is metrizable [52, Theorem 1.4]. Let H_0 be an open noncompact σ-compact subhypergroup of K containing H (see [28, 10.1B] and [55, p. 71, β]). The smallest cardinality of the cover of $K//H$ by compact sets is d. Let E be a compact subset of $K//H$. For $x \in K$, the set

$$(H_0//H) * HxH * (H_0//H)$$

is open and σ-compact in $K//H$. Since E can be covered by a finite number of such sets it is separable by [52, Lemma A.2]. Hence, there is a dense subset T of $K//H$ of cardinality d. Let σ be the normalized Haar measure of H, and let $\phi \in P(K)$. For $f \in L_\infty(K)$, the function $(\sigma * \phi) * f * (\sigma * \phi)^\vee = \sigma * (\phi * f * \check{\phi}) * \sigma$ is continuous and constant on the double cosets of H in K (See the proof of 3.11). Consider

$$A = \{(\sigma * \phi) * f * (\sigma * \phi)^\vee : f \in L_\infty(K)\}$$

as a subspace of $C(K//H)$. Since every function in $C(K//H)$ is determined by its values on T, we have $|C(K//H)| \leq c^d = 2^d$. If m is a TIM on $L_\infty(K)$, then

$$m\big((\sigma * \phi) * f * (\sigma * \phi)^\vee\big) = m(f) \quad \text{for all } f \in L_\infty(K),$$

and hence m can be considered as a continuous linear functional on A. Thus,

$$|\mathrm{TIM}(L_\infty(K))| \leq |A^*| \leq |C(K//H)^*| = c^{2^d} = 2^{2^d}.$$

To see the second statement, we first assume that $G = G(K)$ is open and noncompact. Let L be a σ-compact noncompact open subgroup of G and H a compact normal subgroup of L such that L/H is separable [36, p. 79]. Write

$$(K/H)_r = \{Hx : x \in K\}.$$

Using [28, 10.3B and 10.4B], one can show that, for each $x \in K$, the mapping $Hg \to Hgx$ is continuous ($g \in L$). Also, the set

$$(L/H)x = \{Hgx : g \in L\} = \{Hy : y \in Lx\}$$

is open in $(K/H)_r$. Therefore, every compact set $E \subseteq (K/H)_r$ is separable, and hence there is a dense set in $(K/H)_r$ with cardinality d. If G is compact and open, then $(K/G)_r$ is discrete and $|(K/G)_r| = d$. The rest of the proof now follows as in the group case [36]. □

The following result is due to Chou [6] for the group case.

COROLLARY 5.6. *If K is an infinite discrete amenable hypergroup, then*

$$|\mathrm{IM}(l_\infty(K))| = |\mathrm{IIM}(l_\infty(K))| = |\mathrm{LIM}(l_\infty(K))| = 2^{2^{|K|}}.$$

COROLLARY 5.7. *Let $K = H \vee J$, where H is a compact hypergroup, and J is an infinite discrete hypergroup with $J \cap H = \{e\}$. If K is amenable, then*

$$|\mathrm{LIM}(L_\infty(K))| = |\mathrm{IM}(L_\infty(K))| = |\mathrm{IIM}(L_\infty(K))| = 2^{2^{|J|}}.$$

Proof. Let m be a LIM on $l_\infty(J)$. Write

$$f^1(x) = \int_H f(x * t)\, dt, \qquad x \in K, f \in L_\infty(K).$$

Put $\langle M, f\rangle = \langle m, F\rangle$ where $f^1 = F \circ \pi$ (π is the projection of K onto $K//H = J$). Then M is a LIM on $L_\infty(K)$ since $({}_xf)^1(y) = {}_x(f^1)(y) = {}_{xH}F(yH)$ for all $x, y \in K$. It is easy to see that the mapping $m \to M$ is a bijection of $\mathrm{LIM}(l_\infty(J))$ onto $\mathrm{LIM}(L_\infty(K))$. □

PROPOSITION 5.8. *Let K be a noncompact amenable hypergroup. Then the convex sets* $\mathrm{TLIM}(L_\infty(K))$, $\mathrm{TIM}(L_\infty(K))$ *and* $\mathrm{TIIM}(L_\infty(K))$ *do not have any weak* exposed points or weak* G_δ points (see* [20, *p.* 11–13]).

Proof. Imitate [56, Corollary 3.7].

The next proposition is due to Granirer [21, Proposition 5] for the case when K is a σ-compact locally compact group.

PROPOSITION 5.9. *Let K be a non compact amenable hypergroup. Let $\pi_1(X)[\Gamma_1(X)]$ be the subspace of X spanned by*

$$\{\phi * f - f : \phi \in P(K), f \in X\}$$
$$\left[\{\phi * f - f, h * \check{\psi} - h, f, h \in X, \phi, \psi \in P(K)\}\right],$$

where $X = UC(K)$, $UC_r(K)$, $C(K)$ or $L_\infty(K)$. Then $X/_{\mathrm{cl}\pi_1(X)\oplus \mathbf{C}1}$ and $X/_{\mathrm{cl}\Gamma_1(X)\oplus \mathbf{C}1}$ is not norm separable.

Proof. Choose a sequence $\{f_n\} \subseteq X$ such that $B + \mathrm{cl}_1(X) \oplus \mathbf{C}1$ is dense in X, where B is the linear span of $\{f_n\}$. Let m_0 be a TIM on $L_\infty(K)$ and consider the set

$$M = \{m \in \mathrm{TIM}(L_\infty(K)) : m(f_n) = m_0(f_n), n = 1, 2, \ldots\}.$$

If $m \in M$, then since $m(\mathrm{cl}\,\Gamma_1(X)) = 0$ and $m(1) = 1$, we have $m(f) = m_0(f)$ for all $f \in X$. It then follows that $m = m_0$. This is a contradiction by 5.4. Thus, $X/_{\mathrm{cl}\Gamma_1(X)\oplus \mathbf{C}1}$ and $X/_{\mathrm{cl}\pi_1(X)\oplus \mathbf{C}1}$ are not norm separable. □

Remarks 5.10. There is a natural multiplication on $UC_r(K)^*$ under which it is a Banach algebra: For $f \in UC_r(K)$, $\phi \in UC_r(K)^*$, define $\phi f(x) = \phi({}_x f)$. Then $\phi f \in UC_r(K)$. Indeed,

$$(\phi f)(y) = \int_K \phi f(u)\, \delta_x * \delta_y(u) = \int_K \langle \phi, {}_u f\rangle\, \delta_x * \delta_y(u) = \langle \phi, {}_y({}_x f)\rangle.$$

Hence, if $\{x_\alpha\}$ converges to x in K, then

$$\left\|{}_{x_\alpha}(\phi f) - {}_x(\phi f)\right\|_\infty \leq \|\phi\|\, \|{}_{x_\alpha} f - {}_x f\|_\infty,$$

which converges to zero. Next, define $\phi\psi \in UC_r(K)^*$ ($\psi \in UC_r(K)^*$) by $\langle \phi\psi, f\rangle = \langle \phi, \psi f\rangle$. Then $UC_r(K)^*$ becomes a Banach algebra with a unit (See [19, p. 130–131] or [38, §4]). If K is compact, then $C(K)^* = M(K)$ is semisimple because the left regular representation of K is faithful [28, 6.2I]. If K is noncompact and amenable, then it follows, as in the group case, that the radical $R(UC_r(K)^*)$ of $UC_r(K)$ is not norm separable (see [19, p. 131–132]).

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