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FREDHOLM PROPERTIES AND INTERPOLATION OF FAMILIES OF BANACH SPACES

BY

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1. Introduction

Throughout this paper, D will denote a simply connected domain in the complex plane whose boundary Γ is a rectifiable simple closed curve. Denote by Δ the unit disc $\{w/|w| < 1\}$.

DEFINITION 1.1. *An interpolation family of Banach spaces on* Γ *is a collection* $\{A(\gamma)/\gamma \in \Gamma\}$ *of Banach spaces which satisfies*:

(1) *Every* $A(\gamma)$ *is continuously embedded in a common Banach space* $\mathscr{U}$.

(2) *For every* $a \in \bigcap_{\gamma \in \Gamma} A(\gamma)$, $\|a\|_{A(\gamma)}$ *is a measurable function on* Γ *with respect to* dP_z, *where* $z \in D$ *and* dP_z *is harmonic measure on* Γ *with respect to* z.

(3) *Let*

$$\mathscr{A} = \left\{ a \in \bigcap_{\gamma \in \Gamma} A(\gamma) \Big/ \int_\Gamma \log^+ \|a\|_{A_\gamma}\, dP_z(\gamma) < \infty \right\},$$

where $\log^+ x = \max(\log x, 0)$. $\mathscr{A}$ *is a linear space which is called the log-intersection space for* $\{A(\gamma)/\gamma \in \Gamma\}$. *We assume that* $\mathscr{A}$ *is dense in each* $A(\gamma)$ *and that there exists a measurable function* ψ *on* Γ *which satisfies* $\int_\Gamma \log^+ \psi(\gamma)\, dP_z < \infty$ *such that*

$$\|a\|_{\mathscr{U}} \le \psi(\gamma) \|a\|_{A(\gamma)}$$

for all $a \in \mathscr{A}$ *and* $\gamma \in \Gamma$.

DEFINITION 1.2. *The class* $N^+ = N(\Delta)^+$ *contains all analytic functions in* Δ *for which* $\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$ *are bounded with* $0 \le r < 1$ *and*

$$\lim_{r \to 1^-} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta.$$

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For detailed information on the class N^+, see [8].

We denote by $N^+(D)$ the class of all functions $f(z)$ analytic on D such that $f(\phi(w))$ belongs to the class $N^+ = N^+(\Delta)$, where ϕ is a conformal map from Δ to D. Thus $N^+(D)$ is closed under pointwise addition and multiplication and each $f(z) \in N^+(D)$ possesses a.e. non-tangential limits on Γ. If these non-tangential limits are essentially bounded on Γ, then $f \in H^\infty(D)$, the space of bounded analytic functions on D. A function $f \in N^+(D)$ is termed an outer function in $N^+(D)$, if $f(\phi(w))$ is an outer function in $N^+(\Delta)$.

Let $\mathscr{G}(A(\cdot), \Gamma) = \mathscr{G}(\mathscr{A}) = \mathscr{G}$ be the space of all functions of the form $g(z) = \sum_{j=1}^n \varphi_j(z) a_j$, where $a_j \in \mathscr{A}$ and $\varphi_j \in N^+(D)$, and such that

$$\|g(\cdot)\|_{\mathscr{G}} = \operatorname*{ess\,sup}_{\gamma \in \Gamma} \|g(\gamma)\|_{A\{\gamma\}} < \infty.$$

Note that in general $(\mathscr{G}, \|\cdot\|_{\mathscr{G}})$ is not complete.

DEFINITION 1.3. *For $z \in D$ and $a \in \mathscr{A}$ define*

$$\|a\|_{A\{z\}} = \inf\{\|g\|_{\mathscr{G}} / g \in \mathscr{G},\, g(z) = a\}.$$

The interpolation space $\{A(\gamma)\}\{z\} = A\{z\}$ is defined to be the completion of $(\mathscr{A}, \|\cdot\|_{A\{z\}})$.

In most applications the family of dual spaces $\{A^*(\gamma)\}$ is itself an interpolation family and $(A\{z\})^* = \{A^*(\gamma)\}\{z\}$. In this paper we shall assume that the above duality result holds.

We consider linear operators T mapping $\mathscr{A}$ into $\bigcap_{\gamma \in \Gamma} B(\gamma)$, with $\|Ta\|_{B(\gamma)} \le M(\gamma)\|a\|_{A(\gamma)}$ for all $a \in \mathscr{A}$ and $\gamma \in \Gamma$, where $\{A(\gamma)/\gamma \in \Gamma\}$ and $\{B(\gamma)/\gamma \in \Gamma\}$ are interpolation families on Γ and

$$\int_\Gamma |\log M(\gamma)|\, dP_z(\gamma) < \infty.$$

We will denote the restriction of T to $A\{z\}$ by T_z. A well-known result due to Szegö [11] says:

Let $f(\gamma)$ be a positive dP_z measurable function on Γ such that

$$\int_\Gamma |\log f(\gamma)|\, dP_z(\gamma) < \infty$$

for some (and thus every) $z \in D$. Then there exists a non-vanishing outer function $G(z)$ in $N^+(D)$ whose a.e. non-tangential limits $G(\gamma) = \lim_{z \to \gamma} G(z)$ satisfy $|G(\gamma)| f(\gamma) = 1$ for a.e. $\gamma \in \Gamma$.

Therefore there exists an outer function $G(\cdot)$ such that

$$|G(\gamma)M(\gamma)| = 1 \quad \textit{for } \text{a.e. } \gamma \in \Gamma.$$

DEFINITION 1.4. *Let A and B be Banach spaces, and let T be a linear bounded operator mapping A into B. T is a Fredholm operator if its kernel has a finite dimension and $B = TA \oplus M$, with* $\dim M < \infty$. *The dimension of M is called the codimension of T and is denoted by* $\operatorname{codim} T$. *The index of T is defined by* $i(T) = \dim\ker T - \operatorname{codim} T$.

The question of the stability of Fredholm property when one changes the parameters which determine the interpolation space has been considered by several authors [1], [4], [5], [9], [10], [12], [13]. In [5] it is proved that if $\ker T_s = \{0\}$ and $\operatorname{codim} T_s = d < \infty$, then there exists a ball B centered at s such that $z \in B$ implies that $\ker T_z = \{0\}$ and $\operatorname{codim} T_z = d$. This generalizes the results of Vignati and Vignati who proved the theorem in the case $d = 0$ (see [12]). In this paper we extend the results of [5] in two directions. We prove that if $\dim\ker T_s < \infty$, $\operatorname{codim} T_s = \infty$ and T_s has closed range, then there exists a ball B centered at s such that $z \in B$ implies that $\dim\ker T_z \leq \dim\ker T_s$, $\operatorname{codim} T_z = \infty$ and T_z has closed range. We also eliminate the restriction $\ker T_s = \{0\}$ imposed in [5], i.e., prove that if T_s is Fredholm then there exists a ball $B(s, \delta)$ such that the operators T_z are Fredholm and $i(T_s) = i(T_z)$ for all $z \in B(s, \delta)$.

We recall some results from [5].

DEFINITION 1.5. *Suppose $U(A)$ is a subset of $\mathcal{G}(\mathcal{A})$. Let*

$$U_s = \left\{a \in A\{s\} / \exists f \in U(A), \textit{ such that } f(s) = a\right\}.$$

Define

$$\|a\|_{U_s(A)} = \inf\{\|f\|_{\mathcal{G}} / f(s) = a,\, f \in U(A)\}.$$

Clearly $\|a\|_{A\{s\}} \leq \|a\|_{U_s(A)}$ for all $a \in U_s$.

Let $U_{\{s\}}$ be the completion in $A\{s\}$ of $\{U_s, \|\cdot\|_{U_s}\}$. Clearly $U_{\{s\}} \subset A\{s\}$. If also $\|a\|_{U_s(A)} \leq k\|a\|_{A\{s\}}$ for some k and all $a \in U_s$, we call $U(A)$ an s-Calderón subset (with constant k). If $U(A)$ is an s-Calderón subset and a linear subspace of $\mathcal{G}(\mathcal{A})$, we will say that it is an s-Calderón subspace.

Clearly $U(A)$ is an s-Calderón subspace for some k if and only if $U_{\{s\}}$ is a closed subspace of $A\{s\}$.

THEOREM 1.6. *Let $\{v_1, \ldots, v_d\} \subset \mathcal{G}(\mathcal{A})$ and assume that $\{v_i(s)\}$ are independent. Denote by U the span of $\{v_i\}$ in $\mathcal{G}(\mathcal{A})$. Then there exists $\delta > 0$ such* *that for all* $|z - s| < \delta$, *U is a z-Calderón subspace with a uniform constant k and* $\dim(U_{\{z\}}) = d$.

THEOREM 1.7. *Let* $U(A), V(A)$ *be subspaces of* $\mathcal{G}(\mathcal{A})$, *and let S be an open subset of D. Assume that* $U(A)$ *and* $V(A)$ *are z-Calderón subspaces for all* $z \in S$, *with constant k which is uniform for* $z \in S$. *Then the function*

$$\begin{aligned}\chi_z &= \chi_z(U(A), V(A)) \\ &= \inf\{\|f(z) - g(z)\|_{A\{z\}}/g \in U(A), f \in V(A) \text{ and } \|f(z)\|_{A\{z\}} = 1\}\end{aligned}$$

is continuous in S.

2. Fredholm properties

LEMMA 2.1. *Let E be a closed subspace of* $A\{s\}$ *with infinite codimension. Given any positive integer d, there exists* $\{v_i, i = 1, \dots, d\} \subset \mathcal{G}(\mathcal{A})$ *such that* $\{v_i(s)\}$ *is linearly independent and* $E \cap \operatorname{span}\{v_i(s)\} = \{0\}$.

Proof. Since E has infinite codimension, there exists $M \subset A\{s\}$ with $\dim M = d$ such that $E \cap M = \{0\}$. Assume that $\{e_i, 1 \le i \le d\}$ forms a basis for M with $\|e_i\|_{A\{s\}} = 1$. Let

$$\rho = \rho(e_1, \dots, e_d; E) = \inf\left\{\left\|\sum_{i=1}^{d} c_i e_i - x\right\|_{A\{s\}} \Big/ \max|c_i| = 1,\ x \in E\right\}.$$

Clearly $\rho \le 1$. Since E is a closed subspace and M has finite dimension, then $E \cap M = \{0\}$ implies $\rho > 0$.

Since $\{v(s), v \in \mathcal{G}(\mathcal{A})\}$ is dense in $A\{s\}$, we can find $\{v_i, i = 1, \dots, d\} \subset \mathcal{G}(\mathcal{A})$ with $\|v_i(s) - e_i\|_{A\{s\}} < \rho/2d$. Now suppose $\max\{|c_i|\} = 1$ and let $x \in E$, we have

$$\left\|x - \sum_{i=1}^{d} c_i v_i(s)\right\|_{A\{s\}} \ge \left\|x - \sum_{i=1}^{d} c_i e_i\right\|_{A\{s\}} - \left\|\sum_{i=1}^{d} c_i(v_i(s) - e_i)\right\|_{A\{s\}} > \rho/2.$$

Taking infimum we get $\rho(v_1(s), \dots, v_d(s); E) \ge \rho/2 > 0$. Hence $\{v_i(s)\}$ is linearly independent and $E \cap \operatorname{span}\{v_i(s)\} = \{0\}$.

LEMMA 2.2. *Assume* $\ker T_s = \{0\}$. *If operator* T_z *has closed range and* $\operatorname{codim}(T_s A\{s\}) = \infty$, *then there exists* $\delta > 0$ *such that, for all z with* $|z - s| < \delta$, *the operator* T_z *has closed range and* $\ker T_z = \{0\}$. *Further, given d with* $0 < d < \infty$, *there exists* $\delta_1 > 0$ *such that for all* $|z - s| < \delta_1$ *we also have* $\operatorname{codim} T_z A\{z\} \ge d$.

Proof. Consider the function

$$r(T_z) = \inf\{\|Ta\|_{B\{z\}}/\|a\|_{A\{z\}} = 1\}.$$

Clearly if $r(T_z) > 0$, $\ker T_z = \{0\}$ and the range of T_z is closed. Conversely, if the range of T_z is closed and $\ker T_z = \{0\}$, then by the open mapping theorem $r(T_z) = \|T_z^{-1}\|^{-1} > 0$. Since the range of T_s is closed and $\ker T_s = \{0\}$, we have $r(T_s) > 0$. It is proved in [9], [12] that $r(T_z) > r(T_s)/2 > 0$ for all z which satisfy $|z - s| < \delta$, for some $\delta > 0$. Thus for all those z, $\ker T_z = \{0\}$ and T_z has closed range.

Let $G(\cdot)$ be an outer function such that $|G(\gamma)M(\gamma)| = 1$ for a.e. $\gamma \in \Gamma$. Let $\delta > 0$ be such that $\{z/|z - s| < \delta\} \subset D$ and $r(T_z) > r(T_s)/2$ for all $|z - s| < \delta$. We claim that $G(\cdot)T\mathscr{G}(\mathscr{A})$ is a z-Calderón subspace of $\mathscr{G}(\mathscr{B})$ with constant c, for all z such that $|z - s| < \delta$, where

$$c = 4\|T_s^{-1}\| \sup_{\eta} \left\{\exp\left(\int_\Gamma \log M(\gamma)\, dP_\eta(\gamma)\right)/|\eta - s| < \delta\right\}.$$

In fact, let $y \in (G(\cdot)T\mathscr{G}(\mathscr{A}))(z)$. Then $y \in T_z\mathscr{A}$. Let $x \in \mathscr{A}$ be such that $Tx = y$. Since T_z has closed range, T_z^{-1} is a linear bounded operator defined on $T_z A\{z\}$, and so

$$\|x\|_{A\{z\}} \le \|T_z^{-1}\|\ \|y\|_{B\{z\}}.$$

We choose $g \in \mathscr{G}(\mathscr{A})$ such that $g(z) = x$ and

$$\|g\|_{\mathscr{G}(\mathscr{A})} \le 2\|x\|_{A\{z\}} \le 2\|T_z^{-1}\|\ \|y\|_{B\{z\}}.$$

Then $g(\zeta)/G(z) \in \mathscr{G}(\mathscr{A})$ and $H(\zeta) = G(\zeta)T(g(\zeta)/G(z)) \in G(\cdot)T\mathscr{G}(\mathscr{A})$. Clearly $H(z) = y$, and since $|G(\gamma)M(\gamma)| = 1$ for a.e. $\gamma \in \Gamma$, we have

$$\|H\|_{\mathscr{G}(\mathscr{B})} \le \frac{2\|T_z^{-1}\|\ \|y\|_{B\{z\}}}{|G(z)|} \le c\|y\|_{B\{z\}}.$$

Therefore $G(\cdot)T\mathscr{G}(\mathscr{A})$ is a z-Calderón subspace with constant c for all z such that $|z - s| < \delta$. Therefore:

$$(G(\cdot)T\mathscr{G}(\mathscr{A}))_{\{z\}} = T_z A\{z\}.$$

By Lemma 2.1, there exists $\{v_i, i = 1, \dots, d\} \subset \mathscr{G}(\mathscr{B})$ such that $\{v_i(s)\}$ is independent, and if we let $\operatorname{span}\{v_i(s)\} = M_s$, then $(G(\cdot)T\mathscr{G}(\mathscr{A}))_{\{s\}} \cap M_s = T_s A\{s\} \cap M_s = \{0\}$. Denote by M the space spanned by $\{v_i\}$ in $\mathscr{G}(\mathscr{B})$ and by M_z the space spanned by $\{v_i(z)\}$. By Theorem 1.6, there exists a ball S centered at s, such that M is a z-Calderón subspace with a uniform constant c_1 for all $z \in S$.

Since $(G(\cdot)T\mathscr{G}(\mathscr{A}))_{\{s\}} \cap M_s = \{0\}$ and M_s has finite dimension, we have

$$\chi_s = \chi_s(G(\cdot)T\mathscr{G}(\mathscr{A}), M) > 0.$$

By Theorem 1.7, since $G(\cdot)T\mathscr{G}(\mathscr{A})$ and M are z-Calderón subspaces with a uniform constant when z is close to s, there exists $\delta_1 > 0$ such that $|z - s| < \delta_1$ implies $\chi_z > 0$, which implies $(G(\cdot)T\mathscr{G}(\mathscr{A}))_z \cap M_z = \{0\}$. Therefore $T_z A\{z\} \cap M_z = \{0\}$. By Theorem 1.6, $\dim M_z = d$. Hence the codimension of $T_z A\{z\}$ is not less than d.

THEOREM 2.3. *Assume* $\ker T_s = \{0\}$. *If* T_s *has closed range and* $\operatorname{codim}(T_s A\{s\}) = \infty$, *then there exists* $\delta > 0$, *such that for all* $|z - s| < \delta$, T_z *has closed range and* $\ker T_z = \{0\}$, $\operatorname{codim} T_z A\{z\} = \infty$.

Proof. By Lemma 2.2, there exists a ball B centered at s such that $z \in B$ implies that $\ker T_z = \{0\}$ and T_z has closed range.

Assume that for some $z_0 \in B$, $\operatorname{codim} T_{z_0} = d$ with $0 \le d < \infty$. Denote by S the set of points $z \in B$ such that $\operatorname{codim} T_z = d$. By Theorem 2.7 of [5], S is open. Let $z_1 \in \partial S$ and $z_1 \in B$. Then $\ker T_{z_1} = \{0\}$. If $\operatorname{codim} T_{z_1} = d_1 < \infty$, then by Theorem 2.7 of [5], there exists $B(z_1, \delta_{z_1})$ such that $\operatorname{codim} T_z = d_1$ and $\ker T_z = \{0\}$ for all $z \in B(z_1, \delta_{z_1})$. But $B(z_1, \delta_{z_1}) \cap S \neq \{0\}$, which leads to a contradiction. If $\operatorname{codim} T_{z_1} = \infty$ we get the same contradiction, using Lemma 2.2. Therefore either $S = B$ or $S = \varnothing$. Since $s \notin S$, we do not have $S = B$ and since $z_0 \in S$, we have a contradiction. Therefore $\operatorname{codim} T_z = \infty$ for all $z \in B$.

THEOREM 2.4. *If* T_s *is a Fredholm operator, then there exists* $\delta > 0$, *such that for all* $|z - s| < \delta$, T_z *are Fredholm operators with* $i(T_z) = i(T_s)$.

Proof. Since T_s is a Fredholm operator, then $\ker T_s$ has finite dimension, T_s has closed range and finite codimension d. By Lemma 2.1, there exist $v_i \in \mathscr{G}(\mathscr{B})$, $i = 1, \ldots, d$ such that $B\{s\} = T_s A\{s\} \oplus M$, where M is the space spanned by $\{v_i(s)\}$. Note $v_i(s) \in \mathscr{B}$.

We now define Banach spaces $\tilde{A}(\gamma) = A(\gamma) \oplus M$ with norms

$$\|a + m\|_{\tilde{A}(\gamma)} = \|a\|_{A(\gamma)} + \|m\|_{B(\gamma)}.$$

It is not hard to see that the collection of Banach spaces $\{\tilde{A}(\gamma)/\Gamma\}$ is an interpolation family of Banach spaces on Γ with the *log-intersection space* $\tilde{\mathscr{A}} = \mathscr{A} \oplus M$. We then have $\tilde{A}\{z\} = A\{z\} \oplus M$.

Define a new operator $\bar{T}$ mapping $\tilde{\mathscr{A}}$ into $\bigcap_{\gamma \in \Gamma} B(\gamma)$: $\bar{T}(a + m) = Ta + m$, where $a \in \mathscr{A}$ and $m \in M$. Then

$$\|\bar{T}(a + m)\|_{B(\gamma)} \le (M(\gamma) + 1)\|a + m\|_{\tilde{A}(\gamma)}$$

for all $a + m \in \tilde{A}$ and $\gamma \in \Gamma$. We have

$$\int_\Gamma |\log(M(\gamma) + 1)|\, dP_z(\gamma) < \infty.$$

Clearly $\overline{T}_s$ is onto and its kernel has finite dimension r. Then by Corollary 2.8 in [5], there exists a ball S centered at s such that $z \in S$ implies that $\overline{T}_z$ is onto and its kernel has finite dimension r.

$B_z = \overline{T}_z \tilde{A}\{z\} = T_z A\{z\} + M$. The sum may not be direct. Let $T_z A\{z\} \cap M = M'$ and let $m_1, \dots, m_{r_1}$ be a basis for M'. There exist $a_1, \dots, a_{r_1} \in A\{z\}$ such that $T_z a_j = m_j$. Note that $a_j - m_j \in \ker \overline{T}_z$, but $a_j \notin \ker T_z$. Note also that if $a \in \ker T_z$, then $a \in \ker \overline{T}_z$. Therefore $\dim \ker \overline{T}_z \geq \dim \ker T_z + r_1$. If however $a + m \in \ker \overline{T}_z$ but $T_z a \neq 0$, then $T_z a = -m$ and $T_z a \in M'$. Therefore $\dim \ker \overline{T}_z = \dim \ker T_z + r_1$. Let M_1 be a subspace of M such that $M_1 \oplus M' = M$. Then $B_z = T_z A\{z\} \oplus M_1$ and

$$\begin{aligned}
\operatorname{codim} T_z = \dim M_1 = \dim M - r_1 &= \dim M - \left(\dim \ker \overline{T}_z - \dim \ker T_z\right) \\
&= \dim M - \dim \ker \overline{T}_s + \dim \ker T_z \\
&= \dim M - \dim \ker T_s + \dim \ker T_z \\
&= \operatorname{codim} T_s - \dim \ker T_s + \dim \ker T_z .
\end{aligned}$$

Hence $i(T_z) = i(T_s)$.

References

1. E. Albrecht, *Spectral interpolation*, Operator Theory: Advances and Applications, vol. 14, Birkhauser-Verlag, Basel, 1984, pp. 13–37.
2. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, New York, 1976.
3. A.P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math., vol. 24 (1964), pp. 113–190.
4. W. Cao and Y. Sagher, *Stability of Fredholm properties in interpolation scales*, Ark. Mat., to appear.
5. ______, *Stability in interpolation of families of Banach spaces*, Proc. of Amer. Math. Soc., to appear.
6. R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss, *The complex method for interpolation of operators acting on families of Banach spaces*, Lecture Notes in Mathematics, vol. 779, Springer-Verlag, New York, 1980, pp. 123–153.
7. ______, *A theory of complex interpolation for families of Banach spaces*, Adv. in Math., vol. 33 (1982), pp. 203–229.
8. P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
9. Z. Slodkowski, *A generalization of Vesentini and Wermer' theorem*, Rend. Sem. Mat. Univ. Padova, vol. 75 (1986), pp. 157–171.
10. I.Ya. Sneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled., vol. 9 (1974), pp. 214–229.
11. G. Szegö, *Über die Randwerte einer analytischen Funktion*, Math. Ann., vol. 84 (1921), pp. 482–492.
12. A.T. Vignati and M. Vignati, *Spectral theory and complex interpolation*, J. Funct. Anal., vol. 80 (1988), pp. 383–397.
13. M. Zafran, *Spectral theory and interpolation of operators*, J. Funct. Anal., vol. 36 (1980), pp. 185–204.

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