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LOCAL BOUNDARY REGULARITY OF THE BERGMAN PROJECTION IN NON-PSEUDOCONVEX DOMAINS

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1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth boundary. The Bergman projection P associated to Ω is the orthogonal projection from the space of square-integrable functions on Ω onto the subspace consisting of holomorphic functions. The global or local boundary regularity of the Bergman projection, as well as the boundary extendibility of the Bergman kernel function $K(z, w)$, was proved to have important applications in studying the boundary behavior of biholomorphic and proper holomorphic mappings of Ω [5], [8], [9], [15]. If Ω is a pseudoconvex domain, many results have been obtained as consequences of the $\bar{\partial}$-Neumann theory. For instance, the Bergman projection for a pseudoconvex domain is locally regular, or, satisfies certain pseudolocal estimates at all boundary points of finite type in the sense of D'Angelo [14]. Also the main theorem in [19] states that $K(z, w)$ is smooth in both variables up to the boundary off the boundary diagonal in a strictly pseudoconvex domain. In [3] or [10] the same conclusion has been generalized for weakly pseudoconvex domains of finite type. It has also been shown for smoothly bounded Reinhardt domains [7], which are not necessarily pseudoconvex, that the Bergman projection is globally regular and that the Bergman kernel function behaves nicely on the boundary. Namely, the well-known condition R is satisfied (see Definition 2.1). Thus any derivative of $K(z, w)$ in the z-variable has uniform polynomial growth in the w-variable. See [1] and [4] for some other types of domains that satisfy condition R.

When the smoothly bounded domain is assumed to be arbitrary, little is known about the boundary regularity of the Bergman projection. In [2], Barrett presented a non-pseudoconvex bounded Hartogs domain D with smooth boundary which does not satisfy condition R. Actually, in his example the subspace of bounded holomorphic functions is not dense in the space $H(D)$ of square-integrable holomorphic functions. So there is a smooth function ϕ which is compactly supported in D such that the Bergman projection $P\phi$ of ϕ is not bounded. It is easily seen that for some point w in

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D, there are "bad" boundary points where the Bergman kernel $K(\cdot, w)$ cannot be locally bounded. But, $K(\cdot, w)$ does extend near some "good" boundary points. In fact, it is shown in this paper that for an arbitrary smoothly bounded domain, there are always points, called *extreme* boundary points, such that the Bergman projection associated to this domain is locally regular at them—in the sense that certain pseudolocal estimates hold for the Bergman projection (Theorem 4.1). It will then follow that the Bergman kernel function $K(\cdot, w)$ is locally smooth near these boundary points for any fixed w inside Ω. To describe the most simple example of an extreme point, take any big ball that contains Ω, shrink it until it touches a boundary point of Ω. Then such a point is an extreme boundary point.

It will also be shown that, under the condition of global regularity, if the boundary of a domain is real analytic near a strictly pseudoconvex boundary point of extreme type, then its Bergman kernel function has holomorphic extension past the point.

In the proof of the results, the method developed by Bell in [3] has been adopted. By making use of Bell's idea of comparison of domains, it is possible to apply Catlin's subelliptic estimates [13] at points of finite type for pseudoconvex domains to do the job here.

The main results are proved in §4. In §2 the necessary notation is given and the extreme boundary points are defined. Section 3 is a review of the $\bar{\partial}$-Neumann problem which is the essential tool for carrying out some of the proofs in this paper.

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2. Preliminaries

Throughout this paper, Ω is a smoothly bounded domain in $\mathbf{C}^n$. So there is a real-valued function r, which is smooth (i.e., has continuous derivatives of all orders), such that $\Omega = \{z \in \mathbf{C}^n;\ r(z) < 0\}$ and the gradient of r does not vanish on the boundary $\partial\Omega = \{z \in \mathbf{C}^n;\ r(z) = 0\}$ of Ω. The function r is called the defining function of Ω.

$L^2(\Omega)$ is the usual Hilbert space consisting of all square-integrable functions on Ω with respect to the Lebesgue measure dV in $\mathbf{C}^n \simeq \mathbf{R}^{2n}$. The inner product of any two functions u, v in $L^2(\Omega)$ is $\langle u, v\rangle = \int u\bar{v}\, dV$. The closed subspace consisting of holomorphic functions is denoted by $H(\Omega)$. For any integer $s \geq 0$, the Sobolev space $W^s(\Omega)$ stands for the class of functions having all derivatives in the distribution sense of order less than or equal to s in $L^2(\Omega)$. All $W^s(\Omega)$, where each $s \geq 0$ is an integer, are Hilbert spaces. The inner product $\langle \cdot\, , \cdot \rangle_s$ for $W^s(\Omega)$ is given by

$$\langle u, v\rangle_s = \sum_{|\alpha| \leq s} \langle D^\alpha u, D^\alpha v\rangle, \qquad u, v \in W^s(\Omega).$$

Here D^α, with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2n})$ a multi-index, is a short-hand notation for

$$(\partial^{\alpha_1}/\partial x_1^{\alpha_1}) \cdots (\partial^{\alpha_{2n}}/\partial x_{2n}^{\alpha_{2n}});$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$. The norm $\|\cdot\|_s$ of $W^s(\Omega)$ induced by the inner product of $W^s(\Omega)$ is $\|u\|_s = \langle u, u\rangle_s^{1/2}$ for u in $W^s(\Omega)$. In particular $W^0(\Omega) = L^2(\Omega)$ and $\langle u, v\rangle_0 = \langle u, v\rangle$, $\|u\|_0 = \|u\|$ for u, v in $L^2(\Omega)$. However, when addressing the inner product, or the norm of $L^2(\Omega)$ the notation with no subscripts will always be used.

The closure in $W^s(\Omega)$ of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω is denoted as $W_0^s(\Omega)$. If $s > 0$ is an integer, $W^{-s}(\Omega)$ is by duality a subspace of the distribution space $\mathscr{D}'(\Omega)$ whose elements uniquely extend to be linear continuous functionals on $W_0^s(\Omega)$. If s is an arbitrary real number, $W^s(\Omega)$ can be defined by interpolation [21]. The norm of an element u in $W^{-s}(\Omega)$, $s > 0$ is then

$$\|u\|_{-s} = \sup\{|\langle u, \phi\rangle|;\ \phi \in C_0^\infty(\Omega),\ \|\phi\|_s = 1\}.$$

The notation $\langle\ \cdot\ ,\ \cdot\ \rangle$ now denotes the pairing between the dual spaces which can also be regarded as the action of a distribution on a function. There is no confusion with the L^2-inner product since the usages coincide when both are applicable. Also the dual space $(W^s(\Omega))^*$ of $W^s(\Omega)$ has norm for $u \in (W^s(\Omega))^*$,

$$\|u\|_{-s}^* = \sup\{|\langle u, \phi\rangle|;\ \phi \in C^\infty(\overline{\Omega}),\ \|\phi\|_s = 1\},$$

where $C^\infty(\overline{\Omega})$ is the space of functions with bounded continuous derivatives of all orders in Ω. Certainly $(W^s(\Omega))^* \subset W^{-s}(\Omega)$ and $\|u\|_{-s} \le \|u\|_{-s}^*$ for u in $(W^s(\Omega))^*$. However, if u is harmonic then $\|u\|_{-s}^* \le C\|u\|_{-s}$ with C independent of u [11]. By Sobolev's lemma, the intersection of all the spaces $W^s(\Omega)$, $s \in \mathbf{R}$, is $C^\infty(\overline{\Omega})$.

DEFINITION 2.1. A domain Ω is said to satisfy *condition R*, if the Bergman projection P of Ω maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$. For $z_0 \in \partial\Omega$, the domain Ω is said to satisfy *local condition R* at z_0 if P maps $C^\infty(\overline{\Omega})$ into the subspace of $L^2(\Omega)$ consisting of holomorphic functions which are smooth up to the boundary near z_0.

There are many examples of domains which satisfy condition R, and also domains which satisfy local condition R at certain points. For instance, a smooth bounded pseudoconvex domain satisfies local condition R at all the boundary points of finite type (see Catlin [13]).

It is well-known that the global condition R has many equivalences. For example, Ω satisfies condition R if and only if for every real number $s \ge 0$, there exists $N \geq 0$ such that the Bergman projection P of Ω admits the estimates

$$\|Pu\|_s \leq C\|u\|_{s+N}$$

for any u in $C^\infty(\overline{\Omega})$. Also, it is provided in [7] that condition R holding in Ω is equivalent to the following: for any real number $s \geq 0$ there are positive constants C and m, so that for all $w \in \Omega$, the Bergman kernel function $K(z, w)$ satisfies

$$\|K(\cdot, w)\|_s \leq Cd(w)^{-m},$$

where $d(w) = d(w, \partial\Omega)$ is the distance from w to $\partial\Omega$. Another simple consequence of condition R is that the Bergman kernel function has the property that $K(\cdot, w) \in C^\infty(\overline{\Omega})$ for any w in Ω.

The following fact will be used in proving the main results: there is a linear differential operator Φ^s of finite order with coefficients in $C^\infty(\overline{\Omega})$ such that $P\Phi^s = P$ for each integer $s \geq 0$. Also

$$\Phi^s\colon W^{s+N}(\Omega) \to W_0^s(\Omega) \tag{2.1}$$

is bounded, where $N \geq 0$ is a constant depending on s. The operator Φ^s was first constructed by Bell [5].

The rest of the section will give a description of a special kind of boundary points for any smoothly bounded domain in $\mathbf{C}^n$. These will be the points which possess a certain extremity property and include, for instance, all the boundary points which maximize the distance to any fixed point in $\mathbf{C}^n$. The following is the precise definition.

DEFINITION 2.2. Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth boundary. A point z_0 on the boundary $\partial\Omega$ of Ω is said to be an *extreme* boundary point, or a point of *extreme* type, if there is a bounded pseudoconvex domain D in $\mathbf{C}^n$ such that

(i) D contains Ω and ∂D coincides with $\partial\Omega$ near z_0, and
(ii) z_0 is a point of finite type of D in the sense of D'Angelo [14].

Since the set of finite type points is open [14], it follows that the set of extreme boundary points is open in $\partial\Omega$. For any Ω given, there always exist strictly pseudoconvex domains containing Ω, with boundaries tangential to the boundary of Ω. It can be proved that these tangent points are of extreme type, thus showing the existence of extreme boundary points for any smoothly bounded domain.

PROPOSITION 2.3. *Let Ω_1 be an arbitrary bounded domain with smooth boundary in $\mathbf{C}^n$. Suppose that Ω_2 is a smoothly bounded pseudoconvex domain containing Ω_1, the boundary $\partial\Omega_2$ intersects $\partial\Omega_1$ at a unique point z_0 which is a strictly pseudoconvex boundary point of Ω_2. Then there is a smoothly bounded domain D such that*:

(i) *D is strictly pseudoconvex and contains Ω_1;*

(ii) *there is an open neighborhood V of z_0 so that ∂D coincides with $\partial\Omega_1$ in V.*

Therefore, z_0 is an extreme boundary point of Ω_1 by definition. For any smoothly bounded domain Ω, the *Nebenhülle* of Ω is the interior of the intersection of all pseudoconvex domains containing $\overline{\Omega}$ (see [16]). It is clear that the point z_0 of Proposition 2.3 is in the boundary of the Nebenhülle of Ω_1. The proof of the proposition is easy to see geometrically, it is however lengthy and tedious, is thus omitted.

3. Estimates for the $\bar{\partial}$-Neumann problem

The proof of local regularity of the Bergman projection at boundary points of finite type for a pseudoconvex domain is based on certain pseudolocal estimates of the $\bar{\partial}$-Neumann operator of the form

$$\|\zeta_1\bar{\partial}^*N\bar{\partial}u\|_s \leq C(\|\zeta_2 u\|_s + \|u\|).$$

(Definitions of the notation involved will be given later.) For the purpose here it would be desirable to show the same estimates at extreme boundary points. The goal of this section is therefore to study the $\bar{\partial}$-Neumann problem at boundary points of finite type which will then be applied in the next section to prove the main results.

Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$. The $\bar{\partial}$-operator is a closed and densely defined operator acting on $L^2_{(p,q)}(D)$, the space of (p,q)-forms with square-integrable coefficients on D. Let $\bar{\partial}^*$ be its Hilbert space adjoint. Then $\bar{\partial}^*$ is also closed and densely defined. And the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is surjective to $L^2_{(p,q)}(D)$ from a subspace of $L^2_{(p,q)}(D)$. Therefore if the quadratic form

$$Q(v,w) = \langle\bar{\partial}v,\bar{\partial}w\rangle + \langle\bar{\partial}^*v,\bar{\partial}^*w\rangle$$

is defined for all (p,q)-forms v and w in the domains of $\bar{\partial}$ and $\bar{\partial}^*$, the inverse operator N of $\bar{\partial}\bar{\partial}^* + \bar{\partial}\bar{\partial}^*$, called the Neumann operator [17], satisfies $Q(Nf,g) = \langle f,g\rangle$ for any (p,q)-forms f and g with square-integrable coefficients and with g in the intersection of the domains of $\bar{\partial}$ and $\bar{\partial}^*$. Now suppose that ∂D is smooth in a neighborhood U of a given boundary point z_0

and that a subelliptic estimate holds in U for the $\bar\partial$-Neumann problem. Then there is a positive number $\varepsilon \le 1/2$, which is called the order of the subelliptic estimate, so that

$$\|u\|_\varepsilon^2 \le C\big(Q(u,u) + \|u\|^2\big) \tag{3.1}$$

for every smooth (p,q)-form u which is in the domains of $\bar\partial$ and $\bar\partial^*$ and supported in U.

In [13], Catlin proved a necessary and sufficient condition for a subelliptic estimate to hold in a pseudoconvex domain. Therefore, (3.1) is valid in a neighborhood U of z_0 for all (0, 1)-forms u if and only if $z_0 \in \partial D$ is a point of finite type in the sense of D'Angelo. Fix such a z_0. Let ζ_1 and ζ_2 be a pair of real-valued smooth functions supported in U with $\zeta_2 \equiv 1$ in a neighborhood of the support of ζ_1. The classical a priori estimates for the $\bar\partial$-Neumann problem [17] give that for every smooth (0, 1)-form f in $L^2_{(0,1)}(D)$,

$$\|\zeta_1 Nf\|_{(k+2)\varepsilon}^2 \le C\big(\|\zeta_2 f\|_{k\varepsilon}^2 + \|f\|^2\big), \qquad k = 0, 1, 2, \ldots, \tag{3.2}$$

where the constant C is independent of f. Hence if u is in $C^\infty(D)$ with $\bar\partial u \in L^2_{(0,1)}(D)$, then

$$\|\zeta_1 N\bar\partial u\|_{(k+2)\varepsilon}^2 \le C\big(\|\zeta_2 \bar\partial u\|_{k\varepsilon}^2 + \|\bar\partial u\|^2\big), \qquad k = 0, 1, 2, \ldots .$$

Nevertheless, for the purpose of this paper, it is necessary that the global term $\|\bar\partial u\|^2$ be replaced by $\|u\|^2$.

Proposition 3.1. *For all u in $C^\infty(D)$ with $\bar\partial u \in L^2_{(0,1)}(D)$ and $s \ge 0$,*

$$\|\zeta_1 \bar\partial^* N\bar\partial u\|_{s+\varepsilon}^2 \le C\big(\|\zeta_2 \bar\partial u\|_s^2 + \|u\|^2\big). \tag{3.3}$$

Proof. It follows from (3.1) that,

$$\begin{aligned}
\|\zeta_1 N\bar\partial u\|_\varepsilon^2 &\le C\big(Q(\zeta_1 N\bar\partial u, \zeta_1 N\bar\partial u) + \|\zeta_1 N\bar\partial u\|^2\big) \\
&\le C\big(|Q(N\bar\partial u, \zeta_1^2 N\bar\partial u)| + \|N\bar\partial u\|^2\big) \\
&= C\big(|\langle \bar\partial u, \zeta_1^2 N\bar\partial u\rangle| + \|N\bar\partial u\|^2\big) \\
&\le C\big(\|\zeta_2 \bar\partial u\|^2 + \|N\bar\partial u\|^2\big).
\end{aligned}$$

For any v in $L^2_{(0,1)}(D)$, since Nv is in the domain of $\bar\partial^*$ and $\bar\partial^* N$ is a bounded operator from $L^2_{(0,1)}(D)$ to $L^2(D)$,

$$|\langle N\bar\partial u, v\rangle| = |\langle u, \bar\partial^* Nv\rangle| \le \|u\|\,\|\bar\partial^* Nv\| \le C\|u\|\,\|v\| \le C_\delta\|u\|^2 + \delta\|v\|^2.$$

Putting $v = N\bar\partial u$ and $\delta = 1/2$ in the above inequality gives $\|N\bar\partial u\|^2 \le C\|u\|^2$. Thus $N\bar\partial$ is bounded from $L^2(D)$ to $L^2_{(0,1)}(D)$ and the estimate

$$\|\zeta_1 N\bar\partial u\|_\varepsilon^2 \le C\big(\|\zeta_2\bar\partial u\|^2 + \|u\|^2\big)$$

holds. Now, similar to the proof of (3.2), the following estimates can be obtained by induction:

$$\|\zeta_1 N\bar\partial u\|_{(k+2)\varepsilon}^2 \le C\big(\|\zeta_2\bar\partial u\|_{k\varepsilon}^2 + \|u\|^2\big), \qquad k = 0,1,2,\dots .$$

It is clear from interpolation that

$$\begin{aligned} &\|\zeta_1 N\bar\partial u\|_s^2 \le C\big(\|\zeta_2\bar\partial u\|^2 + \|u\|^2\big) \quad \text{if } 0 \le s \le 2\varepsilon,\\ &\|\zeta_1 N\bar\partial u\|_{s+2\varepsilon}^2 \le C\big(\|\zeta_2\bar\partial u\|_s^2 + \|u\|^2\big) \quad \text{if } s > 0. \end{aligned} \tag{3.4}$$

As in [17], assume that the neighborhood U is contained in a special coordinate system $x_1, x_2, \dots, x_{2n-1}, x_{2n}$ so that $x_{2n} = r$, a local defining function of D near z_0. If u is in $C_0^\infty(\bar D \cap U)$, i.e., u is an element in $C^\infty(\bar D)$ and is supported in U, define the tangential differential operator Λ^t of order t by

$$\widehat{\Lambda^t u}(\xi, x_{2n}) = \big(1 + |\xi|^2\big)^{t/2}\hat u(\xi, x_{2n})$$

where $\hat u(\xi, x_{2n})$ is the tangential Fourier transform of $u(x_1, \dots, x_{2n})$ performed in the first $(2n-1)$ variables. Let η_1 and η_2 be two real-valued functions in $C_0^\infty(U)$ with the property that $\eta_2 \equiv 1$ in a neighborhood of the support of η_1, $\eta_1 \equiv 1$ in a neighborhood of the support of ζ_1, and $\zeta_2 \equiv 1$ in a neighborhood of the support of η_2. Then by the triangle inequality,

$$\|\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2 \le C\Big(\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2 + \big\|(1-\eta_1)\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\big\|^2\Big).$$

Because $(1-\eta_1)\Lambda^{s+\varepsilon}\zeta_1$ is a pseudodifferential operator of order $-\infty$,

$$\big\|(1-\eta_1)\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\big\|^2 \le C\|\bar\partial^* N\bar\partial u\|^2 \le C\|u\|^2. \tag{3.5}$$

And since $\bar\partial\bar\partial^* N\bar\partial = \bar\partial$,

$$\begin{aligned}\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2 &= \langle \eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u, \eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\rangle \\ &= \left\langle N\bar\partial u, \bar\partial\zeta_1(\Lambda^{s+\varepsilon})^*\eta_1^2\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\right\rangle \\ &= \left\langle \eta_2 N\bar\partial u, \bar\partial\zeta_1(\Lambda^{s+\varepsilon})^*\eta_1^2\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\right\rangle \\ &= \langle \eta_2 N\bar\partial u, \zeta_1\eta_1^2\Lambda^{2s+2\varepsilon}\zeta_1\bar\partial\bar\partial^* N\bar\partial u\rangle \\ &\quad + O\left(\|\eta_2 N\bar\partial u\|_{s+\varepsilon}\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|\right) \\ &= \langle \eta_2 N\bar\partial u, \zeta_1\Lambda^{2s+2\varepsilon}\zeta_1\bar\partial u\rangle \\ &\quad + O\left(\|\eta_2 N\bar\partial u\|_{s+\varepsilon}\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|\right),\end{aligned}$$

where a cut-off function η_2 has been slided in the inner products, since one of the terms in the inner product has support contained in the set where $\eta_2 \equiv 1$. Also the fact that the commutator of two pseudodifferential operators has order one less than the sum of their orders is used (see [17, Appendix]). The generalized Schwarz inequality $|\langle u, v\rangle| \le C\|\Lambda^t u\|\,\|\Lambda^{-t}v\|$ for u, v supported in U now yields

$$\begin{aligned}\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2 &\le C\|\eta_2 N\bar\partial u\|_{s+2\varepsilon}^2\|\eta_2\Lambda^s\zeta_1\bar\partial u\|^2 \\ &\quad + O\left(\|\eta_2 N\bar\partial u\|_{s+\varepsilon}\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|\right) \\ &\le C\left(\|\eta_2 N\bar\partial u\|_{s+2\varepsilon}^2 + \|\zeta_2\bar\partial u\|_s^2\right) \\ &\quad + O\left(\|\eta_2 N\bar\partial u\|_{s+\varepsilon}\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|\right) \\ &\le C\left(\|\eta_2 N\bar\partial u\|_{s+2\varepsilon}^2 + \|\zeta_2\bar\partial u\|_s^2\right) \\ &\quad + C_\delta\|\eta_2 N\bar\partial u\|_{s+\varepsilon}^2 + \delta\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2.\end{aligned}$$

Let $\delta = 1/2$. Then

$$\|\eta_1\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2 \le C\left(\|\eta_2 N\bar\partial u\|_{s+2\varepsilon}^2 + \|\zeta_2\bar\partial u\|_s^2\right)$$

for some new constant C. Combining this with (3.5) gives

$$\|\Lambda^{s+\varepsilon}\zeta_1\bar\partial^* N\bar\partial u\|^2 \le C\left(\|\eta_2 N\bar\partial u\|_{s+2\varepsilon}^2 + \|\zeta_2\bar\partial u\|_s^2 + \|u\|^2\right).$$

Apply (3.4) to the term $\|\eta_2 N\bar{\partial}u\|_{s+2\varepsilon}$ with cut-off functions η_2 and ζ_2; then

$$\|\eta_2 N\bar{\partial}u\|^2_{s+2\varepsilon} \le C\left(\|\zeta_2\bar{\partial}u\|^2_s + \|u\|^2\right).$$

Therefore,

$$\|\Lambda^{s+\varepsilon}\zeta_1\bar{\partial}^* N\bar{\partial}u\|^2 \le C\left(\|\zeta_2\bar{\partial}u\|^2_s + \|u\|^2\right).$$

Finally the fact that ∂D is noncharacteristic and that the $\bar{\partial}$-Neumann problem is elliptic over the interior implies (3.3). □

For a sufficiently small neighborhood U of z_0, if ζ_1 and ζ_2 are supported in U as before with $\zeta_2 \equiv 1$ in a neighborhood of the support of ζ_1, (3.3) is actually true for all functions in $L^2(D)$.

PROPOSITION 3.2. *If u is a function in $L^2(D)$ satisfying that $\|\zeta_2\bar{\partial}u\|_s < +\infty$, then $\zeta_1\bar{\partial}^* N\bar{\partial}u$ is in $W^{s+\varepsilon}(D)$ and* (3.3) *is valid for u.*

Proof. From Kohn's formula $P = I - \bar{\partial}^* N\bar{\partial}$ where P is the Bergman projection associated to the domain D and I the identity operator [20], $\bar{\partial}^* N\bar{\partial}$ is the projection of $L^2(D)$ to the orthogonal complement space of $H(D)$ in $L^2(D)$. Thus $\bar{\partial}^* N\bar{\partial}u$ is well defined and is a function in $L^2(D)$ if u is in $L^2(D)$. Without loss of generality, assume that the support of ζ_2 is contained in U, which is the neighborhood of z_0 as in (3.1).

CLAIM. Let u be given as above. For a sufficiently small neighborhood U of z_0 there is a sequence of functions $u_1, u_2, \ldots$ on D which are restrictions of smooth functions on $\mathbf{C}^n$ with the property that $u_j \to u$ in $L^2(D)$ and $\zeta_2\bar{\partial}u_j \to \zeta_2\bar{\partial}u$ in $W^s(D)$ as $j \to +\infty$.

Once the claim is proved, applying (3.3) to $u_j - u_k$ shows that $\{\zeta_1\bar{\partial}^* N\bar{\partial}u_j\}$ is a Cauchy sequence in $W^{s+\varepsilon}(D)$. Since $\zeta_1\bar{\partial}^* N\bar{\partial}u_j$ converges to $\zeta_1\bar{\partial}^* N\bar{\partial}u$ in $L^2(D)$, it follows that $\zeta_1\bar{\partial}^* N\bar{\partial}u_j \to \zeta_1\bar{\partial}^* N\bar{\partial}u$ in $W^{s+\varepsilon}(D)$ and letting $j \to +\infty$ in the estimates (3.3) with u replaced by u_j will imply

$$\|\zeta_1\bar{\partial}^* N\bar{\partial}u\|^2_{s+\varepsilon} \le C\left(\|\zeta_2\bar{\partial}u\|^2_s + \|u\|^2\right) \tag{3.6}$$

for $u \in L^2(D)$.

Proof of the claim. Fix the unit normal vector n at z_0 pointing toward the outside of D. For a positive number α, let D_α be the α-translation of D in the direction n. So $D_\alpha = \{z + \alpha n;\ z \in D\}$. By shrinking U in the subelliptic estimates (3.1) if necessary, assume that the distance between ∂D_α and ∂D in

U is greater than or equal to a constant times α. Let $\{\chi_\alpha, \alpha > 0\}$ be a family of smooth real-valued functions defined in $\mathbf{C}^n$ so that $\chi_\alpha \equiv 1$ in a neighborhood of D and $\chi_\alpha(z) = 0$ if $d(z, D)$, the distance of z to D is greater than or equal to α^2.

Let η_1 and η_2 be another pair of smooth real-valued functions supported in U so that $\eta_2 \equiv 1$ in a neighborhood of the support of η_1 and $\eta_1 \equiv 1$ in a neighborhood of the support of ζ_2, where ζ_1 and ζ_2 are chosen above to be supported in U, so that (3.3) holds.

For α extremely small, consider the family of functions $\{u_\alpha, \alpha > 0\}$ defined by

$$u_\alpha(z) = \eta_2(z)\chi_\alpha(z)u(z - \alpha n) + (1 - \eta_2(z))u(z),$$

where u is a function in $L^2(D)$. After shrinking U if necessary, assume that $\eta_2\bar{\partial}u$ is in $W^s(D)$. By the definition of η_2 and χ_α, the function u_α extends to be defined on $D \cup U$ if $u_\alpha(z)$ is set to equal zero for z in $U \setminus D_\alpha$. It is obvious that $u_\alpha \to u$ in $L^2(D)$ as $\alpha \to 0$. Since

$$\eta_1(z)\bar{\partial}u_\alpha(z) = \eta_1(z)\bar{\partial}\chi_\alpha(z)u(z - \alpha n) + \eta_1(z)\chi_\alpha(z)\bar{\partial}u(z - \alpha n),$$

$\eta_1(z)\bar{\partial}u_\alpha(z)$ is a function in $L^2(U)$ and $(\eta_1\bar{\partial}u_\alpha)(z) = \eta_1(z)\bar{\partial}u(z - \alpha n)$ if $z \in D$, which implies that $\eta_1\bar{\partial}u_\alpha \to \eta_1\bar{\partial}u$ in $W^s(D)$. It is therefore sufficient to prove the claim for u_α in place of u with α small. Now fix such an α.

Choose any sequence of smooth functions $\phi_1, \phi_2, \ldots$ defined in $\mathbf{C}^n$ so that ϕ_j converges to u_α in $L^2(D)$. For δ_j positive and tending to zero as $j \to +\infty$, define

$$u_j = (\eta_1 u_\alpha) * \rho_{\delta_j} + (1 - \eta_1)\phi_j,$$

where $\rho_{\delta_j}(z) = \delta_j^{2n}\rho(z/\delta_j)$ and ρ is a unit test function with support contained in the unit ball. Since $\eta_1 u_\alpha$ has compact support in U, the function u_j is by its definition the restriction of a smooth function on $\mathbf{C}^n$ for every j and

$$\begin{aligned}\|u_j - u_\alpha\| &\leq \left\|(\eta_1 u_\alpha) * \rho_{\delta_j} - \eta_1 u_\alpha\right\| + \left\|(1 - \eta_1)(\phi_j - u_\alpha)\right\| \\ &\leq \left\|(\eta_1 u_\alpha) * \rho_{\delta_j} - \eta_1 u_\alpha\right\| + C\|\phi_j - u_\alpha\| \to 0\end{aligned}$$

as $j \to +\infty$. For δ_j sufficiently small,

$$\begin{aligned}(3.7)\qquad \zeta_2\bar{\partial}u_j &= \zeta_2\Big(\bar{\partial}(\eta_1 u_\alpha) * \rho_{\delta_j} + \bar{\partial}\big((1 - \eta_1)\phi_j\big)\Big) \\ &= \zeta_2\Big(\bar{\partial}(\eta_1 u_\alpha) * \rho_{\delta_j}\Big) = \zeta_2\Big((\eta_1\bar{\partial}u_\alpha) * \rho_{\delta_j}\Big), \quad z \in D.\end{aligned}$$

Now

$$\left(\eta_1\bar{\partial}u_\alpha\right)*\rho_{\delta_j} = \left(\eta_1\bar{\partial}\chi_\alpha u(\cdot - \alpha n)\right)*\rho_{\delta_j} + \left(\eta_1\chi_\alpha\bar{\partial}u(\cdot - \alpha n)\right)*\rho_{\delta_j}$$

and $\bar{\partial}\chi_\alpha$ vanishes in a neighborhood of D. And again if δ_j is sufficiently small then

$$\zeta_2\left(\left(\eta_1\bar{\partial}u_\alpha\right)*\rho_{\delta_j}\right) = \zeta_2\left(\eta_1\bar{\partial}u(\cdot - \alpha n)\right)*\rho_{\delta_j} \quad \text{for } z \in D.$$

Hence (3.7) implies

$$\begin{aligned}\left\|\zeta_2\left(\bar{\partial}u_j - \bar{\partial}u_\alpha\right)\right\|_s &= \left\|\zeta_2\left(\eta_1\bar{\partial}u(\cdot - \alpha n)\right)*\rho_{\delta_j} - \zeta_2\bar{\partial}u(\cdot - \alpha n)\right\|_s \\ &\leq C\left\|\left(\eta_1\bar{\partial}u(\cdot - \alpha n)\right)*\rho_{\delta_j} - \eta_1\bar{\partial}u(\cdot - \alpha n)\right\|_s \to 0\end{aligned}$$

as $j \to +\infty$. The claim is thus proved; so is the proposition. □

The above local density argument works as well for any linear differential operator with smooth coefficients replacing the $\bar{\partial}$ operator.

If the $\bar{\partial}$-Neumann operator on D is further assumed to be globally regular, then a similar argument as above implies that [11, Proposition 3.5] for any $s \geq 0$,

$$\|\zeta_1\bar{\partial}^*N\bar{\partial}u\|_{s+\varepsilon} \leq C\left(\|\zeta_2\bar{\partial}u\|_s + \|u\|^*_{-M}\right) \tag{3.8}$$

for all $u \in L^2(D)$. This is the case, for example, when D is a smoothly bounded pseudoconvex domain with all boundary points being finite type.

4. Main results

The following theorem is a simple consequence of estimates (3.6).

THEOREM 4.1. *For any smoothly bounded domain Ω in $\mathbf{C}^n$, the Bergman projection P of Ω satisfies weak pseudo-local estimates at any extreme boundary point z_0. Namely, there is a neighborhood U of z_0 so that for any pair of real-valued functions ζ_1, ζ_2 in $C_0^\infty(U)$ supported in U with $\zeta_2 \equiv 1$ near the support of ζ_1, and any real number $s \geq 0$,*

$$\|\zeta_1 Pu\|_s \leq C\left(\|\zeta_2 u\|_s + \|u\|\right), \tag{4.1}$$

for all u in $L^2(\Omega)$.

Proof. Let D be a pseudoconvex domain defined in Definition 2.2 which circumscribes Ω so that ∂D coincides with $\partial\Omega$ near z_0. Fix a neighborhood U as in (3.6) and any real-valued functions ζ_1, ζ_2 in $C_0^\infty(U)$ with $\zeta_2 \equiv 1$ in a neighborhood of the support of ζ_1.

Extend the function u and Pu to be defined in D by letting their values outside Ω equal to zero. Then $u - Pu$ is in $L^2(D)$ and is orthogonal to holomorphic functions in Ω, hence orthogonal to holomorphic functions in D which are considered as functions in Ω by restriction. So if P_D is the Bergman projection of D, then

$$P_D(u - Pu) = 0. \tag{4.2}$$

Choose a smooth real-valued function η such that $\eta \equiv 1$ in a neighborhood of the support of ζ_1 and $\zeta_2 \equiv 1$ in a neighborhood of the support of η. It follows from (3.6) that

$$\begin{aligned}\|\zeta_1 Pu\|_s &\leq \|\zeta_1(Pu - P_D Pu)\|_s + \|\zeta_1 P_D u\|_s \\ &\leq C\left(\|\eta\bar{\partial}(Pu)\|_{s-\varepsilon} + \|Pu\|\right) + \|\zeta_1 P_D u\|_s,\end{aligned}$$

where $\varepsilon > 0$ is a small number associated to the subelliptic estimate in (3.1) at z_0. Hence

$$\|\zeta_1 Pu\|_s \leq C\|u\| + \|\zeta_1 P_D u\|_s.$$

The last term in the above estimates is bounded by $C(\|\zeta_2 u\|_s + \|u\|)$, so the result follows. □

An immediate consequence of Theorem 4.1 is the next corollary.

COROLLARY 4.2. *Every smoothly bounded domain satisfies local condition R at all its extreme boundary points.*

Proof. For any u in $C^\infty(\overline{\Omega})$, it follows from (4.1) that $\|\zeta_1 Pu\|_s$ is bounded for all $s \geq 0$. By Sobolev's Lemma, Pu is smooth up to the boundary near the extreme boundary point z_0. Therefore, Ω satisfies local condition R at z_0. □

The simple equation (4.2) in Theorem 4.1, relating the Bergman projections of the two domains, also implies that if D is a smoothly bounded pseudoconvex domain of finite type in $\mathbf{C}^n$, and A is any compact subset of D, then the Bergman projection $P_{D\setminus A}$ of $D \setminus A$ satisfies condition R.

Indeed, if $u \in C^\infty(\overline{D \setminus A})$ then $P_{D\setminus A}u$ extends holomorphically inside D. Corollary 4.2 implies that the function $P_{D\setminus A}u$ extends smoothly near any boundary point of $D \setminus A$. Hence condition R holds for D. This result is proved in [4] in a much more general setting.

More can be said about local condition R. It is easy to show the following proposition which is analogous to the equivalences of global condition R.

PROPOSITION 4.3. *Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ and let z_0 be a boundary point of Ω. Then the following three conditions are equivalent.*

(i) *Ω satisfies local condition R at z_0;*

(ii) *there exists an open neighborhood U of z_0 so that if χ is a smooth function supported in U, then for all $s \geq 0$, and any function $u \in C^\infty(\overline{\Omega})$, the Bergman projection P of Ω admits the estimates*

$$\|\chi P u\|_s \leq C\|u\|_{s+N}, \tag{4.3}$$

for some positive N depending on s;

(iii) *there exists an open neighborhood U of z_0 so that if χ is a smooth function supported in U, then for all $s \geq 0$, the Bergman kernel function $K(z,w)$ satisfies*

$$\|\chi K(\cdot,w)\|_s \leq Cd(w)^{-m}, \quad \textit{all } w \in \Omega, \tag{4.4}$$

for some positive m depending on s.

Proof. It will be shown that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

That (i) implies (ii) follows from the closed graph theorem. Suppose that Ω satisfies local condition R at z_0. It suffices to prove (4.3) for s which is a nonnegative integer. Let $U_1, U_2, \ldots$ be a sequence of open neighborhoods of z_0 with the properties that each U_{k+1} is relatively compact in U_k and that the diameter of U_k tends to zero as $k \to +\infty$. Define a sequence of subspaces $F_1, F_2, \ldots$ of $H(\Omega)$ by

$$F_k = C^\infty\left(\overline{\Omega \cap U_k}\right) \cap H(\Omega),$$

and let $F = \bigcup_{k=1}^\infty F_k$. Each F_k is a Fréchet space with seminorms

$$\left\{p_0 = \|\cdot\|,\, p_s^{(k)} = \|\cdot\|_{W^s(\overline{\Omega \cap U_k})},\, s = 1, 2, \ldots\right\}.$$

If $k \geq j$, F_k can be viewed as containing F_j. Hence F, endowed with the inductive linear topology (the topology defined so that each embedding of F_k into F is continuous) is a reduced and strict inductive limit, is thus a complete Fréchet space.

By assumption the Bergman projection P of Ω maps $C^\infty(\overline{\Omega})$ into F. Since $C^\infty(\overline{\Omega})$ is also a Fréchet space, by applying a form of the closed graph theorem, it can be concluded that there is an integer k such that $P(C^\infty(\overline{\Omega}))$ is contained in F_k and that P: $C^\infty(\overline{\Omega}) \to F_k$ is continuous, provided that P from $C^\infty(\overline{\Omega})$ to F is a closed mapping.

To prove the closedness of P, let $u_n \in C^\infty(\overline{\Omega})$ be so that $u_n \to 0$ in $C^\infty(\overline{\Omega})$ and that $Pu_n \to g$ in F. The set $\Lambda = \{g, Pu_n, n = 1, 2, \ldots, \}$ is bounded in F, so Λ is contained and bounded in F_k for some k. But then $u_n \to 0$ in $L^2(\Omega)$, which implies that $Pu_n \to 0$ in F_k. So $g \equiv 0$ and the graph of P is closed.

The continuity of P: $C^\infty(\overline{\Omega}) \to F_k$ implies that for any integer $s \geq 0$ and u in $C^\infty(\overline{\Omega})$, there exists an N which depends on s,

$$\|Pu\|_{W^s(\Omega \cap U_k)} \leq C\|u\|_{s+N}.$$

This certainly implies (4.3).

The proof that (ii) $\Rightarrow$ (iii) and that (iii) $\Rightarrow$ (i) will be an imitation of the method in proving the global theorems used in [1] and [7].

For each $w \in \Omega$, choose a polydisc centered at w whose polyradius is $\frac{1}{2}d(w)$ in the complex direction normal to $\partial\Omega$ and is uniformly $O(d(w)^{1/2})$ in the complex tangential directions. Let f be a linear isomorphism of this polydisc with the unit polydisc and let ϕ be a smooth radial function compactly supported in the unit polydisc with $\int_{\mathbf{C}^n} \phi \, dV_z = 1$. Let $\phi_w = |\det[f']|^2(\phi \circ f)$, where f' is the complex Jacobian of f. If $u \in H(\Omega)$, then

$$\int_\Omega u\phi_w \, dV_z = \int_{\mathbf{C}^n} (u \circ f^{-1})\phi \, dV_z = (u \circ f^{-1})(0) = u(w).$$

So $P\phi_w = K(\cdot, w)$. And $\|\phi_w\|_{s+N} \leq Cd(w)^{-(s+N+n+1)}$. Thus for any $w \in \Omega$,

$$\|\chi K(\cdot, w)\|_s = \|\chi P\phi_w\|_s \leq C\|\phi_w\|_{s+N} \leq Cd(w)^{-(s+N+n+1)}.$$

So (4.4) follows with $m = s + N + n + 1$.

Finally, suppose (iii) holds. From (2.1), if u is in $C^\infty(\overline{\Omega})$, apply Φ^s to u then $\Phi^s u$ is in $W_0^s(\Omega)$. With $s = m + n + 1$,

$$|\Phi^s u(w)| \leq C\|\Phi^s u\|_s d(w)^m,$$

and by Fubini's theorem

$$\begin{aligned}
\|\chi Pu\|_s &= \left\| \int_\Omega \chi K(\cdot, w)\Phi^s u(w) \, dV_w \right\|_s \\
&\leq C\int_\Omega \|\chi K(\cdot, w)\|_s |\Phi^s u(w)| dV_w \\
&\leq C\int_\Omega d(w)^{-m}\|\Phi^s u\|_s d(w)^m dV_w \\
&\leq C\|\Phi^s u\|_s \leq C\|u\|_{s+N} < +\infty.
\end{aligned}$$

Since s is an arbitrary positive integer and $\chi \equiv 1$ near z_0, it is clear from Sobolev's lemma that Pu extends smoothly up to the boundary near z_0. The proof is thus completed. □

The above proposition and Corollary 4.2 imply the following result.

COROLLARY 4.4. *Assume that z_0 is an extreme boundary point of Ω in $\mathbf{C}^n$. Then for any w fixed in Ω, the Bergman kernel $K(z,w)$ as a function of z is C^∞ smooth up to the boundary near z_0.*

Let $\tilde{\Omega}$ be the envelope of holomorphy of Ω. So $\tilde{\Omega}$ is a Riemann domain over $\mathbf{C}^n$ such that any holomorphic function on Ω can be extended holomorphically to $\tilde{\Omega}$. Since the envelope of a product of two domains is the product of their envelopes, it is clear that the Bergman kernel function $K(z,w)$ extends to be holomorphic in z-variable and antiholomorphic in w-variable in $\tilde{\Omega} \times \tilde{\Omega}$. For brevity, assume $\tilde{\Omega} \subset \mathbf{C}^n$. Now fix w in $\tilde{\Omega}$. From [18, Theorem 1.2.4 and Lemma 5.4.1], there exist a compact set F and an open set O, so that 1) $F \Subset O \Subset \Omega$; 2) $|f(w)| \le \sup_F |f|$, for all $f \in H(\tilde{\Omega})$; 3) $\sup_F |f| \le C\int_O |f|\,dV$. Therefore

$$\left|\frac{\partial^\alpha}{\partial z^\alpha}K(z,w)\right|^2 \le \sup_{\zeta\in F}\left|\frac{\partial^\alpha}{\partial z^\alpha}K(z,\zeta)\right|^2 \le C\int_O\left|\frac{\partial^\alpha}{\partial z^\alpha}K(z,\zeta)\right|^2 dV_\zeta,$$

for any multi-index α. It now follows from the above corollary that the absolute value of $\partial^\alpha K(z,w)/\partial z^\alpha$ is uniformly bounded when z varies in Ω near z_0. So an extension of the previous corollary is proved.

COROLLARY 4.5. *Assume that the envelope of holomorphy $\tilde{\Omega}$ of Ω is contained in $\mathbf{C}^n$. Then the same conclusion in Corollary 4.4 holds for any point $w \in \tilde{\Omega}$.*

Also notice that the above results imply that all the derivatives of $K(\cdot,w)$ vary uniformly for w in compact subsets of Ω. Thus $K(z_0,w)$ is well defined and antiholomorphic in Ω as a function of w.

If the multi-index α is taken to be $(0,\dots,0)$ in the previous inequality, by integrating on Ω with respect to z, then

$$\int_\Omega |K(z,w)|^2 dV_z \le C\int_\Omega\int_O |K(z,\zeta)|^2 dV_\zeta\, dV_z$$
$$= C\int_O\int_\Omega |K(z,\zeta)|^2 dV_z\, dV_\zeta = C\int_O K(\zeta,\zeta)\,dV_\zeta < +\infty.$$

So $K(\cdot,w)$, for all $w \in \tilde{\Omega}$, is an square-integrable function on Ω.

When Ω satisfies global regularity estimates, the extension of $K(z,w)$ in both variables is possible.

THEOREM 4.6. *Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ satisfying condition R. Assume that z_0 is an extreme boundary point of Ω. Then for any point $w_0 \in \overline{\Omega}$, $w_0 \neq z_0$, there are disjoint open neighborhoods U of z_0 and V of w_0, so that*

$$K(z,w) \in C^\infty\big(\overline{(\Omega \cap U)} \times \overline{(\Omega \cap V)}\big).$$

The proof of the theorem differs only in minor part from the proof of Theorem 1 in [3]. From the definition of the extreme boundary points, there is a pseudoconvex domain D containing Ω such that the boundary of D coincides with the boundary of Ω near z_0, which is a point of finite type. When there is an additional assumption that D satisfies condition R, by applying known results an easy proof can be found for this theorem. Also if all the intersection points of ∂D and $\partial\Omega$ are of finite type, the pseudoconvex domain can be altered to become a finite type domain which therefore satisfies condition R. This follows from Catlin's bumping theorem for domains of finite type [12] and the proof of Proposition 2.3.

Proof under the additional assumption. Fix neighborhoods U of z_0 and V of w_0 respectively so that the intersection of $\overline{U}$ and $\overline{V}$ is empty and $D \cap U = \Omega \cap U$. For any multi-index $\beta = (\beta_1, \dots, \beta_n)$, let $D^\beta_{\overline{w}}$ denote $(\partial^\beta/\partial\overline{w}_1^{\beta_1}) \cdots (\partial^\beta/\partial\overline{w}_n^{\beta_n})$. Since $D^\beta_{\overline{w}}K(z,w) \in L^2(\Omega)$ as a function of z, if it is regarded as a function in $L^2(D)$ by zero extension, then

$$P_D\big(D^\beta_{\overline{w}}K(\cdot,w)\big)(z) = D^\beta_{\overline{w}}K_D(z,w),$$

where P_D and $K_D(\cdot,\cdot)$ are the Bergman projection and kernel associated to D. For $w \in \Omega \cap V$ let ζ_1 and ζ_2 be the smooth cut-off functions supported in U as in Theorem 4.1. For any $s \geq 0$, from (3.8) and the fact that $\|\zeta_1 D^\beta_{\overline{w}}K(\cdot,w)\|_{W^s(\Omega)}$ is equal to $\|\zeta_1 D^\beta_{\overline{w}}K(\cdot,w)\|_{W^s(D)}$,

$$\begin{aligned}
(4.5)\quad & \big\|\zeta_1 D^\beta_{\overline{w}}K(\cdot,w)\big\|_{W^s(\Omega)} \\
&\leq \big\|\zeta_1\big(D^\beta_{\overline{w}}K(\cdot,w) - P_D D^\beta_{\overline{w}}K(\cdot,w)\big)\big\|_{W^s(D)} \\
&\quad + \big\|\zeta_1 D^\beta_{\overline{w}}K_D(\cdot,w)\big\|_{W^s(D)} \\
&\leq C\Big(\big\|\zeta_2\bar{\partial}\big(D^\beta_{\overline{w}}K(\cdot,w)\big)\big\|_{W^{s-\varepsilon}(D)} + \big\|D^\beta_{\overline{w}}K(\cdot,w)\big\|^*_{W^{-N}(D)}\Big) \\
&\quad + \big\|\zeta_1 D^\beta_{\overline{w}}K_D(\cdot,w)\big\|_{W^s(D)} \\
&= C\big\|D^\beta_{\overline{w}}K(\cdot,w)\big\|^*_{W^{-N}(D)} + \big\|\zeta_1 D^\beta_{\overline{w}}K_D(\cdot,w)\big\|_{W^s(D)}
\end{aligned}$$

where N is an arbitrary positive constant, ε is as before, and C does not depend on w. Since Ω satisfies condition R and $D_{\bar{w}}^{\beta}K(\cdot, w)$ has support in $\overline{\Omega}$ and is holomorphic in Ω it follows that

$$\| D_{\bar{w}}^{\beta}K(\cdot, w) \|^{*}_{W^{-N}(D)} \leq \| D_{\bar{w}}^{\beta}K(\cdot, w) \|^{*}_{W^{-N}(\Omega)} \leq C \| D_{\bar{w}}^{\beta}K(\cdot, w) \|_{W^{-N}(\Omega)}.$$

If N is chosen big enough the last term is uniformly bounded for $w \in V$. Then so is the first term above. The second term following the equal sign in (4.5) can be treated in different cases. If w_0 is in D, then the neighborhood V of w_0 can be so chosen that $V \Subset D$. Then clearly the second term is uniformly bounded. If w_0 is on the boundary ∂D, since D satisfies condition R, by Theorem 2 of [3] (or [10]) the function $K_D(z, w)$ extends in both variables up to the boundary off the boundary diagonal. So it is still uniformly bounded for $w \in V$. Hence the proof is concluded by applying Sobolev's lemma. □

As in Theorem 3 of [3], the weak pseudolocal estimates at extreme boundary points can be strengthened if the global condition R is satisfied.

THEOREM 4.7. *Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ satisfying condition R. Assume that z_0 is an extreme boundary point of Ω. Then the Bergman projection P of Ω satisfies strong pseudolocal estimates at z_0. Namely, there is a neighborhood U of z_0, so that for any pair of smooth real-valued functions ζ_1 and ζ_2 supported in U and $\zeta_2 \equiv 1$ in a neighborhood of the support of ζ_1, for any $s \geq 0$ and an arbitrary integer $N > 0$,*

$$\|\zeta_1 Pu\|_s \leq C(\|\zeta_2 u\|_s + \|u\|^{*}_{-N}),$$

for all u in $L^2(\Omega)$.

Proof. It follows from Theorem 1 that

$$\begin{aligned}\|\zeta_1 Pu\|_s &\leq \|\zeta_1 P(\zeta_2 u)\|_s + \|\zeta_1 P((1-\zeta_2)u)\|_s \\ &\leq C(\|\zeta_2 u\|_s + \|\zeta_2 u\|) + \|\zeta_1 P((1-\zeta_2)u)\|_s \\ &\leq C\|\zeta_2 u\|_s + \|\zeta_1 P((1-\zeta_2)u)\|_s.\end{aligned}$$

Since

$$\zeta_1(z)P((1-\zeta_2)u)(z) = \int_{\Omega} \zeta_1(z)K(z,w)(1-\zeta_2(w))u(w)\, dV_w,$$

and the support of ζ_1, which is contained in U, does not meet the support of

$1-\zeta_2$, Theorem 4.6 clearly implies that for any positive integer N,

$$\|\zeta_1 P((1-\zeta_2 u))\|_s \leq C\|u\|^*_{-N}.$$

Therefore,

$$\|\zeta_1 Pu\|_s \leq C(\|\zeta_2 u\|_s + \|u\|^*_{-N})$$

for all u in $C^\infty(\overline{\Omega})$. □

Observe that in view of the proof of Theorem 1 in [3], it is possible to show the same smooth extension result for the kernel function in Theorem 4.6 for both w_0 and z_0 being extreme, without the global regularity assumption of the Bergman projection.

If the boundary of a pseudoconvex domain D is real analytic near a strictly pseudoconvex boundary point z_0, then the $\bar{\partial}$-Neumann operator applied to any $(0,1)$-form u with square-integrable coefficients, whose support is a positive distance away from z_0, gives a function Nu which extends to be real analytic near z_0 (see [22] and [23]). From this, some analytic extension results of the Bergman kernel have been derived in [6]. In the rest of the section, such behavior of the Bergman kernel $K(\cdot,\cdot)$ associated to an arbitrary smoothly bounded domain Ω will be studied at strictly pseudoconvex boundary points of extreme type, without assuming the global pseudoconvexity.

THEOREM 4.8. *Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ which satisfies condition R. Assume that z_0 is a strictly pseudoconvex boundary point of extreme type and that the boundary $\partial\Omega$ is real analytic near z_0. Then for any open neighborhood V of z_0, there exists a neighborhood U of z_0, with $U \Subset V$, so that the Bergman kernel $K(z,w)$ extends to be in $C^\infty(U \times (\overline{\Omega}\setminus V))$ as a function which is holomorphic in z and anti-holomorphic in w on $U \times (\Omega \setminus \overline{V})$.*

Proof. Let D be a pseudoconvex domain containing Ω so that ∂D and $\partial\Omega$ coincides near z_0. Let V be an arbitrary neighborhood of z_0. Without loss of generality, assume that there is another open neighborhood W of z_0, with $V \Subset W$ and $D \cap W = \Omega \cap W$. From Theorem 4.5, there exists a neighborhood U of z_0 so that $K(\cdot,\cdot) \in C^\infty((\overline{\Omega \cap U}) \times (\overline{\Omega \setminus V}))$ with $U \Subset V$.

Fix χ, a real-valued function in $C^\infty(\mathbf{C}^n)$ which is compactly supported in W and equal to one in a neighborhood of V. Then $h \to \Phi^s((1-\chi)h)$ defines a bounded linear operator from the subspace of holomorphic functions in $W^{s+N}(\Omega)$ into $W_0^s(\Omega)$, where Φ^s is defined in (2.1). For any multi-index α, set

$$\psi_\alpha(\cdot,w) = \chi\frac{\partial^\alpha}{\partial\overline{w}^\alpha}K(\cdot,w) + \Phi^2\left((1-\chi)\frac{\partial^\alpha}{\partial\overline{w}^\alpha}K(\cdot,w)\right), \qquad w \in \Omega\setminus\overline{V}.$$

Then $\psi_\alpha(\cdot, w) \in W^2(\Omega)$ can be considered as in $W^2(D)$ by zero extension. If P_D and $K_D(\cdot, \cdot)$ are the Bergman projection and the kernel function associated to D, then clearly

$$\frac{\partial^\alpha}{\partial \overline{w}^\alpha} K_D(\cdot, w) = P_D(\psi_\alpha(\cdot, w)).$$

Kohn's formula $P_D = I - \bar{\partial}^* N \bar{\partial}$ implies that

$$\psi_\alpha(\cdot, w) = \frac{\partial^\alpha}{\partial \overline{w}^\alpha} K_D(\cdot, w) + \bar{\partial}^* N \bar{\partial} \psi_\alpha(\cdot, w).$$

Since z_0 is a strictly pseudoconvex point of D, after shrinking U if necessary, by Theorem 1 of [6], $\partial^\alpha K_D(\cdot, w)/\partial \overline{w}^\alpha$ extends to be in $C^\infty(U \times (\overline{\Omega} \setminus V))$ and to be holomorphic in z and anti-holomorphic in w in $U \times (\overline{\Omega} \setminus V)$. And

$$\bar{\partial}\psi_\alpha(\cdot, w) = (\bar{\partial}\chi)\frac{\partial^\alpha}{\partial \overline{w}^\alpha} K(\cdot, w) + \bar{\partial}\left(\Phi^2\left((1-\chi)\frac{\partial^\alpha}{\partial \overline{w}^\alpha} K(\cdot, w)\right)\right),$$

which is equal to zero in a neighborhood of V. Hence $\bar{\partial}^* N \bar{\partial} \psi_\alpha(\cdot, w)$ extends to be holomorphic past z_0 for each w fixed. Moreover, the family of functions

$$\left\{\bar{\partial}\psi_\alpha(\cdot, w); w \in \Omega \setminus V\right\}$$

is uniformly bounded in W^1 norm on the set U. The same Baire category argument in [6] shows that after again shrinking U if needed, $\bar{\partial}^* N \bar{\partial} \psi_\alpha(\cdot, w)$ extends holomorphically to U. Also for $w \in \Omega \setminus V$,

$$\left\| \bar{\partial}^* N \bar{\partial} \psi_\alpha(\cdot, w) \right\|_{L^2(\Omega \cup U)} \leq C \left\| \bar{\partial} \psi_\alpha(\cdot, w) \right\|_{W^1(\Omega)} \leq C < +\infty,$$

where the constants are independent of $w \in \Omega \setminus V$. Therefore

$$\chi \frac{\partial^\alpha}{\partial \overline{w}^\alpha} K(\cdot, w) = \psi_\alpha(\cdot, w) - \Phi^2\left((1-\chi)\frac{\partial^\alpha}{\partial \overline{w}^\alpha} K(\cdot, w)\right)$$

is uniformly bounded in any compact subset of U. Since the multi-index α is arbitrary, from Sobolev's lemma $K(z, w) \in C^\infty(U \times (\overline{\Omega} \setminus V))$ and is holomorphic in z and anti-holomorphic in w in $U \times (\overline{\Omega} \setminus V)$. □

Also as before, for w in $\tilde{\Omega}$, the envelope of Ω, the same argument as above shows that $K(\cdot, w)$ extends to be analytic past all strictly pseudoconvex extreme boundary points.

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