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DIMENSION, VOLUME, AND SPECTRUM OF A RIEMANNIAN MANIFOLD

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1. Introduction

We consider the Laplace operator Δ defined on a Riemannian manifold M. In local coordinates we have;

$$\Delta f = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\, \frac{\partial f}{\partial x^j} \right),$$

where g_{ij} is the metric tensor, $g^{ik}g_{kj} = \delta^i_j$, and $g = \det(g_{ij})$.

The spectrum of Δ on M is the set of eigenvalues for the eigenvalue problem given by the equation

$$\Delta\phi = \lambda\phi,$$

where in case M has boundary we require that $\phi = 0$ on ∂M. In the latter case the spectrum is called the Dirichlet spectrum. We will sometimes be interested in the case where the manifold with boundary of interest is (the closure of) a relatively compact connected domain D in a complete Riemannian manifold X. Since M (or $\overline{D}$) is assumed to be compact the spectrum is given by a sequence of nonnegative numbers;

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \uparrow \infty$$

counted with multiplicity.

Note. As a matter of convention we will refer to the (Dirichlet) spectrum of Δ on M simply as the spectrum of M. Also, all manifolds referred to in this paper will be assumed to be connected and all domains will be assumed to have smooth boundary.

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The spectrum of a manifold determines certain aspects of its geometry. For instance, Weyl's formula,

$$\lim_{k\to\infty}\frac{\lambda_k}{k^{2/n}}=\frac{c_n}{\mathrm{Vol}(M)^{2/n}},$$

shows that the spectrum determines both the volume and the dimension, n, of the manifold. One may ask how many eigenvalues it takes to determine the dimension. This is one of the main topics of the present paper. For determining the dimension of a manifold from a finite part of its spectrum we will need some restriction on the class of manifolds under consideration. This is demonstrated by the following simple counterexample.

Counterexample. Let M be any compact Riemannian manifold and let $\{\lambda_i(M)\}$ be its spectrum. Also, let $S^1(\varepsilon)$ denote the circle of radius ε. Suppose that we try to determine the dimension of M by using N eigenvalues where N does not depend on any geometric assumption on M. But this must fail since $M \times S^1(\varepsilon)$ has eigenvalues $\{\lambda_i(M) + 4\pi n^2/\varepsilon^2\}$ and so, for ε small enough, the first N eigenvalues of M and $M \times S^1(\varepsilon)$ agree. However, M and $M \times S^1(\varepsilon)$ differ in dimension. Thus some a priori geometric assumptions must be made on the class of manifolds M if we are to get a theorem.

A related question, discussed in [D-L1], [D-L2], [L], and [T], is the following. Given ε, how many eigenvalues will suffice to provide an approximation of the volume of the manifold (or domain) accurate to within ε? The answer should depend on certain allowable a priori data to be specified.

There exists an asymptotic formula for the trace of the heat kernel which gives the following expression involving the spectrum.

$$\sum_{j=1}^{\infty}\varepsilon^{-\lambda_j t}\sim(4\pi t)^{-n/2}\mathrm{Vol}(M)$$

as $t \downarrow 0$. Now, from this we once again see that the volume and dimension of a manifold are determined by the spectrum but we are not any closer to answering the questions that we have raised about using a finite number of eigenvalues. The problem is partially that we have to get good geometric control on the error in these asymptotic formulas. Much has been done to achieve this in [D-L1] and [D-L2] and we will be relying heavily on variations of the analysis found therein. Among the results to be proven here we have the following theorems.

THEOREM 1. *Let M be a connected compact Riemannian manifold without boundary such that the sectional curvature of M is bounded from above by $b > 0$ and from below by $\kappa > 0$. Let $c > 0$ be a lower bound on the injectivity radius of M. Then there exists a number N such that the dimension of M is* *determined by the first N eigenvalues from the spectrum of M. Furthermore, N depends only on* κ, b, c *and* $\mathcal{M}(d) = \max\{j\colon \lambda_j < d\}$, *where d depends only on* κ, *and* $\lambda_1(M)$.

THEOREM 2. *Let D be a connected convex Euclidean domain. Then the dimension of D is determined by the first N Dirichlet eigenvalues where N depends only on* λ_1, *and* $\mathcal{M}(\lambda_1\beta)$ *and where* β *depends on* λ_1 *and* λ_2.

We will also, as a matter of course, obtain the following result which is the closed manifold version of Theorem 4.3 of [D-L1].

THEOREM 3. *Let M be a connected compact Riemannian manifold without boundary and of dimension n. Let the Ricci curvature be bounded from below by* $-a(n-1) \leq 0$ *and the sectional curvature bounded from above by* $b > 0$. *Let* $c > 0$ *be a lower bound for the injectivity radius* $i(M)$ *of M. Then given* $\varepsilon > 0$ *there are numbers* δ *and* $N(\delta)$ *such that*

$$\left|(4\pi\delta)^{n/2}\sum_{i=1}^{N} e^{-\lambda_i\delta} - \operatorname{Vol} M\right| < c_5\delta < \varepsilon$$

for $N \geq N(\delta)$, *and where* $\delta, N(\delta)$ *depend only on* ε, *a bound on* diam M, dim M, a, b, *and* c. *Also* c_5 *depends only on* diam M, dim M, a, b, *and* c.

Note. We will see below in Lemma 1 that in a certain sense diam M can be bounded using a finite number of eigenvalues.

Remark. Theorems 1, 2, and 3 will follow from somewhat more general statements to be proven in the sequel (Theorems 6, 7, and 8 below).

2. Volume and a finite part of the spectrum

In this section we give a few *known* results about obtaining arbitrarily accurate approximations of a manifold's volume from a finite number of its eigenvalues. This will serve to make the discussion of Section 1 more concrete. Also, these ideas will be of use to us later when we attack the problem of determining the dimension of a manifold from a finite number of eigenvalues. The first result applies to the class of convex Euclidean domains of fixed dimension and is due to P. Li and S.T. Yau [T].

THEOREM 4 (Li and Yau). *Let D be a convex domain in* $\mathbf{R}^n$. *Given* $\varepsilon > 0$ *there exists an* N_0 *depending only on* ε, n, λ_1 *and* $\mathcal{M}(\beta\lambda_1)$ *such that for* $k > N_0$

$$\left|\frac{\lambda_k}{k^{2/n}} - \frac{c_n}{(\operatorname{Vol}(D))^{2/n}}\right| < \varepsilon$$

where $\beta > 8\pi^{-2}n(n+4)$ *and* $\mathcal{M}(\beta\lambda_1) = \max\{j\colon \lambda_j \leq \beta\lambda_1\}$.

Notice that $\mathcal{M}(\lambda_1\beta)$ is determined by a finite part of the spectrum. It is worth pointing out that $\mathcal{M}(\lambda_1\beta)$ really occurs in the proof as part of an upper bound for the out radius of the domain:

$$R_{\text{out}} \leq \frac{\pi \mathcal{M}(\lambda_1\beta)}{\sqrt{\lambda_1}}. \tag{1.2}$$

The above theorem can be looked at as saying that one can take crude information about a manifold (say R_{out}) and use this to determine the number of eigenvalues needed to give arbitrarily accurate information about a geometric invariant of the manifold (in this case volume). For purposes of this paper let us call the set of all Riemannian manifolds satisfying a given set of assumptions (such as curvature bounds) a geometric class. In some sense, a theorem like the one above tells us something about how much information is obtainable from a *finite* part of the spectrum. The more inclusive the class to which the theorem applies the more information is thereby shown to be contained in some finite part of the spectrum.

Although based on heat kernel asymptotics instead of Weyl's asymptotic formula, the following theorem is analogous to Theorem 4 and is a good example of the type of theorem to be found in [D-L1] and [D-L2]. Before we state the theorem we need a definition.

DEFINITION. A domain D in a Riemannian manifold is *weakly convex* if whenever two points in ∂D are connected by a unique minimizing geodesic segment then that segment is contained in the closure of D.

THEOREM 5. *Let D be a smooth weakly convex domain in a complete Riemanian manifold X of dimension n. Suppose further that D is contained in a geodesic ball of radius R say $B(p, R)$. Let $\varepsilon > 0$ be given. Then for $\delta > 0$ sufficiently small and a positive integer $N(\delta)$ one has*

$$\left| (4\pi\delta)^{n/2} \sum_{i=1}^{N(\delta)} e^{-\lambda_i\delta} - \operatorname{Vol} D \right| < c_4\delta^{1/2} < \varepsilon.$$

Furthermore, δ, $N(\delta)$ and c_4 depend only upon n, R, an upper bound for the sectional curvatures of X, a lower bound for the Ricci curvatures of X, and a lower bound for the convexity radius of X. Actually, the convexity radius and curvature bounds need only apply on say $B(p, R+1)$.

Now we will prove some inequalities similar to those used to prove Theorem 5 above but for the case of a compact manifold without boundary.

Then we will prove the closed manifold version of Theorem 5. For later use, we will need to modify our inequalities so that the constants that arise depend only on an upper bound for the dimension of the manifold rather than the precise dimension. We also get a slightly modified version of Theorem 5 above where we only need some (possibly crude) a priori bound on dimension instead of the precise dimension. This will all be straightforward but these versions will be needed in the next section when we come to determining the dimension from a finite part of the spectrum.

In our most general setting we will need to assume a bound on dimension. This is analogous to the bound on the out-radius which sometimes must be assumed in the theorems on approximating the volume mentioned above. Just as there are cases where we can determine an out-radius bound either by using a finite number of eigenvalues or by using curvature assumptions there will be cases where we will similarly be able to omit even the bound on dimension (thus arriving at Theorems 1 and 2).

Theorem 3 will follow from the following theorem together with the diameter bound given by Lemma 1 below.

THEOREM 6. *Let M be a compact Riemannian manifold without boundary and with Ricci curvature bounded below by $-a(\dim M - 1)$ and sectional curvatures bounded from above by b where $a, b \geq 0$. Assume that $\dim M \leq n_0$ and that the injectivity radius of M is bounded from below by $c > 0$. Then given $\varepsilon > 0$ there exist numbers δ and $N(\delta)$ such that for $N \geq N(\delta)$,*

$$\left|(4\pi\delta)^{\dim M/2} \sum_{i=1}^{N} e^{-\lambda_i\delta} - \operatorname{Vol} M\right| < c_5\delta < \varepsilon,$$

and for $l > \dim M$,

$$\left|(4\pi\delta)^{l/2} \sum_{i=1}^{N} e^{-\lambda_i\delta}\right| < 4\pi c_5\delta^2 + c_6(4\pi\delta) < \varepsilon,$$

where δ, $N(\delta)$, c_5 and c_6 depend only on a, b, n_0, c and a bound for the diameter of M.

We also have the following.

THEOREM 6′. *Let D be a smooth weakly convex domain contained in a complete Riemannian manifold X. Let $n = \dim X \leq n_0$. Assume that D is contained in a ball $B(p, r_0)$ such that on $B(p, r_0 + 1)$, the Ricci curvature is bounded below by $-a(\dim X - 1)$, the sectional curvature is bounded above by b and the injectivity radii are bounded from below by c. Then given $\varepsilon > 0$*

there exists a δ, *and* $N(\delta)$ *such that for* $N \geq N(\delta)$

$$\left|(4\pi\delta)^{\dim D/2}\sum_{i=1}^{N} e^{-\lambda_i\delta} - \operatorname{Vol} D\right| < c_5\delta^{1/2} < \varepsilon,$$

and for $l >$ dim X,

$$\left|(4\pi\delta)^{l/2}\sum_{i=1}^{N} e^{-\lambda_i\delta}\right| < 4\pi c_5\delta^{3/2} + c_6(4\pi\delta)^{1/2} < \varepsilon,$$

where δ, $N(\delta)$, *and* c_5 *depend only on* a, b, n_0, c *and* r_0.

Remark. There is a subtle point that needs to be brought out regarding the statements of Theorems 6 and 6′. In using these theorems to get Theorem 3 (and the analog for weakly convex domains) we will just assume dim $M = n = n_0$. However, in applying these theorems in Section 4 below the dimension n is an unknown to be determined by finite spectral means. There we will only assume an upper bound on dimension, say n_0. In this case the form of the lower bound on Ricci curvature is somewhat inappropriate since it explicitly involves the exact dimension n. For this reason we will instead make the stronger assumption that the sectional curvature is bounded from below by $-a$. The statements of Theorems 6 and 6′ are nonetheless true and have this unusual form only because of the double use to which we put them.

3. Inequalities

For the proofs of Theorem 6 and 6′ we will need some inequalities which we record as propositions. Let trace $E_M(t)$ and trace $E_D(t)$ be the traces of the heat kernels on M and D respectively. Here, we use the Dirichlet heat kernel on D.

PROPOSITION 1. *Let* M *be a compact Riemannian manifold with Ricci curvature bounded below by* $-a(\dim M - 1)$ *and sectional curvature bounded above by* b. *Also let the injectivity radius be bounded below by* $c > 0$. *Then if* $\dim M \leq n_0$ *we have*

$$\begin{aligned}&\left|\operatorname{trace} E_M(t) - (4\pi t)^{-n/2}\operatorname{Vol} M\right|\\ &\qquad \leq t^{-n/2}\hat{\beta}_{11}e^{\hat{\beta}_{12}\sqrt{a}c}e^{-\hat{\beta}_{13}c^2/t} + \hat{\beta}_{13}(a+b)t^{-n/2+1}\operatorname{Vol} M + dt\operatorname{Vol} M\end{aligned}$$

where all the $\hat{\beta}$ *depend only on* n_0 *and where* d *depends only on* n_0, a, b, *and* c.

PROPOSITION 2. *Let D be a smooth relatively compact domain in a complete Riemannian manifold X of dimension $n = \dim X \leq n_0$. Let the Ricci curvature of X be bounded below by $-a(\dim X - 1)$ and the sectional curvature bounded above by b. Let $h \geq 0$ be an upper bound for the mean curvature of ∂D. Let $c > 0$ be a lower bound for the injectivity radius of X. Then*

$$\begin{aligned}
&\left|\operatorname{trace} E_D(t) - (4\pi t)^{-n/2}\operatorname{Vol} D\right| \\
&\quad \leq \hat{\alpha}_1 \frac{1}{\sqrt{a}} \int_0^\infty e^{\hat{\alpha}_2 s} e^{-\hat{\alpha}_3 s^2/at}\, ds \left(1 + \left|\frac{h}{\sqrt{a}}\right|\right)^{n-1} t^{-n/2}\operatorname{Vol}\partial D \\
&\qquad + \hat{\alpha}_4 (a+b) t^{-n/2+1}\operatorname{Vol} D + dt\operatorname{Vol} D,
\end{aligned}$$

where $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\alpha}_4$ depend only on n_0 and d depends only on a, b, c and n_0. Also, if D is contained in a geodesic ball $B(p, r_0)$ of radius r_0 then the curvature and injectivity radius bounds need only hold on $B(p, r_0 + 1)$.

Proof of Proposition 1. Let E_b^n denote the heat kernel of a sphere of dimension $n = \dim \mathrm{X}$ ($\leq n_0$) and constant curvature b. Then by Lemma 3.1 of [D-L1] we have

$$E_b^n(t, x, x) \leq (4\pi t)^{-n/2} + \beta_1(n) b t^{-n/2+1} + \beta_2(n) b^{n/2+1} t$$

where $\beta_1(n)$ and $\beta_2(n)$ are constants depending only on n. Now we make our first simple modification. Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be defined by $\hat{\beta}_i = \max_{n \leq n_0}\{\beta_i(n)\}$. Also let $\hat{C}$ be given by $\hat{C} = \max\{b, 1\}^{n_0/2+1}$. Then we have

$$E_b^n(t, x, x) \leq (4\pi t)^{-n/2} + \hat{\beta}_1 b t^{-n/2+1} + \hat{\beta}_2 \hat{C} t.$$

Now since $E_b^n(t, x, y)$ is really a function of $\operatorname{dist}(x, y)$ we can transplant E_b^n to M so that $E_b^n(t, x, y)$ is defined in a neighborhood of the diagonal $x = y$ in $M \times M$. We have

$$E_M(t, x, x) \leq E_b^n(t, x, x) + d_0(n) t, \tag{8}$$

where $d_0(n)$ depends only on n, b, and c. This is proven in [D-Li2]. Now by replacing $d_0(n)$ by $\max_{n \leq n_0}\{d_0(n)\}$ in (8) and combining with (7) we have

$$E_M(t, x, x) \leq (4\pi t)^{-n/2} + \hat{\beta}_1 b t^{-n/2+1} + dt$$

where d depends only on b, c and n_0. If we integrate this over M we get

$$\operatorname{trace} E_M(t) \leq (4\pi t)^{-n/2}\operatorname{Vol} M + \hat{\beta}_1 b t^{-n/2+1}\operatorname{Vol} M + d\operatorname{Vol} Mt. \tag{9}$$

Now we need a lower bound on $E_M(t, x, x)$. Let $x \in M$ and $0 < r \le c$. In [C-Y] it is proven that

$$E_M(t, x, x) \ge E_{-a,r}(t, x, x),$$

where $E_{-a,r}$ is the transplant of the Dirichlet heat kernel on a ball of radius r on the simply connected complete space of constant negative curvature $-a$. This transplant is once again defined on an open set containing the diagonal in $M \times M$.

We now need a lower bound on $E_{-a,r}(t, x, x)$. Proposition 2.4 of [D-L1] gives the result we need. The proof is a familiar application of Duhamel's principle. After application of this proposition we have

$$E_M(t, x, x) \ge (4\pi t)^{-n/2} - \beta_3(n) e^{\beta_4(n)\sqrt{a}\, r} e^{-\beta_5(n) r^2/t} t^{-n/2} - \beta_6(n) a t^{-n/2+1}.$$

where the $\beta_i(n)$ depend on n. Once again we make a modification by letting

$$\hat{\beta}_i = \max_{n \le n_0}\{\beta_i(n)\} \quad \text{for } i = 3, 4, \text{ and } 6$$
$$\hat{\beta}_5 = \min_{n \le n_0}\{\beta_5(n)\}.$$

Then we get

$$E_M(t, x, x) \ge (4\pi t)^{-n/2} - \hat{\beta}_3 e^{\hat{\beta}_4\sqrt{a}\, r} e^{-\hat{\beta}_5 r^2/t} t^{-n/2} - \hat{\beta}_6 a t^{-n/2+1}$$

where the $\hat{\beta}_i$ depend only on n_0. Integrate this over M to get

$$\text{(10)} \qquad \operatorname{trace} E_M(t) \ge (4\pi t)^{-n/2} \operatorname{Vol} M - \hat{\beta}_6 a t^{-n/2+1} \operatorname{Vol} M - \hat{\beta}_3 t^{-n/2} e^{\hat{\beta}_4\sqrt{a}\, r} e^{-\hat{\beta}_5 r^2/t} \operatorname{Vol} M.$$

Our upper and lower bounds combine to give Proposition 1.

Proposition 2 is proved similarly but not quite as easily since we have a boundary to consider. Fortunately, it is essentially proved in [D-L1], being an obvious modification of the Theorem 3.5 found therein. We will simply refer the reader to that paper. We now finish the proof of Theorems 6 and 6′.

Proof of Theorems 6 *and* 6′. The right hand sides of the inequalities of Propositions 1 and 2 are of orders $O(t^{1-n/2})$ and $O(t^{1/2-n/2})$ respectively. In [D-Li2] it is shown that $\lambda_i \ge c_5 i^{2/n}$ for all $i \ge m$ where c_5 and m depend only on the allowable geometric data a, b, c, and n_0. Now let $\varepsilon > 0$ be given. By Proposition 1 above there is a constant c_4 depending only on n_0, a, b, c, and Vol M such that

$$\left|(4\pi t)^{n/2} \operatorname{trace} E_M(\delta) - \operatorname{Vol} M\right| \le c_4 \delta < \varepsilon/2.$$

Now by standard volume comparison theory [B-C] we can use diam M instead of Vol M in our list of things that c_4 depends on since the latter, Vol M, can be bounded in terms of n_0, a, and diam M (see Lemmas 2 and 3 below).

We may assume $\delta < 1$. Choose $N(\delta)$ so that

$$\sum_{i=N(\delta)+1}^{\infty} e^{-\lambda_i\delta} \le \sum_{i=N(\delta)+1}^{\infty} e^{c_5 i^{2/n}\delta} \le c_6\delta < \varepsilon/2$$

and apply the triangle inequality to get

$$\left|(4\pi\delta)^{n/2}\sum_{i=1}^{N(\delta)} \varepsilon^{-\lambda_i\delta} - \operatorname{Vol} M\right| \le c_6\delta + \left|(4\pi\delta)^{n/2}\operatorname{trace} E_M(\delta) - \operatorname{Vol} M\right| \le \varepsilon.$$

The first part of Theorem 6 now follows. The second part follows from the first part. Namely, use the triangle inequality on the first part and multiply by $(4\pi\delta)^{(l-\dim M)/2}$ to get

$$\left|(4\pi\delta)^{l/2}\sum_{i=1}^{i=N(\delta)} e^{-\lambda_i\delta}\right| < 4\pi c_5\delta(4\pi\delta)^{(l-\dim M)/2} + c_6 \operatorname{Vol} M(4\pi\delta)^{(l-\dim M/2)}.$$

To get from here to the needed inequality we use the usual bound on Vol M and also restrict δ to be less than $1/4\pi$ so that

$$(4\pi\delta)^{(l-\dim M)/2} \le (4\pi\delta)^{1/2}.$$

The proof of Theorem 6′ is analogous to that of Theorem 6 except that we use proposition 2 instead of Proposition 1. We will leave out the details since they do not differ significantly from the analysis leading up to Theorem 4.3 of [D-L1].

Next we bound the diameter (and hence the volume) of M in terms of a finite number of eigenvalues and the allowable data n_0, and a. In the following three lemmas we assume the same hypotheses as in Theorem 6.

LEMMA 1. *Let M be compact without boundary. There exists a number d_1 depending only on n_0, and a such that*

$$\operatorname{diam} M \le \mathscr{M}(d_1) + 1$$

where $\mathscr{M}(s) = \max\{j : \lambda_j \le s\}$.

Proof of Lemma 1. Let r_0 be the diameter of M. We may assume that $r_0 > 1$. Choose a geodesic segment γ of length r_0 connecting x_1 to x_2. There exists at least $k = [r_0]$ nondegenerate, non-intersecting geodesic balls $B_1, B_2, \ldots, B_k$ of radius $1/2$ with centers on γ. Here, $[\cdot]$ is the greatest integer function. Thus by the Poincaré minimum principle $\lambda_k(M) \leq \max_i \lambda_1(B_i)$.

Now by S.Y. Cheng's eigenvalue comparison theory [C2] we have $\lambda_1(B_i) \leq d$ where d depends only on dim M and a. We can clearly replace d above by some $d_1 \geq d$ depending only on n_0 and a. Thus we have $\lambda_k(M) \leq d_2$ so that $k \leq \mathscr{M}(d_1)$. Now use the fact that $r_0 - 1 \leq [r_0] = k$ to get $r_0 \leq \mathscr{M}(d_2) + 1$.

Now we can use the above bound on r_0 to bound the volume of M. Let $B_{-a}(r_0)$ be a geodesic ball of radius r_0 in a simply connected manifold of constant curvature $-a$. Then by standard volume comparison, [B-C], we have the following fact which we record as a lemma.

LEMMA 2.

$$\operatorname{Vol} M \leq \operatorname{Vol} B_{-a}(r_0) \leq \beta_{20}(n)(\beta_{21}(n))^{r_0}$$

where $\beta_{20}(n)$ and $\beta_{21}(n)$ depend only on $n =$ dim M and a. Also if $a = 0$ we may replace the last term above by $\beta_{22}(n)r_0^n$.

By following our usual practice of replacing n dependence by n_0 dependence by setting $\hat{\beta}_i = \max_{n \leq n_0}\{\beta_i(n)\}$ etc. we obtain the following lemma from Lemmas 1 and 2.

LEMMA 3. *Let $r_0 =$ diam M then*

$$\operatorname{Vol} M \leq \hat{\beta}_{20}(\hat{\beta}_{21})^{r_0} \leq d_*$$

and in case $a = 0$

$$\operatorname{Vol} M \leq \hat{\beta}_{22}(\max\{r_0, 1\})^{n_0} \leq d_*,$$

where in either case d_* depends only on n_0, a and $\mathscr{M}(d_1)$. Also, $\hat{\beta}_{20}$, $\hat{\beta}_{21}$, and $\hat{\beta}_{22}$ depend only on n_0.

Also, there is a number d depending on a, n_0, c and b such that Vol $M \geq d$.

4. Determining the dimension of a manifold from a finite number of eigenvalues

In this section we apply the results of the previous section to show that, for certain large classes of manifolds or domains, there is a fixed finite number of eigenvalues that determine the dimension of a given manifold. We first prove a theorem that requires a crude a priori bound on the dimension of manifolds in the class and then we will exhibit cases where the bound on dimension can be eliminated from the hypotheses thereby proving Theorems 1 and 2.

Notation. Let $\mathscr{C} = \mathscr{C}(n_0, a, b, c, r_0)$ denote the class of compact Riemannian manifolds M whose dimension is less than or equal to n_0 and whose diameter is less than or equal to r_0 and whose sectional curvature K_M and injectivity radius $i(M)$ satisfy the bounds $-a \leq K_M \leq b$ and $0 < c \leq i(M)$ where $a, b \geq 0$.

THEOREM 7. *Let $\mathscr{C}(n_0, a, b, c, r_0)$ be given. Then there exists a number N such that for any $M \in \mathscr{C}(n_0, a, b, c, r_0)$, the dimension* $\dim M$ *is determined by the first N eigenvalues of M. In fact as the proof will show N is given constructively in terms of n_0, a, b, c and r_0.*

Also, an upper bound r_0 for the diameter of a manifold M can be determined from a finite number of eigenvalues. Precisely, there exists a number d_1 depending only on n_0, and a such that

$$\operatorname{diam} M \leq \mathscr{M}(d_1) + 1$$

where $\mathscr{M}(s) = \max\{j : \lambda_j \leq s\}$.

Notation. Let $\mathscr{X} = \mathscr{X}(n_0, a, b, c, r_0)$ denote the class of complete Riemannian manifolds X of dimension $\dim X \leq n_0$ such that X contains a geodesic ball of radius $r_0 + 1$ on which the sectional curvature is bounded below by $-a$ and above by b and on which the injectivity radii are greater than or equal to $c > 0$. For convenience we will refer to such a ball as *distinguished in X*.

THEOREM 8. *Given a class of manifolds $\mathscr{X}(n_0, a, b, c, r_0)$ as above, there exists a number N such that for all $X \in \mathscr{X}$ and D any weakly convex domain contained in a ball $B(p, r_0) \subset X$ with $B(p, r_0 + 1)$ distinguished in X, the dimension* $\dim D = \dim X$ *is determined by the first N Dirichlet eigenvalues of D.*

Remark. Informally stated, Theorems 7 and 8 above simply say that the dimension of a manifold or domain can be determined from its first N eigenvalues, where N depends only on the bounds n_0, a, b, c, and r_0. Also, notice that if in Theorem 8 the curvature and injectivity radius bounds given by the numbers a, b, and c hold on all of a given complete Riemannian manifold then every bounded ball of radius $r_0 + 1$ is distinguished in X. Thus we may use any weakly convex domain D as long as we take r_0 to be an upper bound on the out-radius of D.

Proof of Theorems 7 *and* 8. We have gathered enough facts so that the proof goes quickly. First consider Theorem 7. The last statement of Theorem 7 is just Lemma 1. Define

$$S_k^l(t) = (4\pi t)^{l/2} \sum_{i=1}^{k} e^{-\lambda_i t}.$$

By Theorem 6 there exist δ and N such that for any $M \in \mathscr{C}$, if $l = \dim M$ then $|S_k^l(t) - \operatorname{Vol} M| \leq \varepsilon$ for all $k \geq N$ and $0 < t \leq \delta$ and if $l > \dim M$ then $|S_k^l(t)| \leq \varepsilon$ for all $k \geq N$ and $0 < t \leq \delta$.

Now by the last inequality of Lemma 3 there is a number d such that $\operatorname{Vol} M \geq d$ uniformly for all $M \in \mathscr{C}$. Thus if we let $\varepsilon = d/10$ then

$$\dim M = \max\{l \colon S_N^l(\delta) \geq d/5\}.$$

This proves Theorem 7.

Now the proof of Theorem 8 is analogous except that we use Theorem 6′ instead of Theorem 6.

Proof of Theorems 1 *and* 2. Theorems 1 and 2 follow from Theorems 7 and 8 once we show that the stricter hypotheses of these theorems allow us to give appropriate bounds on dimension thus allowing us to get rid of n_0 from our hypotheses. Also, for Theorem 2 we need an appropriate bound on the out radius of the domain. This latter bound is the content of Lemma 5 of the next section where we take a closer look at the Euclidean case. Also notice that in applying Theorem 8 to get Theorem 2 we may take $c = \infty$, and $a = b = 0$.

For Theorem 1 we are assuming that sectional curvature is bounded from below by $\kappa > 0$. Notice that we may take $a = 0$ in using Theorem 7. It follows from Lichnerowicz's theorem [Lic] that $\lambda_1(M) \geq \dim(M)\kappa$ and hence $\dim(M) \leq \lambda_1(M)/\kappa$.

For Theorem 2 we use Theorem 8 with $a = b = 0$, $c = \infty$, and r_0 an upper bound for the out-radius of the given convex Euclidean domain D. To bound the dimension we use the following inequality:

$$\dim(D) \leq \frac{4\lambda_1}{\lambda_2 - \lambda_1}. \tag{11}$$

This inequality follows immediately from the eigenvalue gap estimate of Payne, Polya, and Weinberger and its generalizations [P-P-W], [C1].

Notice that it is crucial that this last inequality *not* involve the volume or diameter of M or our argument would become circular since we need the bound on dimension to bound the volume (Lemma 3).

5. A closer look at Euclidean domains

Let α_{n-1} denote the $n - 1$ dimensional volume of $S^{n-1} \subset \mathbf{R}^n$ and also let

$$\tilde{m} = \mathscr{M}\left(\lambda_1(D)\frac{2n(n+5)}{\pi^2}\right)$$

for a given domain $D \subset \mathbf{R}^n$

THEOREM 9. *Let D be a convex domain in $\mathbf{R}^n$. Then there exists a function $N(t) = N(t, n, \lambda_1(D), \tilde{m})$ depending only on the indicated quantities such that*

$$\left|(4\pi t)^{n/2}\sum_{i=1}^{N(t)} e^{-\lambda_i t} - \operatorname{Vol} D\right| \le C(n, \lambda_1(D), \tilde{m})\sqrt{t}$$

where

$$C(n, \lambda_i(D), \tilde{m}) = \left(\pi^{n-1/2}\alpha_{n-1}e^{n/2}\left(\frac{\tilde{m}}{\sqrt{\lambda_1(D)}}\right)^{n-1} + 1\right).$$

The point of Theorem 9 is the explicitness of the constant $C(n, \lambda_1, \tilde{m})$. Thus we may use this to get a better handle on the N appearing in Theorem 2. To do this we need a version of Theorem 9 in which the constant on the right hand side does not depend on the dimension N. We will indicate how this is done after proving Theorem 9.

We will need some simple lemmas. The following lemma is standard and has an elementary proof which we leave out.

LEMMA 4. *Let $\alpha_{n-1} = nw_n$ denote the $(n-1)$-dimensional volume of $S^{n-1} \subset \mathbf{R}^n$ where w_n is the volume of the unit ball $B^n \subset \mathbf{R}^n$. Then if $D \subset \mathbf{R}^n$ is convex and if R is the radius of a ball containing D. Then*

$$\operatorname{Vol}\partial D \le \alpha_{n-1}R^{n-1}$$

Consider the counting function $\mathscr{M}(s) = \max\{j\colon \lambda_j(D) \le s\}$. Then we have:

LEMMA 5. *Let D be as above with $\lambda_1(D)$ its first Dirichlet eigenvalue, and let*

$$\tilde{m} = \mathscr{M}\left(\lambda_1(D)\frac{2n(n+5)}{\pi^2}\right).$$

Then

$$R_0 \leq \frac{\pi \tilde{m}}{\sqrt{\lambda_1}(D)}$$

where R_0 is the out-radius of D.

Proof. We use the fact that $\pi^2/4R_{\text{in}}^2 \leq \lambda_1(D)$ where R_{in} is the radius of the largest ball contained in D. This fact is proved in [T] and a more general version of it is found in [L-Y]. Thus there is a ball B of radius $r_1 = \pi/2\sqrt{\lambda_1}$ contained in D. Let x_0 be its center. Since ∂D is compact we can find $y \in \partial D$ such that $r_2 \equiv \text{dist}(x_0, y) = \sup_{x \in \partial D} d(x_0, x)$. It is not hard to see that one can fit at least $k = [r_2/2r_1] + 1$ balls of radius $r_1/2$ into the convex hull of $B \cup \{y\}$ where $[\cdot]$ is the greatest integer function. Let these balls be denoted by $b_1, b_2, \ldots, b_k$.

Now let $\phi_1, \phi_2, \ldots,$ be a complete orthonormal sequence of Dirichlet eigenfunctions corresponding to the λ_k. Let f_i be the first Dirichlet eigenfunction of b_i extended to zero on $D - b_i$. Then we can find a_i not all zero such that $f = \sum_{i=1}^k a_i f^i$ $(\in H_0^1(D))$ is orthogonal to $\phi_1, \ldots \phi_{k-1}$. Thus by the Poincaré minimum principle

$$\lambda_k(D) \leq \frac{\int_D |\nabla f|^2}{\int_D f^2} \leq \frac{\sum a_i^2 \int_{b_i} |\nabla f_i|^2}{\sum a_i^2 \int_{b_i} f_i^2} \leq \sup_i \frac{\int |\nabla f_i|^2}{\int_{b_i} f_i^2} \leq \lambda_1(b_i)$$

Now if $k \geq \tilde{m}$ then by the definition of $\tilde{m}$ and r_1 we have

$$\frac{2n(n+5)}{4r_1} = \lambda_1(D)\frac{2n(n+5)}{\pi^2} \leq \lambda_k(D) \leq \lambda_1(b_i) = \frac{\lambda_1(b_1)}{r_1^2}$$

but $\lambda_1(b_1) \leq n(n+4)/2$. Thus we arrive at the contradiction $n(n+5)/2 \leq n(n+4)/2$. Hence

$$\tilde{m} > k \geq \left[\frac{r_2}{2r_1}\right] + 1$$

and

$$\tilde{m} > \frac{r_2}{2r_1} \geq \frac{R_0}{\pi}\sqrt{\lambda_1(D)}\,.$$

The result now follows.

LEMMA 6. *Let $D \subset \mathbf{R}^n$ be convex as above. Then*

$$\left|(4\pi t)^{n/2}\sum_{i=1}^{\infty} - \operatorname{Vol} D\right| \le \alpha_{n-1}e^{n/2}\left(\frac{\pi\tilde{m}}{\sqrt{\lambda_1(D)}}\right)^{n-1}\sqrt{\pi t}$$

where $\alpha_{n-1} = \operatorname{Vol}(S^{n-1})$.

Proof. The analysis in this proof uses familiar classical ideas and should be compared to the analysis found in [D], [M], and also [A-N]. Let $p(x, y, t)$ denote the Dirichlet kernel of $e^{-t\Delta}$ on D and $k(x, y, t)$ the Euclidean heat kernel on $\mathbf{R}^n$.

$$k(x, y, t) = (4\pi t)^{-n/2}\exp\left(-\frac{|x-y|^2}{4t}\right).$$

A standard argument using the maximum principle gives

$$k(x, y, t) \ge p(x, y, t).$$

Now if we apply the maximum principle to $k(x, y, t) - p(x, y, t)$ considered as a function of x and t then

$$\begin{aligned} k(x, y, t) - p(x, y, t) &\le \max_{(x,s)\in D\times[0,t]}\{k(x, y, s) - p(x, y, s)\} \\ &= \max_{(x,s)\in \partial D\times[0,t]}\{k(x, y, s) - p(x, y, s)\} \\ &\le \max_{(x,s)\in \partial D\times[0,t]} k(x, y, s) \\ &\le \max_{s\in[0,t]}(4\pi s)^{-n/2}\exp\left(\frac{-l(y)^2}{4s}\right) \end{aligned}$$

where $l(y) = \operatorname{dist}(y, \partial D)$.

Now

$$(4\pi s)^{-n/2}\exp\left(\frac{-l(y)^2}{4s}\right)$$

attains a maximum at $s = l(y)^2/2n$ and is increasing for $s \le l(y)^2/2n$. Thus for all $t \le l(y)^2/2n$ we have

$$0 \le k(x, y, t) - p(x, y, t) \le (4\pi t)^{-n/2}\exp\left(\frac{-l(y)^2}{4t}\right).$$

Now if $t > l(y)^2/2n$ we have

$$0 \leq k(x,y,t) - p(x,y,t) \leq k(x,y,t) \leq (4\pi t)^{-n/2}$$
$$\leq (4\pi t)^{n/2} e^{n/2} e^{-l(y)^2/4t}$$

thus for any $t > 0$ we have

$$0 \leq k(x,y,t) - p(x,y,t) \leq (4\pi t)^{-n/2} e^{n/2} e^{-l(y)^2/4t}$$

we set $x = y$ and integrate over D to obtain

$$0 \leq \operatorname{Vol} D (4\pi t)^{-n/2} - \sum_{i=1}^{n} \exp(-\lambda_i(D)t) \leq (4\pi t)^{-n/2} e^{n/2} \int_D e^{-l(x)^2/4t}\, dV.$$

Now if we let D_l denote the set $\{x \in D\colon \operatorname{dist}(x, \partial D) \leq l\}$ and ∂D_l its boundary we have

$$\int_D e^{-l(x)^2/4t}\, dV = \int_0^{l_{\max}} \int_{\partial D_l} e^{-l^2/4t}\, dA_l\, dl$$

where dA_l is the $(n-1)$-dimensional surface element ∂D_l and $l_{\max} = \sup\{\operatorname{dist}(x, \partial D)\colon x \in D\}$. Now since D is convex we have

$$\operatorname{Vol}(\partial D_l) \leq \operatorname{Vol}(\partial D)$$

thus

$$\int_0^{l_{\max}} \int_{\partial D_l} e^{-l^2/4t}\, dV_l\, dl \leq \operatorname{Vol} \partial D \int_0^{\infty} e^{-l^2/4t}\, dl = \operatorname{Vol} \partial D \sqrt{\pi t}$$

combining we get

$$0 \leq \operatorname{Vol}(D) - (4\pi t)^{n/2} \sum_{i=1}^{n} \exp(-\lambda_i(D)t) \leq e^{n/2} \sqrt{\pi t} \operatorname{Vol} \partial D$$

$$\leq \alpha_{n-1} e^{n/2} R_0^{n-1} \sqrt{\pi t} \leq \alpha_{n-1} e^{n/2} \left(\frac{\pi \tilde{m}}{\sqrt{\lambda_1(D)}} \right)^{n-1} \sqrt{\pi t}$$

where we used Lemma 4 and Lemma 5.

Proof of Theorem 9. Let $\{\lambda_i(n, \tilde{m}, \lambda_1(D))\}$ denote the sequence of Dirichlet eigenvalues of the Euclidean ball of radius $\pi \tilde{m}/\sqrt{\lambda_1(D)}$. Then by domain monotonicity of eigenvalues and Lemma 5 we have

$$\sum_{i=N+1}^{\infty} \exp(-\lambda_i(D)t) \leq \sum_{i=N+1}^{\infty} \exp(-\lambda_i(n, \tilde{m}, \lambda_1(D))t)$$

for any N. In fact since the last sum is just the tail of the sum occurring in trace of $e^{-t\Delta}$ for Δ acting on the Euclidean ball of radius $\pi\tilde{m}/\sqrt{\lambda_i(D)}$ we may choose

$$N(t) = N(t, n, \tilde{m}, \lambda_1(D))$$

depending only on the indicated quantities such that

$$(4\pi t)^{n/2} \sum_{i=N(t)+1}^{\infty} \exp(-\lambda_1(n, \tilde{m}, \lambda_1(D))t) \leq \sqrt{t}$$

Thus we have

$$\begin{aligned}
&\left|(4\pi t)^{n/2} \sum_{i=1}^{N(t)} e^{\lambda_i(D)t} - \operatorname{Vol} D\right| \\
&\leq \left|(4\pi t)^{n/2} \sum_{i=1}^{\infty} e^{-\lambda_i(D)t} - \operatorname{Vol} D\right| + (4\pi t)^{n/2} \sum_{i=N(t)+1}^{\infty} e^{-\lambda_i(D)t} \\
&\leq \left(\pi^{n-1/2}\alpha_{n-1}e^{n/2}\left(\frac{\tilde{m}}{\sqrt{\lambda_1(D)}}\right)^{n-1} + 1\right)\sqrt{t}
\end{aligned}$$

and we are done.

As mentioned, the point of Theorem 9 is the explicitness of the constant $C(n, \lambda_1, \tilde{m})$. We may use this to get a better handle on the N appearing in Theorem 2. To do this we need a version of Theorem 9 in which the constant on the right hand side does not depend on the dimension n. In view of equation (11) the first thing to do is let $n_0 = 4\lambda_1/(\lambda_2 - \lambda_1)$ and then replace $\tilde{m} = \mathscr{M}(\lambda_1 2n(n+5)/\pi^2)$ by $\hat{m} = \mathscr{M}(\lambda_1 2n_0(n_0+5)/\pi^2)$ so that

$$r_0 \leq \frac{\pi\hat{m}}{\sqrt{\lambda_1}}$$

where r_0 is the out radius of D. Here we have used Lemma 5, the fact that $n \leq n_0$, and the consequent fact that $\tilde{m} \leq \hat{m}$. Next define

$$\tilde{C}(\lambda_1, \lambda_2, \hat{m}) = \left(\pi^{n_0-1}\alpha e^{n_0/2}\left(\hat{m}/\sqrt{\lambda_1}\right)^{n_0-1} + 1\right)$$

where $\alpha = \max_{n \le n_0}\{\alpha_{n-1}\}$ and $n_0 = 4\lambda_1/(\lambda_2 - \lambda_1)$. In fact making these replacements all through the proof of Theorem 9 we see that the following is true.

THEOREM 10. *Let D be a convex domain in a Euclidean space. Then there exists a function $N(t) = N(t, \lambda_1(D), \lambda_2(D), \hat{m})$ depending only on the indicated quantities such that*

$$\left|(4\pi t)^{\dim D/2} \sum_{i=1}^{N(t)} e^{-\lambda_i t} - \operatorname{Vol} D\right| \le \hat{C}(\lambda_1, \lambda_2, \hat{m})\sqrt{t}\,.$$

Concluding comments

We know from the counterexample of Section 1 that there is no finite number N of eigenvalues that suffices to determine the dimension of a Riemannian manifold uniformly for the class of all Riemannian manifolds. Now, given a geometric class of manifolds described without explicit reference to dimension, when can the dimension be determined from a finite part of the spectrum? In retrospect we see that we will obtain a theorem roughly whenever the dimensions of the manifolds of the given geometric class can be bounded appropriately. If our geometric class is defined in part by explicitly requiring a uniform volume bound then we may proceed along the lines of the proof of Theorem 7 (or 8) to get a result without needing anything like Lemma 3. If, on the other hand, we do not include a volume bound in defining our geometric class (the approach taken in this paper) then we will need something like Lemma 3. In this case we are obliged to look for dimension bounds that do not involve the volume. This is because Theorem 3 bounds volume in terms of a bound on dimension. The reason for not allowing volume bounds in the hypothesis of a theorem giving a finite spectral determination of dimension is that volume has units of (length)$^{\dim M}$ which seems somewhat inappropriate. Clearly, one can hope for many variations of these results where the geometric classes under consideration are varied.

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