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ON SOME STRANGE SUMMATION FORMULAS

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1. Introduction

During the last two years the first named author used symbolic algebra programs and long hours of computer experiments to formulate several infinite series identities. Some of his conjectures were communicated to other mathematicians as informal letters, and were circulated among interested parties. In particular [10] and [11] contained several conjectures in the form of identities reminiscent of Ramanujan's work.

Some of the series relevant to this work are

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\sqrt{b^2+\pi^2 n^2}\right) = \frac{\pi^2}{4}\left(\frac{\sin b}{b} - \frac{\cos b}{3}\right), \tag{1.1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\sqrt{n^2\pi^2 - 9}\right) = -\frac{\pi^2}{12 e^3}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+\frac{1}{2}} \frac{\sin\sqrt{b^2+\pi^2(n+1/2)^2}}{\sqrt{b^2+\pi^2(n+1/2)^2}} = \frac{\pi}{2}\frac{\sin b}{b}, \tag{1.3}$$

$$\sum_{-\infty}^{\infty} \frac{\cos\left[\sqrt{(n\pi+\phi+a/b)(n\pi+\phi+ab)}\,\right]}{(-1)^n(n\pi+\phi)^2} = \cot\phi\frac{\cos a}{\sin\phi} - \frac{(b+1/b)\sin a}{2\sin\phi}. \tag{1.4}$$

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and

$$(1.5)\quad \sum_{-\infty}^{\infty} \frac{(n\pi + \phi + a)^2 + 3a^2}{(n\pi + \phi - 2a)^2 (n\pi + \phi + 2a)^2} \frac{\sin\left(a - \dfrac{3a^2}{n\pi + \phi + a}\right)}{\sin\left(\phi + a + \dfrac{3a^2}{n\pi + \phi + a}\right)}$$

$$= \frac{\cot(\phi - 2a) - \cot(\phi + 2a)}{4\sin(\phi + 2a)}.$$

Observe that the special $b = \pi$ in (1.1) can be written in the exotic form [10]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{\pi}{n + \sqrt{n^2 + 1}}\right) = \frac{\pi^2}{12}.$$

Also note that (1.5) can be written in the more compact form

$$(1.5\text{a})\quad \sum_{-\infty}^{\infty} \left[(n\pi + f + z)^{-2} + 3(n\pi + f - 3z)^{-2}\right] \cot\left(f + \frac{3z^2}{f + n\pi}\right)$$

$$= 3\cot(f + z)\csc^2(f - 3z) + \csc^2(f + z)\cot(f - 3z).$$

Some of Gosper's earlier computer generated formulas led to very interesting identities, see for example [7] and [8] and for q analogues see [9]. The communications [10] and [11] contain additional identities which have yet to be proven.

In this paper we use Fourier transforms to prove (1.1), (1.2), and (1.3) and extend their validity from trigonometric functions to Bessel functions. In addition we prove (1.4) and (1.5), which seem to be the deepest formulas in this group. We also prove several evaluations of similar infinite series. We also point out the connection between series identities of the type (1.1)–(1.5) and formulas discovered by Ramanujan [1].

On the surface the aforementioned formulas do not seem to involve Bessel functions, but a Bessel function $J_0(x)$ appears in our proofs of (1.1)–(1.3), then gets integrated out. Bessel functions, whether appearing explicitly or implicitly, are behind the existence of these series summations. Recall that a Bessel function of the first kind, [2] or [22], $J_\nu(x)$, is

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{n!\,\Gamma(n + \nu + 1)}.$$

In Section 2 we first give a proof of the following generalization of (1.3)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1/2} \frac{J_\nu\left(\sqrt{b^2+\pi^2(n+1/2)^2}\right)}{\left[b^2+\pi^2(n+1/2)^2\right]^{\nu/2}} = \frac{\pi}{2} b^{-\nu} J_\nu(b), \qquad b>0, \quad \mathrm{Re}(\nu) > -\frac{1}{2}, \tag{1.6}$$

where $J_\nu(x)$ is a Bessel function of the first kind. The relationships (14) and (15) of §7.11 in [2] are

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \qquad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \tag{1.7}$$

respectively. Now (1.6) generalizes (1.3) because (1.6) reduces to (1.3) when $\nu = 1/2$. The case $\nu \to -1/2$ of (1.6) is a critical case and the series in (1.6) is not absolutely convergent when $\nu = -1/2$. In fact (1.1) can be thought of as the critical limiting case, $\nu \to -1/2$ of (1.6). We do not know how to compute directly the limit of (1.6) as $\nu \to -1/2$, but we proved (1.1) and (1.2) by essentially letting $\nu \to -1/2$ in the formulas used to prove (1.6). We generalize (1.1) in a different direction to (3.5). It is of interest to note that if $\nu = m + 1/2$, $m = 0, 1, 2, \ldots$, then, as we shall see in Section 2, (1.6) will take the form

$$\sum_{j=0}^{m} \frac{(-1)^j (2m-j)!}{j!\,(m-j)!} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{\sin(Z_n + j\pi/2)}{Z_n^{2m-j+1}} = \frac{\pi}{2} \sum_{j=0}^{m} \frac{(-1)^j (2m-j)!}{j!\,(m-j)!} \frac{\sin(b + j\pi/2)}{b^{2m-j+1}} \tag{1.8}$$

where Z_n is

$$Z_n := \sqrt{b^2 + \pi^2(n+1/2)^2}. \tag{1.8a}$$

Observe that (1.8) involves only polynomials and trigonometric functions. In fact (1.8) is just the mth derivative with respect to b of (1.6).

It is still possible to further generalize (1.6) to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1/2)^{2k-1}} \frac{J_\nu\left(\sqrt{b^2+\pi^2(n+1/2)^2}\right)}{\left[b^2+\pi^2(n+1/2)^2\right]^{\nu/2}} = (-1)^k b^{-\nu} \frac{(4\pi)^{2k-3/2}}{(2k-1)!} \sum_{m=0}^{k-1} \binom{2k-1}{2m} B_{2k-2m-1}\left(\frac{1}{4}\right) \times (8b)^{-m} \Gamma(m+1/2) J_{\nu+m}(b). \tag{1.9}$$

The polynomials $\{B_m(x)\}$ in (1.9) are the Bernoulli polynomials. Clearly (1.6) is the special case $k = 1$ of (1.9). A proof of the identity (1.9) will be given in Section 2. The Bernoulli polynomials of odd degrees evaluated at $x = 1/4$ are related to Euler numbers via

$$E_n = (-1)^{n+1} \frac{4^{2n+1}}{2n+1} B_{2n+1}(1/4), \qquad n \geq 0,$$

[20, (14.3) p. 24]. A straightforward calculation establishes the following equivalent and more elegant form of formula (1.9)

(1.9a)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1/2)^{2k-1}} \frac{J_\nu\left(\sqrt{b^2+\pi^2(n+1/2)^2}\right)}{\left[b^2+\pi^2(n+1/2)^2\right]^{\nu/2}}$$

$$= 2b^{-\nu}(-1)^{k-1}\pi^{2k-1} \sum_{m=0}^{k-1} \frac{(2b)^{-m}}{m!\,(2k-2m-2)!} E_{2k-2m-2} J_{\nu+m}(b).$$

In Section 3 we shall prove (1.1) and (1.2) and also generalize them to Bessel functions. Our generalization is formula (3.5). In Section 4 we shall give yet another generalization of (1.6) which is different from (1.9). This generalization is formula (4.1). In finding this generalization we were guided by the fact that the quantities $n\pi$ and $(n+1/2)\pi$ appearing in (1.1) and (1.2), (1.6), and (1.9) are the positive zeros of $J_{1/2}(z)$ and $J_{-1/2}(z)$, respectively.

One attractive special case of (4.1) is

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1/2)}{(n+1/2)^2 - z^2} \frac{\sin\sqrt{b^2+\pi^2(n+1/2)^2}}{\sqrt{b^2+\pi^2(n+1/2)^2}} \tag{1.10}$$

$$= \frac{\pi}{2} \frac{1}{\cos(\pi z)} \frac{\sin\sqrt{b^2+\pi^2 z^2}}{\sqrt{b^2+\pi^2 z^2}},$$

where $z \neq (n+1/2)\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Clearly (1.3) is the special case $z = 0$ of (1.10).

In Section 5 we give a proof of (1.4) based on a Mittag-Leffler expansion and a somewhat elaborate use of Fourier sine and cosine transforms. We also prove (1.5) using Mittag-Leffler expansion and the so called ${}_2H_2$ summation formula,

(1.11)
$$ {}_2H_2(a,b;c,d;1) := \sum_{-\infty}^{\infty} \frac{(a)_n(b)_n}{(c)_n(d)_n} = \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)}, $$

$\mathrm{Re}(c+d-a-b) > 1$, where the shifted factorial $(s)_n$ is

$$ (s)_n = \Gamma(s+n)/\Gamma(s), \qquad s \neq 0, -1, -2, \dots. \tag{1.12} $$

The sum (1.11) is stated correctly on page 181 of [21] but is stated incorrectly in the list of formulas collected in its Appendix III, see (III.28), page 245 in [21]. The sum (1.11) is usually referred to Dougall's sum. For a recent generalization of Dougall's sum we refer the interested reader to Osler's work [18].

Further computer experiments led to the following infinite series and infinite products identities:

$$ \sum_{n=0}^{\infty} \frac{n^2\pi^2+\phi^2}{(n^2\pi^2-\phi^2)^2}(-1)^n \cos\sqrt{n^2\pi^2+a^2-\phi^2} = \frac{\cos\sqrt{a^2-\phi^2}}{2\phi^2} + \frac{a\cos a\cot\phi + \phi\sin a}{2a\sin\phi}, \tag{1.13} $$

$$ \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{2\left[\left(n+\frac{1}{2}\right)^2\pi^2-\phi^2\right]^2} \cos\left(\sqrt{\left(n+\tfrac{1}{2}\right)^2\pi^2+a^2-\phi^2}\right) = \frac{a\tan\phi\cos a - \phi\sin a}{4\pi a\phi\cos\phi}, \tag{1.14} $$

(1.15)
$$ \prod_{n=1}^{\infty}\left[\frac{1}{2}+\frac{1}{4}\csc\left(\frac{\pi}{6}+\frac{a}{(-2)^n}\right)\right] = a^{-1}\tan\left(a+\frac{\pi}{6}\right) - \frac{1}{2a}\sec\left(a+\frac{\pi}{6}\right), $$

$$ \sum_{n=0}^{\infty} \frac{E_{2n}(\frac{1}{3})(-1)^n x^{2n+1}}{(2^{2n+1}+1)(2n+1)!} = \sqrt{3}\log\left[\frac{\sin\left(\frac{\pi}{3}+\frac{x}{6}\right)}{\sin\left(\frac{\pi}{3}-\frac{x}{6}\right)}\right], \tag{1.16} $$

and

$$(1.17)\qquad \sum_{-\infty}^{\infty} \frac{\cot\left(\dfrac{(n\pi + a)(n\pi + b)}{n\pi + (a+b)/2}\right)}{(n\pi + a)(n\pi + b)} = \frac{\cot^2 a - \cot^2 b}{2(b-a)}.$$

$$(1.18)\qquad \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}}{(n+1/2)^2} \cos\sqrt{a^2 + (2n+1)^2\frac{\pi^2}{16}} = \frac{\pi^2 \cos a}{2\sqrt{2}}.$$

The polynomials $\{E_n(x)\}$ in (1.16) are the familiar Euler polynomials. The identities (1.13), (1.14), (1.17) and (1.18) were conjectured in [11]. It may be of interest to indicate that (1.16) has the equivalent form

$$(1.16\text{a})\qquad \sum_{n=0}^{\infty} \frac{E_{2n}(\frac{1}{3})(-1)^n x^{2n+1}}{(2^{2n+1}+1)(2n+1)!} = 2\sqrt{3}\tanh^{-1}\left(\frac{\tan(x/6)}{\sqrt{3}}\right).$$

The correspondence [11] also contains the additional conjecture

(1.19)

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1/2)}{(n+1/3)^2(n+2/3)^2} \cos\left(\pi\sqrt{(n+1/6)(n+5/6)}\right) = \pi^2 e^{-\pi\sqrt{3}/6},$$

which is the special case $\phi = \pi/6$ and $a^2 = -\pi^2/12$ of (1.14). It is trivial to see that the right hand side of (1.17) can be written in the form

$$\frac{\sin(a-b)\sin(a+b)}{2(b-a)\sin^2 a \sin^2 b}.$$

Section 6 contains proofs of (1.13)–(1.16). In Section 7 we prove (1.17) and (1.18). Formula (1.18) is particularly attractive because it involves a sieving process, which may be connected to character sums. The connection with character sums is still under investigation, despite great skepticism by the first author. In Section 8 we point out a connection between certain series identities of Ramanujan and some of our results. In particular we indicate briefly alternate proofs of (1.6) and related identities. We also mention a connection with [14]. One key idea that Gosper conjectured and employed to arrive at sums like (1.1)–(1.5) is the so called nonlocal derangement identity, namely

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} a_j \prod_{\substack{k=1\\k\neq j}}^{n} \left(1 + \frac{a_k}{b_j - b_k}\right). \tag{1.20}$$

It is assumed that the b_j's in (1.20) are distinct. Two proofs of (1.20) are known to date. In Section 8 we will give a proof of (1.20) and indicate briefly how some new identities follow from (1.20). Some of the identities in this work were first discovered by making special choices of the a_k's and b_k's in (1.20). The rest of the original proofs involved tricky series rearrangements and limiting processes.

Section 9 consists of a study of a continuous analog of (4.1) and its applications. For example we prove that the functions

$$f(x) := \frac{b^{\nu/2} K_\nu(\sqrt{b}\,) x^{\nu/2} K_\nu(\sqrt{x}\,)}{2^{\nu-1}\Gamma(\nu)(x+b)^{\nu/2} K_\nu(\sqrt{x+b}\,)} \tag{1.21a}$$

and

$$g(x) := \frac{b^{\nu/2} x^{\nu/2} I_\nu(\sqrt{x+b}\,)}{2^{\nu}\Gamma(\nu+1)(x+b)^{\nu/2} I_\nu(\sqrt{b}\,) I_\nu(\sqrt{x}\,)} \tag{1.21b}$$

are Laplace transforms of infinitely divisible probability distributions, when $\nu > 0$. Here $K_\nu(x)$ and $I_\nu(x)$ are modified Bessel functions [2], [22],

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}, \tag{1.22}$$

$$K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \tag{1.23}$$

$$\nu \neq \text{integer}, \quad K_n(z) = \lim_{\nu \to n} K_\nu(z), \text{ for integer } n.$$

We then use the complete monotonicity of the functions in (1.21a) and (1.21b) to establish inequalities for modified Bessel functions [4], [22]. In addition we prove the positivity of an integral with an oscillatory kernel. This is stated as Theorem 9.12. The paper ends with a Section 10 containing some concluding remarks and a discussion of some formulas of F. Oberhettinger related to (1.1), (1.3), (1.6) and (4.1).

The referee drew our attention to the work of Forrester [6]. Forrester used Cauchy's theorem to prove few main summation theorems. He then showed how generalizations of Ramanujan's formulas follow from his main results. We found no overlap between our results and those of Forrester.

2. Proofs of (1.6) and (1.9)

Our proofs use the Fourier series expansions

(2.1)
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1/2}\cos[(n+1/2)x\pi] = \begin{cases} \pi/2, & x\in(-1,1), \\ -\pi/2, & x\in(-2,-1)\cup(1,2), \end{cases}$$

[20], and

$$\sum_{n=1}^{\infty}\frac{\sin 2n\pi t}{n^{2k-1}} = \frac{(2\pi)^{2k-1}\Psi_{2k-1}(t)}{2(-1)^k(2k-1)!}, \tag{2.2}$$

[20, (8.61) p. 16], [15, (4) p. 243] where the Ψ_n's are related to the Bernoulli polynomials via, [20, p. 15],

$$\Psi_n(t) := B_n(t-\lfloor t\rfloor), \qquad n\geq 1. \tag{2.3}$$

Proof of (1.6). Entry (50) in §1.13 of [2] is

(2.4)
$$\sqrt{\frac{\pi}{2}}\, b^{\nu}J_{\nu+1/2}\left(\sqrt{b^2+y^2}\right)(b^2+y^2)^{-(\nu+1/2)/2}$$
$$= \int_0^1 (1-x^2)^{\nu/2}J_\nu\left(b\sqrt{1-x^2}\right)\cos(xy)\,dx, \qquad \operatorname{Re}\nu>-1, \quad b>0.$$

In (2.4) replace ν by $\nu - 1/2$ and y by $(n+1/2)\pi$, then multiply the result by $(-1)^n/(n+1/2)$ and sum for $n = 0, 1, \ldots$, to see that the left hand side of (1.6) is

$$\sqrt{\frac{2}{\pi}}\int_0^1 (1-x^2)^{(\nu-1/2)/2}J_{\nu-1/2}\left(b\sqrt{1-x^2}\right)b^{-\nu+1/2}$$
$$\times\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1/2}\cos[(n+1/2)\pi x]\,dx,$$

which, in view of (2.1) and Sonnine's first integral [22, §12.11]

$$(2.5)\quad J_{\alpha+\beta+1}(z) = \frac{2^{-\beta}z^{\beta+1}}{\Gamma(\beta+1)}\int_0^1 x^{2\beta+1}(1-x^2)^{\alpha/2} J_\alpha\left(z\sqrt{1-x^2}\right)dx,$$

$$\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1,$$

establish (1.6).

To see how (1.8) follows from (1.6) we may use the relationships (23) and (26) in §7.5 involving Lommel polynomials in [22]. Then we see that (1.6) becomes

$$\begin{aligned}
&\sum_{j=0}^{\lfloor m/2\rfloor} \frac{(-4)^j(2m-2j)!}{(2j)!(m-2j)!}\sum_{n=0}^{\infty}\frac{\sin Z_n}{Z_n^{2m-2j+1}}\frac{(-1)^n}{n+1/2} \\
&\qquad -2\sum_{j=0}^{\lfloor (m-1)/2\rfloor}\frac{(-4)^j(2m-2j-1)!}{(2j+1)!(m-1-2j)!}\sum_{n=0}^{\infty}\frac{\cos Z_n}{Z_n^{2m-2j}}\frac{(-1)^n}{n+1/2} \\
&= \frac{\pi}{2}\left[\sum_{j=0}^{\lfloor m/2\rfloor}\frac{(-4)^j(2m-2j)!}{(2j)!(m-2j)!}\frac{\sin b}{b^{2m-2j+1}}\right. \\
&\qquad \left. -2\sum_{j=0}^{\lfloor (m-1)/2\rfloor}\frac{(-4)^j(2m-2j-1)!}{(2j+1)!(m-1-2j)!}\frac{\cos b}{b^{2m-2j}}\right],
\end{aligned}$$

and after rearranging the terms on the right side we obtain (1.8).

Proof of (1.9). As in the proof of (1.6) we apply (2.4) to express the left hand side of (1.9) in the form

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\, b^{-\nu+1/2}\int_0^1 (1-x^2)^{(\nu-1/2)/2} J_{\nu-1/2}\left(b\sqrt{1-x^2}\right) \\
&\qquad \times \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1/2)^{2k-1}}\cos[(n+1/2)\pi x]\,dx.
\end{aligned}$$

Let us denote the sum in the above integral by $H_k(x)$, that is

$$(2.6)\qquad H_k(x) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1/2)^{2k-1}}\cos[(n+1/2)\pi x].$$

Now split the sum in (2.2) into a sum over odd n and a sum over even n and obtain

$$\frac{(2\pi)^{2k-1}\Psi_{2k-1}(t)}{2(-1)^k(2k-1)!} = \left(\sum_{n \text{ odd}} + \sum_{n \text{ even}}\right)\frac{\sin(2n\pi t)}{n^{2k-1}}$$
$$= \sum_{n=1}^{\infty}\frac{\sin(2n\pi(2t))}{2^{2k-1}n^{2k-1}} + \sum_{n=1}^{\infty}\frac{\sin((2n-1)2\pi t)}{(2n-1)^{2k-1}}.$$

Then apply (2.2) to establish

$$\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi t]}{(2n-1)^{2k-1}} = \frac{(-1)^k(4\pi)^{2k-1}}{2[(2k-1)!]}\left[\Psi_{2k-1}(t/2) - 2^{1-2k}\Psi_{2k-1}(t)\right].$$

This implies

$$H_k(x) = \frac{(-1)^k(2\pi)^{2k-1}}{2(2k-1)!}\left[2^{2k-1}\Psi_{2k-1}\left(\frac{1}{2}t\right) - \Psi_{2k-1}(t)\right], \qquad t := (1+x)/2. \tag{2.7}$$

Therefore the left hand side of (1.9) is

$$\frac{(-1)^k(2\pi)^{2k-3/2}}{(2k-1)!}b^{-\nu+1/2}\int_0^1(1-x^2)^{(\nu-1/2)/2}J_{\nu-1/2}\left(b\sqrt{1-x^2}\right)$$
$$\times\left(2^{2k-1}\Psi_{2k-1}\left(\frac{x+1}{4}\right) - \Psi_{2k-1}\left(\frac{x+1}{2}\right)\right)dx. \tag{2.8}$$

Using the functional equations

$$B_n(2x) = 2^{n-1}(B_n(x+1/2) + B_n(x)) \text{ and } B_n(1-x) = (-1)^n B_n(x),$$

[20, p. 13], together with (2.3) and the fact that in (2.8), x lies in $[0,1]$, we simplify the quantity in square brackets in (2.7) to

$$2^{2k-2}\left[B_{2k-1}((1-x)/4) + B_{2k-1}((1+x)/4)\right].$$

We then apply (2.3) and

$$B_n(u+v) = \sum_{j=0}^{n}\binom{n}{j}u^j B_{n-j}(v), \tag{2.9}$$

with $v = 1/4$ and $u = \pm x/4$ to write the left hand side of (1.9) in the form

$$\sqrt{\frac{2}{\pi}}\, b^{-\nu+1/2} \frac{(-1)^k (4\pi)^{2k-1}}{2(2k-1)!} \sum_{m=0}^{k-1} \binom{2k-1}{2m} B_{2k-2m-1}\left(\frac{1}{4}\right)(-4)^{-2m}$$
$$\cdot \int_0^1 x^{2m}(1-x^2)^{(\nu-1/2)/2} J_{\nu-1/2}\left(b\sqrt{1-x^2}\right) dx$$

and then the Sonine first integral (2.5) enables us to reduce the above expression to the right hand side of (1.9). This completes the proof of (1.9).

3. Proofs and generalizations of (1.1) and (1.2)

Our proofs use entry (47) of §1.13 in [2], namely

$$(3.1)\quad (b^2+y^2)^{-1/2} \sin\left[a(b^2+y^2)^{1/2}\right] = \int_0^a J_0\left[b(a^2-x^2)^{1/2}\right]\cos(xy)\, dx,$$

and

$$(3.2)\qquad \Psi_{2n}(t) = 2(-1)^{n-1}(2n)! \sum_{m=1}^{\infty} \frac{\cos(2\pi m t)}{(2\pi m)^{2n}}$$

(see [15, §82, (3)] or [20, (8.62) p. 16]). Note that (3.2) is the derivative of (2.2).

Proof of (1.1). Differentiate (3.1) with respect to a and then set $a = 1$ to obtain

$$(3.3)\quad \cos\sqrt{b^2+y^2} = \cos y - b\int_0^1 (1-x^2)^{-1/2} J_1\left[b\sqrt{1-x^2}\right]\cos(xy)\, dx,$$

since $(J_0(z))' = -J_1(z)$ [2, §7.2.8]. Therefore the left hand side of (1.1) is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - b\int_0^1 \frac{J_1\left[b\sqrt{1-x^2}\right]}{\sqrt{1-x^2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(\pi n x)\, dx.$$

The sum of the first and second series in the above expression are $\pi^2/6$ and $\pi^2\Psi_2((x+1)/2)$, respectively. Combining these facts with

$$\Psi_2(x) = x^2 - x + 1/6 \quad \text{for } 0 \le x < 1,$$

we reduce the left hand side of (1.1) to

$$\frac{\pi^2}{6} + \pi^2 \frac{b}{4} \int_0^1 \sqrt{1-x^2}\, J_1\left(b\sqrt{1-x^2}\right) dx$$
$$- \pi^2 \frac{b}{6} \int_0^1 (1-x^2)^{-1/2} J_1\left(b\sqrt{1-x^2}\right) dx.$$

The first integral equals $J_{3/2}(b)\Gamma(1/2)/\sqrt{(2b)}$, by (2.5). For the second integral we appeal to entry (34) in §8.5 of [4] and find its value to be $(b/2)\,{}_1F_2(1; 2, 3/2; -b^2/4)$. Finally using the three term recursion relation

$$J_{\nu+1}(z) = (2\nu/z)J_\nu(z) - J_{\nu-1}(z), \tag{3.4}$$

(1.6) and the observation

$$\begin{aligned} {}_1F_2(1; 2, 3/2; -b^2/4) &= \sum_{n=0}^{\infty} \frac{(-b^2/4)^n}{(3/2)_n (2)_n} \\ &= 2\sum_{n=0}^{\infty} \frac{(-b^2)^n}{(2n+2)!} = \frac{2}{b^2}[1 - \cos b], \end{aligned}$$

we establish (1.1) and the proof is complete.

Proof of (1.2). Formula (1.1) will continue to hold when b is purely imaginary and (1.2) is only the special case $b = 3i$ of (1.1). To prove (1.2) for purely imaginary b one simply repeats the proof of (1.1) since (3.1) holds for purely imaginary b as a specialization of the second finite integral of Sonine [22, §12.13].

We now generalize (1.1) to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} \cos\left(\sqrt{n^2\pi^2 + b^2}\right) &= \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} B_{2k} \\ &+ \frac{(-1)^{k-1}\pi^{2k}}{(2k)!} \sum_{j=0}^{k} \binom{2k}{2j} (1 - 2^{2j-1}) B_{2j} \\ &\times \left[\Gamma(k - j + 1/2)(b/2)^{j-k+1/2} J_{k-j-1/2}(b) - 1\right], \end{aligned} \tag{3.5}$$

where the B_j's are Bernoulli numbers.

Proof of (3.5). We start with (3.3) and obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} \cos\sqrt{n^2\pi^2 + b^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - b\int_0^1 [1-x^2]^{-1/2} J_1\left[b\sqrt{1-x^2}\right] \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} \cos(n\pi x)\, dx \\
&= \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} B_{2k} - b\int_0^1 [1-x^2]^{-1/2} \\
&\quad \times J_1\left(b\sqrt{1-x^2}\right) \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} \Psi_{2k}((x+1)/2)\, dx \\
&= \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} B_{2k} - b\frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} \\
&\quad \times \sum_{j=0}^{2k} \binom{2k}{j} B_j\left(\frac{1}{2}\right) 2^{j-2k} \int_0^1 J_1(bu)(1-u^2)^{(2k-j-1)/2}\, du,
\end{aligned}$$

where we used (2.3), (2.9) and (3.2). Here again entry (34) of §8.5 in [4] comes to our rescue and the integral on the extreme right hand side is

$$\frac{b}{4} \frac{\Gamma(k + (1-j)/2)}{\Gamma(k + (3-j)/2)} {}_1F_2(1; 2, k + (3-j)/2; -b^2/4).$$

On the other hand [20], $B_j(1/2) = 0$ for odd j and $B_{2j}(1/2) = -(1 - 2^{1-2j})B_{2j}$, the B_{2j}'s being Bernoulli numbers. Therefore the left hand side of (3.5) is

(3.6)
$$\begin{aligned}
&\frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} B_{2k} + \frac{b^2(-1)^k \pi^{2k}}{4(2k)!} \\
&\quad \times \sum_{j=0}^{k} \binom{2k}{2j} B_{2j} \frac{2 - 2^{2j}}{2k - 2j + 1} {}_2F_1(1; 2, k - j + 3/2; -b^2/4).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&-\frac{b^2/4}{k - j + 1/2} {}_1F_2(1; 2, k - j + 3/2; -b^2/4) \\
&= {}_0F_1(-; k - j + 1/2; -b^2/4) - 1 \\
&= \Gamma(k - j + 1/2)(b/2)^{j-k+1/2} J_{k-j-1/2}(b) - 1.
\end{aligned}$$

The above identity shows that the left hand side of (3.5) as given by (3.6) reduces to the right hand side of (3.5) and this proves (3.5).

4. A generalization of (1.6)

The zeros of the even function $x^{-\nu}J_\nu(x)$ are all real and simple, see Chapter 15 in [22]. Let

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \dots$$

be the positive zeros of $x^{-\nu}J_\nu(x)$. We shall prove

$$\sum_{n=1}^{\infty} \frac{j_{\mu,n}^{1+\mu}/J_{\mu+1}(j_{\mu,n})}{j_{\mu,n}^2 - z^2} \frac{J_{\mu+\nu+1}\sqrt{b^2 + j_{\mu,n}^2}}{\left(b^2 + j_{\mu,n}^2\right)^{(\mu+\nu+1)/2}} \tag{4.1}$$
$$= \frac{z^\mu}{2J_\mu(z)}[b^2 + z^2]^{-(\mu+\nu+1)/2} J_{\mu+\nu+1}\left(\sqrt{b^2+z^2}\right),$$

for $\mu > -1$, $\nu > -1$, $b \ge 0$, $z \ne \pm j_{\mu,n}$, $n = 1, 2, \dots$, and $\mathrm{Re}(z^2 + b^2) > 0$. Note that (1.6) indicates that (4.1) reduces to (1.6) when $\mu = -1/2$ and $z = 0$. Also note that (1.10) is the case $\mu = -1/2$ and $\nu = 0$ of (4.1).

Proof of (4.1). We start with Sonine's second integral, (4) of §7.7 in [2],

$$\int_0^1 J_\mu(ax) J_\nu\left(b\sqrt{1-x^2}\right) x^{\mu+1}(1-x^2)^{\nu/2}\, dx \tag{4.2}$$
$$= a^\mu b^\nu (a^2 + b^2)^{-(\mu+\nu+1)/2} J_{\mu+\nu+1}\left(\sqrt{a^2+b^2}\right),$$

$a > 0$, $b > 0$, $\mathrm{Re}(\mu) > -1$, $\mathrm{Re}(\nu) > -1$. Therefore upon choosing $a = j_{\mu,n}$, multiplying both sides by $1/[(j_{\mu,n})^2 - z^2)$ and summing over n we find the left hand side of (4.1) equals

$$b^{-\nu} \int_0^1 J_\nu\left(b\sqrt{1-x^2}\right) x^{\mu+1}(1-x^2)^{\nu/2} \sum_{n=1}^{\infty} \frac{j_{\mu,n}/J_{\mu+1}(j_{\mu,n})}{j_{\mu,n}^2 - z^2} J_\mu(x j_{\mu,n})\, dx.$$

The series under the integral sign can be summed by (59) of §7.15 in Erdélyi et al [2]. Its sum is $J_\mu(xz)/[2J_\mu(z)]$. Therefore the left hand side of (4.1) is

$$\left[2b^\nu J_\mu(z)\right]^{-1} \int_0^1 J_\nu\left(b\sqrt{1-x^2}\right) x^{\mu+1}(1-x^2)^{\nu/2} J_\mu(xz)\, dx.$$

The latter integral is Sonine's second integral (4.2) and the above expression reduces to the right hand side of (4.1). This completes the proof.

5. Proofs of (1.4) and (1.5)

We now give a proof of (1.4) that uses (3.3) and the Mittag-Leffler expansion

(5.1)
$$\frac{\cos[x(z+A)]}{\sin z} = \sum_{-\infty}^{\infty}(-1)^n\cos[x(n\pi+A)]\left\{\frac{1}{z-n\pi}+\frac{1}{n\pi}\right\},$$
$$0 \le x \le 1.$$

Note that the identity (5.1) can be established by the general procedure outlined in §7.4, pages 134–135 in [23]. Differentiating (5.1) with respect to z, then letting $z = -\phi$, we get

(5.2)
$$\frac{\cos[x(\phi-A)]}{\sin^2\phi}\cos\phi + \frac{x\sin[x(\phi-A)]}{\sin\phi} = \sum_{-\infty}^{\infty}\frac{\cos[x(n\pi+A)]}{(-1)^n(\phi+n\pi)^2}, \qquad 0 \le x \le 1.$$

Proof of (1.4). We first observe that by expanding both sides in (3.3) in powers of b, one finds that (3.3) holds for purely imaginary b. In (3.3) we replace b by iB then set

(5.3) $\quad y = n\pi + \phi + ac$ and $B = a|b - 1/b|/2$ where $c = (b + 1/b)/2$.

Summing over n from $-\infty$ to ∞ one can then see that the left hand side of (1.4) is

$$\sum_{-\infty}^{\infty}\frac{\cos\sqrt{(n\pi+\phi+ac)^2-B^2}}{(-1)^n(n\pi+\phi)^2}$$
$$= \sum_{-\infty}^{\infty}\frac{\cos(n\pi+\phi+ac)}{(-1)^n(n\pi+\phi)^2}$$
$$+ B\int_0^1(1-x^2)^{-1/2}I_1\left(B\sqrt{1-x^2}\right)\sum_{-\infty}^{\infty}\frac{\cos[x(n\pi+\phi+ac)]}{(-1)^n(n\pi+\phi)^2}\,dx.$$

Recall the definition of $I_\nu(z)$ in (1.22). Both series in the right hand side of the above equality can be summed by (5.2). The result is that the above expression is

$$\frac{\cos(ac)}{\cos\phi}\cot\phi - \frac{\sin(ac)}{\sin\phi}$$
$$+ B\int_0^1 (1-x^2)^{-1/2} I_1\left(B\sqrt{1-x^2}\right)\left[\frac{\cos(acx)}{\sin\phi}\cot\phi - \frac{x\sin(acx)}{\sin\phi}\right]dx,$$

which is nothing but the derivative with respect to u evaluated at $u = 1$ of the integral

$$\int_0^u I_0\left(B\sqrt{u^2-x^2}\right)\left[\frac{\cos(acx)}{\sin\phi}\cot\phi - \frac{x\sin(acx)}{\sin\phi}\right]dx. \tag{5.4}$$

The integral in (5.4) is a sum of two integrals and both integrals can be evaluated from Sonine's second integral, (4.2). Thus the quantity (5.4) is

$$\frac{\cos\phi}{\sin^2\phi}\frac{\sin\left(u\sqrt{a^2c^2-B^2}\right)}{\sqrt{a^2c^2-B^2}} - \frac{acu\sqrt{u}}{(a^2c^2-B^2)^{3/4}\sin\phi}J_{3/2}\left(u\sqrt{a^2c^2-B^2}\right)$$

and its derivative at $u = 1$ is the right hand side of (1.4), since $(z^\nu J_\nu(Z))' = z^\nu J_{\nu-1}(z)$, [2, §7.2.8] and $J_{1/2}(z) = (\pi z/2)^{-1/2}\sin z$, see (1.7).

Proof of (1.5). We set $c = a + 1$ and $d = b + 1$ in (1.11) and apply the reflection formula

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z).$$

The result is

$$\sum_{-\infty}^{\infty} \frac{1}{(\alpha+n\pi)(\beta+n\pi)} = \frac{\sin(\beta-\alpha)}{(\beta-\alpha)\sin\alpha\sin\beta}, \tag{5.5}$$

and by differentiation we establish the relationships

$$\sum_{-\infty}^{\infty} \frac{1}{(\alpha+n\pi)(\beta+n\pi)^2} = \frac{\sin(\beta-\alpha)}{(\beta-\alpha)^2\sin\alpha\sin\beta} - \frac{1}{(\beta-\alpha)\sin^2\beta}, \tag{5.6}$$

(5.7)
$$\sum_{-\infty}^{\infty} \frac{1}{(\alpha+n\pi)^2(\beta+n\pi)^2}$$
$$= \frac{1}{(\beta-\alpha)^2\sin^2\beta} + \frac{1}{(\beta-\alpha)^2\sin^2\alpha} - \frac{2\sin(\beta-\alpha)}{(\beta-\alpha)^3\sin\alpha\sin\beta}.$$

Observe that the left hand side of (1.5) is

$$\begin{aligned}(5.8)\qquad &\sin(\phi+2a)\sum_{-\infty}^{\infty}\frac{(n\pi+\phi+a)^2+3a^2}{(n\pi+\phi+2a)^2(n\pi+\phi-2a)^2}\\ &\times\cot\left(\phi+a+\frac{3a^2}{n\pi+\phi+a}\right)\\ &-\cos(\phi+2a)\sum_{-\infty}^{\infty}\frac{(n\pi+\phi+a)^2+3a^2}{(n\pi+\phi+2a)^2(n\pi+\phi-2a)^2}.\end{aligned}$$

Now using

$$(n\pi+\phi+a)^2+3a^2=(n\pi+\phi-2a)^2+6a(n\pi+\phi-2a)+9a^2,$$

we see that the second sum in (5.8) is

$$\begin{aligned}&\sum_{-\infty}^{\infty}\frac{1}{(n\pi+\phi+2a)^2}+\sum_{-\infty}^{\infty}\frac{6a}{(n\pi+\phi-2a)(n\pi+\phi+2a)^2}\\ &+\sum_{-\infty}^{\infty}\frac{12a^2}{(n\pi+\phi-2a)^2(n\pi+\phi+2a)^2},\end{aligned}$$

and the three series appearing in the above expression can be summed by (5.5), (5.6) and (5.7). The result, after some simplification, is

$$\begin{aligned}(5.9)\qquad &\cot(\phi+2a)\sum_{-\infty}^{\infty}\frac{(n\pi+\phi+a)^2+3a^2}{(n\pi+\phi+2a)^2(n\pi+\phi-2a)^2}\\ &=\frac{\cot(\phi+2a)}{4\sin^2(\phi+2a)}+\frac{3\cot(\phi+3a)}{4\sin^2(\phi-2a)}.\end{aligned}$$

In view of (5.8) and (5.9) we reduce proving (1.5) to proving the following identity

(5.10)

$$\begin{aligned}&\sum_{-\infty}^{\infty}\frac{(n\pi+\phi+a)^2+3a^2}{(n\pi+\phi+2a)^2(n\pi+\phi-2a)^2}\cot\left(\phi+a+\frac{3a^2}{n\pi+\phi+a}\right)\\ &=\frac{\cot(\phi-2a)}{4\sin^2(\phi+2a)}+\frac{3\cot(\phi+2a)}{4\sin^2(\phi-2a)}.\end{aligned}$$

We now prove (5.10). The limiting case $\beta \to \alpha$ of (5.5) and the Mittag-Leffler expansion

$$\cot x = \lim_{M\to\infty} \sum_{m=-M}^{M} (m\pi + x)^{-1},$$

imply

$$\text{(5.11)} \quad \frac{\cot(\phi - 2a)}{\sin^2(\phi + 2a)} + \frac{3\cot(\phi + 2a)}{\sin^2(\phi - 2a)} = \lim_{M\to\infty} \sum_{n=-M}^{M} \sum_{m=-M}^{M} s_{m,n}$$

where

$$s_{m,n} = (n\pi + \phi + 2a)^{-2}(m\pi + \phi - 2a)^{-1} + 3(n\pi + \phi - 2a)^{-2}(m\pi + \phi + 2a)^{-1}.$$

On the other hand the algebraic identities

$$\frac{1}{(n\pi + \phi + 2a)(m\pi + \phi - 2a)} = \frac{1}{(m\pi + \phi + a)(n\pi + \phi + a) + 3a^2}\frac{n\pi + \phi + a}{n\pi + \phi + 2a} + \frac{1}{(m\pi + \phi + a)(n\pi + \phi + a) + 3a^2}\frac{3a}{m\pi + \phi - 2a}$$

and

$$\frac{1}{(n\pi + \phi + 2a)(m\pi + \phi - 2a)} = \frac{1}{(m\pi + \phi + a)(n\pi + \phi + a) + 3a^2}\frac{n\pi + \phi + a}{n\pi + \phi - 2a} - \frac{1}{(m\pi + \phi + a)(n\pi + \phi + a) + 3a^2}\frac{a}{m\pi + \phi + 2a}$$

enable us to express $s_{m,n}$ in the convenient form

$$\begin{aligned} s_{m,n} = {} & \frac{(n\pi + \phi + 2a)^{-2}}{m\pi + \phi + a + 3a^2/(n\pi + \phi + a)} + \frac{3a/[(m\pi + \phi - 2a)(n\pi + \phi + 2a)]}{(m\pi + \phi + a)(n\pi + \phi + a) + 3a^2} \\ & + \frac{3(n\pi + \phi - 2a)^{-2}}{m\pi + \phi + a + 3a^2/(n\pi + \phi + a)} - \frac{3a/[(m\pi + \phi + 2a)(n\pi + \phi - 2a)]}{(m\pi + \phi + a)(n\pi + \phi + a) + 3a^2}. \end{aligned}$$

Observe that the second and fourth terms in the above expression for $s_{m,n}$ are anti-symmetric in m and n. Hence its double sum over m and n vanishes. On the remaining two sums we sum over m first then n. The result of this calculation and (5.11) show that the right hand side of (5.10) is

$$\frac{1}{4}\sum_{-\infty}^{\infty}\left[\frac{1}{(n\pi+\phi+2a)^2}+\frac{3}{(n\pi+\phi-2a)^2}\right]\cot\left(\phi+a+\frac{3a^2}{n\pi+\phi+a}\right),$$

which after some simplification reduces to the left hand side of (5.10). This completes the proof of (1.5).

6. Proofs of (1.13)–(1.16)

We first show how (1.13) and (1.14) follow from (1.4).

Proof of (1.13). Apply the partial fraction decomposition

$$\frac{n^2\pi^2+\phi^2}{(n^2\pi^2-\phi^2)^2}=\frac{1}{2(n\pi+\phi)^2}+\frac{1}{2(-n\pi+\phi)^2}$$

to express (1.13) in the equivalent form

$$(6.1)\quad \sum_{-\infty}^{\infty}\frac{(-1)^n}{(n\pi+\phi)^2}\cos\left(\sqrt{n^2\pi^2+a^2-\phi^2}\right)=\frac{a\cos a\cot\phi+\phi\sin a}{a\sin\phi}.$$

Now (6.1) is (1.4) where b and b^{-1} are the roots of the equation $ax^2+2\phi x+a=0$ and the proof of (1.13) is complete.

Proof of (1.14). In (1.4) set $\phi=\Psi+\pi/2$ and assume that b and b^{-1} are the roots of $ax^2+2\Psi x+a=0$. The result is

$$(6.2)\quad \sum_{-\infty}^{\infty}\frac{\cos\left(\sqrt{(n+1/2)^2\pi^2+a^2-\Psi^2}\right)}{(-1)^n[(n+1/2)\pi+\Psi]^2}=\frac{\Psi\sin a-a\tan\Psi\cos a}{a\cos\Psi}.$$

Express the left hand side of (6.2) is a sum of two series according as $n\ge 0$ or $n<0$. In the latter series replace n by $-n-1$, so the new n is nonnegative. Now combine the two sums and replace Ψ by ϕ. After some simplification one establishes (1.14).

Proof of (1.15). We start with the trigonometric identity

$$\frac{1}{4}(2+\csc\theta) = \frac{\sin(2\theta)+\cos\theta}{2\sin(2\theta)} = \frac{\sin(2\theta)+\sin\left(\frac{\pi}{2}-\theta\right)}{2\sin(2\theta)}$$
$$= \frac{\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\cos\left(\frac{3\theta}{2}-\frac{\pi}{4}\right)}{\sin(2\theta)}.$$

In the above identity replace θ by a $(-2)^{-n}+\pi/6$ then multiply the results for $n=1,2,\ldots$. Observe that a repeated application of the trigonometric identity $\sin(2x)=2\sin x\cos x$ leads to

$$\prod_{n=1}^{\infty}\cos(x/2^n) = \lim_{N\to\infty}\left\{\left(\prod_{n=1}^{N}\cos(x/2^n)\right)\frac{2^N\sin(x/2^N)}{2^N\sin(x/2^N)}\right\} = \frac{\sin x}{x}.$$

Now the above analysis yields

$$\prod_{n=1}^{\infty}\left\{\frac{1}{2}+\frac{1}{4}\csc((-2)^{-n}a+\pi/6)\right\} \tag{6.3}$$
$$= \frac{2\sin(3a/2)}{3a}\prod_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{3}-(-2)^{-n-1}a\right)}{\sin\left(\frac{\pi}{3}-(-2)^{1-n}a\right)}.$$

The infinite product on the right hand side of the above equation is of the form $\prod a_{j+2}/a_j$, hence it converges to $(a_0a_1)^{-1}\lim a_na_{n+1}$. Here $a_n = \sin(a(-2)^{-n-1}+\pi/3)$. Therefore the infinite product on the right hand side of (6.3) converges to

$$\frac{3}{4}\csc\left(\frac{\pi}{3}-a\right)\csc\left(\frac{\pi}{3}+\frac{a}{2}\right).$$

Applying the trigonometric identities

$$\sin(3\theta) = -\sin(3\theta+\pi) = -(\sin(\theta+\pi/3))\left[4\cos^2(\theta+\pi/3)-1\right]$$
$$= -(\sin(\theta+\pi/3))[1+\cos(2\theta+2\pi/3)]$$
$$= (\sin(\theta+\pi/3))[-1+\cos(2\theta+\pi/6)],$$

with $\theta = a/2$ we establish (1.15) after some algebraic manipulations.

Recall that the Euler polynomials $E_n(x)$ and the Bernoulli polynomials $B_n(x)$ have the exponential generating functions [15, (1) p. 50, (7) p. 250]

$$\frac{2e^{xt}}{1+e^t} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n, \tag{6.4}$$

$$B(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n. \tag{6.5}$$

Our proof of (1.16) uses the functional equations

$$E_{n-1}(x) = \frac{2}{n}\left[B_n(x) - 2^n B_n(x/2)\right], \tag{6.6}$$

$$B_n(mx) = m^{n-1} \sum_{r=0}^{m-1} B_n(x + r/m). \tag{6.7}$$

Formula (6.7) is in §86, page 252 in [15] while (6.6) follows from (6.4) and (6.5).

Proof of (1.16). The key idea is to show that

$$\frac{E_{2n}\left(\frac{1}{3}\right)}{2^{2n+1}+1} = -\frac{2}{2n+1} B_{2n+1}\left(\frac{1}{3}\right). \tag{6.8}$$

Once (6.8) is established we then identify the left hand side of (1.16) as

$$-2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} B_{2n+1}\left(\frac{1}{3}\right)$$

which, in view of (6.5), is

$$i \int_0^{ix} \left[B\left(\frac{1}{3}, -u\right) - B\left(\frac{1}{3}, u\right)\right] \frac{du}{u}.$$

On the other hand (6.5) yields

$$\frac{B\left(\frac{1}{3}, u\right) - B\left(\frac{1}{3}, -u\right)}{u} = \frac{\sinh(u/6)}{\sinh(u/2)},$$

and the left hand side of (1.16) has the integral representation

$$-\int_0^x \frac{\sin(t/6)}{\sin(t/2)}\, dt,$$

which, since $\sin(3x) = [4\cos^2 x - 1]\sin x = [1 + 2\cos(2x)]\sin x$, shows that the left hand side of (1.16) is

$$\int_0^x \frac{dt}{1 + 2\cos(t/3)} = \frac{3}{2}\int_{-x}^x \frac{du}{1 + 2\cos u}.$$

Using $2\arctan x = \ln\{(1+x)/(1-x)\}$ and the evaluation

$$\int_0^x \frac{du}{1 + 2\cos u} = \frac{1}{\sqrt{3}}\,\mathrm{Arctan}\left[\frac{\sqrt{3}\sin x}{2 + \cos x}\right]$$

we reduce the left hand side of (1.16) to

$$\begin{aligned}\frac{\sqrt{3}}{2}\ln\left[\frac{2 + \cos(x/3) + \sqrt{3}\sin(x/3)}{2 + \cos(x/3) - \sqrt{3}\sin(x/3)}\right] &= \frac{\sqrt{3}}{2}\ln\left[\frac{1 + \cos((x-\pi)/3)}{1 + \cos((x+\pi)/3)}\right]\\ &= \frac{\sqrt{3}}{2}\ln\left[\frac{2\cos^2((x+\pi)/6)}{2\cos^2((x-\pi)/6)}\right].\end{aligned}$$

The last expression can be easily reduced to the right hand side of (1.16). Thus it only remains to establish (6.8). Set $m = 2$ and $x = 1/6$ in (6.7) to get

$$B_{2n+1}(1/3) = 2^{2n}[B_{2n+1}(2/3) + B_{2n+1}(1/3)],$$

then set $x = 1/3$ in $B_n(1-x) = (-1)^n B_n(x)$ to obtain

$$2^{2n}B_{2n+1}(1/6) = (1 + 2^{2n})B_{2n+1}(1/3).$$

Finally the above equation and (6.6) establish (6.8). This concludes the proof of (1.16).

7. Proofs of (1.17) and (1.18)

The proof of (1.17) depends on tricky series rearrangements and is similar to our proof of (1.5).

Proof of (1.17). Use the Mittag-Leffler expansion for the cotangent function, see the equation between (5.10) and (5.11), to get

$$\cot^2 x = \lim_{M\to\infty}\sum_{n=-M}^{M}\sum_{m=-M}^{M}[(x+n\pi)(x+m\pi)]^{-1}. \tag{7.1}$$

Then express $[(n\pi + a)(m\pi + a)]^{-1}$ as

$$[(n\pi + a)(m\pi + b)]^{-1} + (b-a)[(n\pi + a)(m\pi + b)(m\pi + a)]^{-1}$$

and write a similar expression after the interchanges $a \leftrightarrow b$, $m \leftrightarrow n$. These considerations lead to

(7.2)
$$\begin{aligned}\frac{\cot^2 a - \cot^2 b}{b-a} &= \lim_{M\to\infty} \sum_{n=-M}^{M} \sum_{m=-M}^{M} \{[(a+n\pi)(b+n\pi)(b+m\pi)]^{-1} \\ &\qquad + [(a+n\pi)(b+n\pi)(b+m\pi)]^{-1}\}.\end{aligned}$$

Clearly

$$\begin{aligned}&\left(n\pi + \frac{a+b}{2}\right)\left(m\pi + \frac{a+b}{2}\right) - \left(\frac{a-b}{2}\right)^2 \\ &\qquad = (a+n\pi)\left(m\pi + \frac{a+b}{2}\right) - \frac{a-b}{2}(a+m\pi).\end{aligned}$$

By multiplying and dividing $[(a + n\pi)(b + m\pi)(a + m\pi)]^{-1}$ by the above quantity we get

$$\begin{aligned}&[(a+n\pi)(b+m\pi)(a+m\pi)]^{-1} \\ &\quad = \frac{[(b+m\pi)(a+m\pi)]^{-1}}{n\pi + \frac{a+b}{2} - \frac{\frac{1}{4}(a-b)^2}{m\pi + (a+b)/2}} \\ &\qquad - \frac{a-b}{2} \frac{[(a+n\pi)(b+m\pi)]^{-1}}{\left(n\pi + \frac{a+b}{2}\right)\left(m\pi + \frac{a+b}{2}\right) - \left(\frac{a-b}{2}\right)^2}.\end{aligned}$$

We also obtain a similar formula with the interchanges $a \leftrightarrow b$, $m \leftrightarrow n$. Using (7.2) and the above manipulations we obtain

$$\frac{\cot^2 a - \cot^2 b}{b-a} = \lim_{M\to\infty} \sum_{n=-M}^{M} \sum_{m=-M}^{M} \{A_{m,n}(a,b) + A_{n,m}(a,b)\},$$

where

$$A_{m,n}(a,b) := \frac{[(a+m\pi)(b+m\pi)]^{-1}}{n\pi + \frac{a+b}{2} - \frac{(a-b)^2/4}{m\pi + (a+b)/2}}.$$

Since the summand $A_{m,n}(a,b)$ is symmetric in m and n then

$$\frac{\cot^2 a - \cot^2 b}{b-a} = 2\lim_{M\to\infty} \sum_{n=-M}^{M} \sum_{m=-M}^{M} A_{m,n}(a,b). \tag{7.3}$$

The limit $M \to \infty$ of the sum over n in the above expression is

$$\frac{1}{(a+m\pi)(b+m\pi)} \cot\left(\frac{a+b}{2} - \frac{(a-b)^2/4}{m\pi + (a+b)/2}\right)$$

and the result of (7.3) is now equivalent to (1.17).

Proof of (1.18). It is clear that $(-1)^{n(n+1)/2}$ equals 1 if $n \equiv 0$ or 3 (mod 4) but is -1 when $n \equiv 1$ or 2 (mod 4). Now write the left hand side of (1.19) as a sum of four series depending on the residue of n modulo 4. Apply (3.3) with b replaced by a to see that the left hand side of (1.18) is

(7.4)

$$\begin{aligned}
&\frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \left[\frac{1}{(4k+1/2)^2} + \frac{1}{(4k+3/2)^2} + \frac{1}{(4k+5/2)^2} + \frac{1}{(4k+7/2)^2}\right] \\
&\quad - a\int_0^1 (1-x^2)^{-1/2} J_1\left(a\sqrt{1-x^2}\right) \\
&\quad \times \left\{ \sum_{k=0}^{\infty} \sum_{r=0}^{3} (-1)^{r(r+1)/2} \frac{\cos\left[x\left(2k\pi + (2r+1)\frac{\pi}{4}\right)\right]}{(4k+r+1/2)^2} \right\} dx.
\end{aligned}$$

The first term in the above expression is a multiple of the sum of the squares of the reciprocals of the positive odd integers. It is $\pi^2/(2\sqrt{2})$. Furthermore the observation

$$(-1)^{r(r+1)/2} = \frac{\cos\left(\frac{\pi}{4}(8k+2r+1)\right)}{\cos(\pi/4)}$$

and the trigonometric identity $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

imply

$$\sum_{k=0}^{\infty}\sum_{r=0}^{3}(-1)^{r(r+1)/2}\frac{\cos\left[x\left(2k\pi+(2r+1)\frac{\pi}{4}\right)\right]}{(4k+r+1/2)^2}$$
$$=\sqrt{2}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi x}{4}(2k+1)\right)\cos\left(\frac{\pi}{4}(2k+1)\right)}{(k+1/2)^2}$$
$$=\frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi}{4}(x+1)(2k+1)\right)}{(k+1/2)^2}$$
$$+\frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi}{4}(x-1)(2k+1)\right)}{(k+1/2)^2}.$$

Recall that the piecewise polynomial $\Psi_2(t)$ of (2.3) satisfies [20]

$$\Psi_2(t)=4\sum_{n=1}^{\infty}\frac{\cos(2\pi nt)}{(2\pi n)^2},$$

hence we have

$$\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi}{4}(x+1)(2k+1)\right)}{(2k+1)^2}$$
$$=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{4}(x+1)\right)}{n^2}-\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{4}(x+1)\right)}{(2n)^2}$$
$$=\pi^2\Psi_2((x+1)/8)-(\pi^2/4)\Psi_2((x+1)/4),$$

and a similar formula with $x+1$ replaced by $x-1$. On the other hand

$$\Psi_n(t)=\Psi_n(t+1),\quad \Psi_n(t)=\Psi_n(1-t)$$

imply $\Psi_n(-t)=\Psi_n(t)$ for $0<t<1$. Therefore we proved

$$\sum_{k=0}^{\infty}\sum_{r=0}^{3}(-1)^{r(r+1)/2}\frac{\cos\left[x\left(2k\pi+(2r+1)\frac{\pi}{4}\right)\right]}{(4k+r+1/2)^2}$$
$$=\frac{\pi^2}{\sqrt{2}}\left\{4\Psi_2\left(\frac{x+1}{8}\right)+4\Psi_2\left(\frac{1-x}{8}\right)-\Psi_2\left(\frac{x+1}{4}\right)-\Psi_2\left(\frac{1-x}{4}\right)\right\}.$$

The function $\Psi_2(t)$ is $t^2 - t + 1/6$ if $0 \leq t \leq 1$. Recall that the left hand side of (1.18) is the expression (7.4). The above calculations show that the left hand side of (1.18) is

$$\frac{\pi^2}{2\sqrt{2}} - \frac{\pi^2 a}{2\sqrt{2}} \int_0^1 (1 - x^2)^{-1/2} J_1\left(a\sqrt{1 - x^2}\right) dx.$$

Finally the value of the integral in the above expression can be found from (3.3) and the above expression will reduce to the right hand side of (1.18). This completes the proof of (1.18).

8. A Ramanujan connection and a nonlocal derangement

B.C. Berndt [1] has been editing the published Notebooks of Ramanujan. The first two parts have appeared in book form already. Section 9 of Chapter 4 of the Notebooks contains a very interesting formal iteration procedure of Ramanujan. As an illustration Berndt [1, pp. 95–98] shows how Ramanujan formally iterated the functional relationship $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$ to obtain the series expansion

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}\{f(2n + 1) + f(-2n - 1)\} = \frac{\pi}{2} f(0). \tag{8.1}$$

Observe that (1.6) is simply (8.1) if

$$f(z) = \left[b^2 + \pi^2 z^2/4\right]^{-\nu/2} J_\nu\left(\sqrt{b^2 + \pi^2 z^2/4}\right). \tag{8.2}$$

Berndt then shows that (8.1) holds in the form

$$\lim_{N \to \infty} \sum_{k=-N}^{N} \frac{(-1)^k}{2k + 1} f(2k + 1) = \frac{\pi}{2} f(0)$$

if $f(z)$ is an entire function of z and there exists a sequence r_n, $1 \leq n < \infty$ of positive numbers tending to ∞ such that

$$\frac{1}{2\pi i} \int_{|z|=r_n} \frac{f(z)\, dz}{z \cos(\pi z/2)} = o(1), \quad \text{as } n \to \infty. \tag{8.3}$$

One can then give an alternate proof of (1.6) by verifying the existence of a sequence of positive numbers r_n satisfying the above conditions given by Berndt [1]. Berndt also pointed out that the choice of circular contours is not strictly necessary and any appropriate sequence of contours may be chosen.

It is possible to extend (8.1) by replacing $2n + 1$ by a multiple of $j_{\nu,n}$, if $\nu > -1$. We only need to replace (8.3) by the condition

$$\frac{1}{2\pi i}\int_{|z|=r_n} \frac{z^{\nu-1}f(z)}{J_\nu(z)}\,dz = o(1), \quad \text{as } n \to \infty. \tag{8.4}$$

Since $z^{-\nu}J_\nu(z)$ is an entire function and $(z^{-\nu}J_\nu(z))' = -z^{-\nu}J_{\nu+1}(z)$, [2, §7.2.8], a calculation of residues enables us to replace (8.1) by the more general expansion

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(j_{\nu,n+1})^{\nu-1}}{J_{\nu+1}(j_{\nu,n+1})}\{f(j_{\nu,n+1}) + f(-j_{\nu,n+1})\}& \\ = \frac{\Gamma(\nu+1)}{2^\nu}f(0), \qquad \nu > -1.&\end{aligned} \tag{8.5}$$

Note that $(-1)^n J_{\nu+1}(j_{\nu,n+1}) > 0$, when $\nu > -1$ because the positive zeros of $J_\nu(z)$ and $J_{\nu+1}(z)$ interlace. Thus the coefficients of the expression in curled brackets in (8.5) alternate in sign as they do in (8.1). In fact (4.1) is a representation of the quotient on the right hand side, after replacing z^2 by z, as a Stieltjes transform $\int(z+t)^{-1}\,d\sigma(t)$, where $d\sigma$ is a discrete measure.

The series summation formula (4.1) is reminiscent of the sums in [14] involving the modified Bessel functions $I_\nu(x)$. In fact an expansion theorem of entire functions $F(z)z^{\nu/2}/I_\nu(z)$ in terms of the values of F at $-(j_{\nu,m})^2$ is stated as Theorem 6.4 in [14]. The work [14] was concerned with infinite divisibility of certain probability distributions. It contains integral representations of several quotients of modified Bessel functions and confluent hypergeometric functions as Stieltjes transforms of discrete or continuous measures.

We now state and prove the following mild generalization of the nonlocal derangement formula (1.20).

THEOREM 8.6. *Let $c, b_1, b_2, \ldots, b_n$ be distinct complex numbers, then*

$$\begin{aligned}\prod_{k=1}^{n} \frac{x + a_k - b_k}{x - b_k} &= \prod_{k=1}^{n} \frac{c + a_k - b_k}{c - b_k} \\ &\quad + \sum_{k=1}^{n} \frac{a_k(x-c)}{(b_k - c)(x - b_k)} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{b_k + a_j - b_j}{b_k - b_j}.\end{aligned} \tag{8.7}$$

Proof. Let $c \notin \{b_1, b_2, \dots, b_n\}$ and set

$$f(x) := \prod_{k=1}^{n} (x + a_k - b_k), \qquad w(x) := (x - c) \prod_{k=1}^{n} (x - b_k).$$

By Lagrange interpolation on the nodes $c, b_1, b_2, \dots, b_n$ we first represent $f(x)$ as a finite series.

$$f(x) = f(c) \prod_{k=1}^{n} \frac{(x - b_k)}{(c - b_k)} + \sum_{k=1}^{n} f(b_k) \frac{w(x)}{(x - b_k) w'(b_k)}.$$

Since

$$f(b_k) = a_k \prod_{1 \le j \ne k \le n} (b_k + a_j - b_j),$$

we write the above expression of $f(x)$ in the form

$$\prod_{k=1}^{n} \frac{x + a_k - b_k}{x - b_k}$$
$$= \prod_{k=1}^{n} \frac{c + a_k - b_k}{c - b_k} + \sum_{k=1}^{n} \frac{a_k (x - c)}{(b_k - c)(x - b_k)} \prod_{\substack{j=1 \\ j \ne k}}^{n} \frac{b_k + a_j - b_j}{b_k - b_j}.$$

This completes the proof.

COROLLARY 8.8. *If $b_1, b_2, \dots, b_n$ be distinct complex numbers, then we have*

$$\prod_{k=1}^{n} \left(1 + \frac{a_k}{x - b_k}\right) - 1 = \sum_{k=1}^{n} \frac{a_k}{x - b_k} \prod_{\substack{j=1 \\ j \ne k}}^{n} \left(1 + \frac{a_j}{b_k - b_j}\right). \tag{8.9}$$

Proof. Let $c \to \infty$ in (8.7).

Observe that the nonlocal derangement formula (1.20) follows by equating the coefficients of $1/x$ in the Laurent series expansion of (8.9) about $x = \infty$. It is not difficult to see that (8.7) and (8.9) are equivalent.

Richard Askey pointed out that (8.9) is in [16] and is credited to Jet Wimp in a private communication to Steven Milne. Milne and Wimp were only interested in the case $x = 0$. More complicated proofs of the case $x = 0$ of (8.9) were given earlier by J. Louck and L. Biedenharn and by S. Milne. For details, interesting applications, and references see [16, p. 67].

As we saw earlier the nonlocal derangement formula follows from equating coefficients of $1/x$ in the Laurent expansion of both sides of (8.9). By equating coefficients of higher powers of $1/x$ in (8.7) or (8.9) one would expect to discover other identities similar to the nonlocal derangement. This is indeed the case. The coefficient of $1/x^2$ in (8.9) leads to

$$\sum_{1\le k<j\le n} a_k a_j + \sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k b_k \prod_{\substack{j=1\\ j\neq k}}^{n} \left[1 + a_j/(b_k - b_j)\right]. \tag{8.10}$$

The special case $a_k = 2b_k$ is the curious identity

$$\left(\sum_{k=1}^{n} b_k\right)^2 = \sum_{k=1}^{n} b_k^2 \prod_{\substack{j=1\\ j\neq k}}^{n} \left(\frac{b_j + b_k}{b_k - b_j}\right). \tag{8.11}$$

We now establish two unusual looking formulas implied by the nonlocal derangement identity (1.20), see (8.12), (8.13), and (8.14) below. Choose $a_n = -a/(4n^2)$, $b_n = 1/n$, $n > 0$. In this case (1.20) becomes

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{j^2} &= \sum_{j=1}^{n} \frac{1}{j^2} \prod_{\substack{k=1\\ k\neq j}}^{n} \left(1 - \frac{ja/4}{k(k-j)}\right) = \sum_{j=1}^{n} \frac{1}{j^2} \prod_{\substack{k=1\\ k\neq j}}^{n} \left(\frac{k^2 - kj - ja/4}{k(k-j)}\right) \\
&= \sum_{j=1}^{n} \frac{1}{j^2} \prod_{\substack{k=1\\ k\neq j}}^{n} \left(\frac{k\left(1 - \frac{j}{k} - \frac{1}{4}aj/k^2\right)}{k-j}\right) \\
&= \sum_{j=1}^{n} \frac{1}{j^2} \frac{n!}{j} \frac{(-1)^{j-1}}{(j-1)!\,(n-j)!} \left(\frac{-4}{a}\right) \\
&\quad \times \prod_{k=1}^{n} \left(1 - \frac{j + \sqrt{j(j+a)}}{2k}\right)\left(1 - \frac{j - \sqrt{j(j+a)}}{2k}\right).
\end{aligned}$$

In the last product multiply the kth factor by $e^{j/k}$ and compensate by replacing 4 by

$$4\exp\left(\sum_{n\ge k\ge 1} j/k\right).$$

Now let $n \to \infty$ and obtain

(8.12)
$$-\frac{a\pi^2}{12} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j!\,j} \frac{1}{\Gamma\left(1 - \frac{j+\sqrt{j(j+a)}}{2}\right)\Gamma\left(1 - \frac{j-\sqrt{j(j+a)}}{2}\right)}.$$

In deriving (8.12) we used the infinite product representation of the gamma function and the fact that $\sum_{n \geq k \geq 1} 1/k - \log n \to \gamma$, the Euler constant, as $n \to \infty$.

If we multiply and divide the gamma functions in (8.12) by

$$\Gamma\left(\frac{j+\sqrt{j(j+a)}}{2}\right)\Gamma\left(\frac{j-\sqrt{j(j+a)}}{2}\right)$$

then use the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ we establish the identity

(8.13)
$$\sum_{j=1}^{\infty} \left(1 - (-1)^j \cos\left(\sqrt{j(j+a)}\,\pi\right)\right) \frac{\Gamma\left(\frac{j+\sqrt{j(j+a)}}{2}\right)\Gamma\left(\frac{j-\sqrt{j(j+a)}}{2}\right)}{j!\,j} = -\frac{\pi^4 a}{12}.$$

In both (8.12) and (8.13) the condition $\operatorname{Re} a < 4$ is needed for convergence of the infinite series involved. Similarly the choices $a_n = -a/(4n(n+b))$, $b_n = 1/n$, $n > 0$, lead to the relationship

(8.14)
$$\sum_{k=1}^{\infty} \left[1 - (-1)^k \cos(D_k \pi)\right] \frac{\Gamma(\frac{1}{2}(k - D_k))\Gamma(\frac{1}{2}(k + D_k))}{(k-1)!\,(k+b)^2} = -\frac{\pi^2 a}{2b} H(b), \qquad \operatorname{Re} a < 4,$$

where

(8.15)
$$D_k := \sqrt{k^2 + ak + ab}, \qquad H(b) = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+b}\right].$$

It is clear that $H(b)$ is related to the Ψ function ($\Psi(z) := \Gamma'(z)/\Gamma(z)$) by $H(b) = \gamma + \Psi(b+1)$.

9. Continuous analogues and some infinitely divisible distributions

A function defined on $(0,\infty)$ is called completely monotonic if and only if

$$(-1)^n \frac{d^n f(x)}{dx^n} \geq 0 \quad \text{for } x \in (0,\infty).$$

A probability measure $d\mu$ is infinitely divisible if and only if for every positive integer n there is a probability measure $d\mu_n$ such that $d\mu$ is an n-fold convolution of $d\mu_n$, Feller [5]. When $d\mu$ is supported on a subset of $[0,\infty)$ then $d\mu$ is infinitely divisible if and only if

(9.1)

$$\int_0^\infty e^{-xt}\, d\mu(t) = e^{-h(x)}, \qquad h(0) = 0, \quad \text{and } h'(x) \text{ is completely monotonic.}$$

THEOREM 9.2. *The functions* $f(x)$ *and* $g(x)$ *in* (1.21a) *and* (1.21b) *are Laplace transforms of infinitely divisible probability distributions when* $\nu > 0$.

Proof. In order to show that (9.1) holds we need to show that $f(0+) = g(0+) = 1$ and that $-f'(x)/f(x)$ and $-g'(x)/g(x)$ are completely monotonic. This is the case because the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ are positive for positive x. To see that $f(0+) = g(0+) = 1$ we apply the definitions, [2, §7.2.2],

(9.3)

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}, \qquad K_\nu(z) := \frac{\pi}{2\sin(\pi\nu)}[I_{-\nu}(z) - I_\nu(z)],$$

when $\nu \neq 1, 2, \ldots$. If ν is a positive integer we apply the definition in [2, §7.2.5] instead. The differential recurrence relation

$$(x^{-\nu} I_\nu(x))' = x^{-\nu} I_{\nu+1}(x),$$

[2, §7.11] imply

$$-\frac{g'(x)}{g(x)} = \frac{I_{\nu+1}(\sqrt{x})}{\sqrt{x}\, I_\nu(x)} - \frac{I_{\nu+1}(\sqrt{x+b})}{\sqrt{x+b}\, I_\nu(\sqrt{x+b})},$$

and the complete monotonicity of $-g'(x)/g(x)$ now follows from the

Mittag-Leffler expansion [2, §7.9]

$$\frac{I_{\nu+1}(\sqrt{x})}{\sqrt{x}\,I_\nu(\sqrt{x})} = \sum_{n=1}^{\infty} \frac{2}{x + j_{\nu,n}^2}. \tag{9.4}$$

The complete monotonicity of $-f'(x)/f(x)$ similarly follows from $(x^\nu K_\nu(x))' = -x^\nu K_{\nu-1}(x)$ [2, §7.11] and the continuous analogue of (9.4)

$$\frac{K_{\nu-1}(\sqrt{x})}{\sqrt{x}\,K_\nu(\sqrt{x})} = \frac{4}{\pi^2}\int_0^\infty \frac{t^{-1}}{x+t^2}\left\{J_\nu^2(t) + Y_\nu^2(t)\right\}^{-1} dt. \tag{9.5}$$

This completes the proof.

The integral representation (9.5) was implicit in a paper of Grosswald [12]. The present form of (9.5) is due to Ismail who also used it extensively. Details and references are in [14]. For interesting infinitely divisible random variables and various completely monotonic quotients of modified Bessel functions we refer the interested reader to [14] and [19]. Among other things reference [19] gives probabilistic settings for the random variables discussed in [14] and it also establishes their infinite divisibility. We do not know a natural probabilistic set up that leads to the probability distributions of Theorem 9.2. It is likely that they are hitting time distributions for a Brownian motion between two planes a distance b apart.

Before identifying the probability measures whose Laplace transforms are $f(x)$ and $g(x)$ we point out a simple application of Theorem 9.2. Recall that $e^{-h(x)}$ is completely monotonic if $h'(x)$ is completely monotonic, hence $1 = f(0) > f(\infty)$. Furthermore note that

$$K_\nu(x) \approx [\pi/(2x)]^{1/2} e^{-x} \quad \text{as } x \to \infty.$$

This gives the known inequality

$$x^\nu K_\nu(x) < 2^{\nu-1}\Gamma(\nu), \qquad x > 0.$$

Unfortunately the corresponding relations for the function $g(x)$, that is $1 > g(\infty)$ and

$$I_\nu(x) \approx (2\pi x)^{-1/2} e^x \quad \text{as } x \to \infty,$$

only imply the trivial inequality

$$I_\nu(x) > (x/2)^\nu/\Gamma(\nu+1), \qquad x > 0.$$

We now compute the probability measures in Theorem 9.2. It is not possible to find the inverse Laplace transform of f and g directly so we use the Stieltjes transform. The representation and inversion theorems for the Stieltjes transform are the following.

THEOREM 9.6. (Representation Theorem). *If*
(i) *$F(z)$ is analytic for $|\arg z| < \pi/\alpha$ for some α, $0 < \alpha < 1$, and*
(ii) *$F(z) = o(1)$ as $z \to \infty$ and $zF(z) = o(1)$ as $z \to 0$, uniformly in every closed sector $|\arg z| \le \pi/\alpha'$, $\alpha' > \alpha$,*
then

$$(9.7)\quad F(z) = \frac{1}{\pi}\int_{0+}^{\infty} \frac{dt}{x+t}\,\frac{1}{2\pi i}\int_C \frac{zF(te^z)e^{z/2}}{\pi^2+z^2}\,dz, \qquad x \in (0,\infty),$$

where C is a closed rectifiable closed curve going around $[-i\pi, i\pi]$ in the positive direction and lying completely in the strip $|\mathrm{Im}\, z| < \pi/\alpha$.

THEOREM 9.8. (The Inversion Theorem). *If $F(z)$ is a Stieltjes transform, say*

$$F(z) = \int_0^{\infty} \frac{d\mu(t)}{z+t},$$

and if μ is normalized by $\mu(0) = \mu(0+)$; $\mu(t) = [\mu(t+) + \mu(t-)]/2$, $t > 0$; then $\mu(t)$ is given by

$$\mu(v) - \mu(u) = \lim_{s\to 0+} \frac{1}{2\pi i}\int_u^v \{F(-t-is) - F(-t+is)\}\,dt.$$

For references see [14].

THEOREM 9.9. *The integral representation*

$$(9.10)\qquad \frac{z^{\nu/2}K_\nu(\sqrt{z})}{(z+b)^{\nu/2}K_\nu(\sqrt{z+b})} - 1 = \frac{1}{\pi}\int_0^{\infty} \frac{w(t)}{z+t}\,dt,$$

holds for $\nu > 0$, $|\arg z| < \pi$, *where*

(9.11)

$$w(t) := \begin{cases} \dfrac{\pi}{2}\left(\dfrac{t}{b-t}\right)^{\nu/2} \dfrac{J_\nu(\sqrt{t})}{K_\nu(\sqrt{b-t})}, & 0 < t < b, \\ \left(\dfrac{t}{t-b}\right)^{\nu/2} \dfrac{J_\nu(\sqrt{t})Y_\nu(\sqrt{t-b}) - Y_\nu(\sqrt{t})J_\nu(\sqrt{t-b})}{J_\nu^2(\sqrt{t-b}) + Y_\nu^2(\sqrt{t-b})}, & b < t < \infty, \end{cases}$$

Proof. The function $z^\nu K_\nu(z)$ is analytic in the z plane cut along the negative real axis. For $\nu > 0$ the aforementioned function has only finitely many zeros and none of them lie in the closed sector $|\arg z| \le \pi/2$, [22, §15.7]. Furthermore (9.3) and the asymptotic relationship

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}\, e^{-z} \quad \text{as } |z| \to \infty,\ |\arg z| < \frac{3}{2}\pi$$

[22, §7.23] show that the left hand side of (9.10) satisfy assumptions (i) and (ii) of the representation theorem, Theorem 9.6. We then apply the inversion theorem, Theorem 9.8, to the left hand side of (9.10) and use the analytic continuation formula

$$e^{\pm i\pi\nu/2}K_\nu(ze^{\pm i\pi/2}) = -\frac{\pi}{2}(Y_\nu(z) + iJ_\nu(z)),$$

[2, §7.2.2], to establish (9.11) after some elementary manipulations. This completes the proof of Theorem 9.9.

Now recall Bernstein's Theorem, that is a function is completely monotonic if and only if it is the Laplace transform of a positive measure [5]. Next observe that Theorem 9.2, the identity

$$\int_0^\infty \frac{w(t)}{x+t}\, dt = \int_0^\infty e^{-xu} \int_0^\infty e^{-ut} w(t)\, dt\, du,$$

and (9.10) identify the inverse Laplace transform of the left hand side of (9.10) as the integral

$$\int_0^\infty e^{-xt} w(t)\, dt.$$

This establishes the following theorem.

THEOREM 9.12. *The inequality*

$$\int_0^\infty e^{-xt} w(t)\,dt > 0,$$

holds for $x > 0$.

Observe that $w(t)$ is an oscillatory function with infinitely many zeros, so it is somewhat surprising that its Laplace transform is positive.

10. Some formulas of Oberhettinger and remarks

It may be of interest to point out that the summation formulas (1.6)–(1.9) can be thought of as discrete analogues of a discontinuous integral of Sonine, namely

$$\begin{aligned}(10.1)\qquad &\int_0^\infty J_\mu(bt)(t^2+z^2)^{-\nu/2} J_\nu\left(a\sqrt{t^2+z^2}\right) t^{\mu+1}\,dt \\ &= \begin{cases} 0 & (a < b) \\ \dfrac{b^\mu}{a^\nu}\left(\dfrac{\sqrt{a^2-b^2}}{z}\right)^{\nu-\mu-1} & (a > b) \end{cases}\end{aligned}$$

of which the interesting special case $b \to 0$ is

$$(10.2)\qquad \int_0^\infty (t^2+z^2)^{-\nu/2} J_\nu\left(a\sqrt{t^2+z^2}\right) t^{2\mu+1}\,dt = \frac{2^\mu \Gamma(\mu+1)}{a^{\mu+1} z^{\nu-\mu-1}} J_{\nu-\mu-1}(az),$$

Watson [22, §13.47]. Formula (10.1) holds if a and b are positive and $\mathrm{Re}(\nu) > \mathrm{Re}(\mu) > -1$, while (10.2) holds for $a \ge 0$, and $\mathrm{Re}(\nu - 1/2) > \mathrm{Re}(2\mu) > -2$.

Mizan Rahman kindly pointed out that (1.6) is in the table of series and products prepared by Hansen [13]. In fact Hansen mentions the more general formulas involving ultraspherical (Gegenbauer) polynomials $\{C_n^{(a)}(x)\}$ [2].

(10.3)

$$\begin{aligned}&\sum_{k=1}^\infty \frac{(-1)^k}{k^2-b^2}(x^2+k^2)^{-a/2} C_{2n}^{(a)}\left[x(k^2+x^2)^{-1/2}\right] J_{2n+a}\left[y(k^2+x^2)^{1/2}\right] \\ &\quad= \frac{(2a)_{2n}}{2(2n)!} b^{-2} x^{-a} J_{2n+a}(xy) \\ &\qquad - \frac{\pi}{2b}\frac{(x^2+b^2)^{-a/2}}{\sin(\pi b)} C_{2n}^{(a)}\left[x(x^2+b^2)^{-1/2}\right] J_{2n+a}\left[y(x^2+b^2)^{1/2}\right],\end{aligned}$$

$0 < y < \pi$, $\mathrm{Re}(a) > 0$, and

(10.4)

$$\sum_{k=1}^{\infty} \frac{(-1)^k(2k+1)}{(2k+1)^2 - b^2}\left[(2k+1)^2 + x^2\right]^{-a/2} C_{2n}^{(a)}\left\{x\left[x^2 + (2k+1)^2\right)\right]^{-1/2}\right\}$$
$$\cdot J_{2n+a}\left\{\left[y(2k+1)^2 + x^2\right]^{1/2}\right\}$$
$$= \frac{\pi(x^2+b^2)^{-a/2}}{4\cos(\pi b/2)} C_{2n}^{(a)}\left\{x\left[(2k+1)^2 + x^2\right]^{-1/2}\right\} J_{2n+a}\left[y(x^2+b^2)^{1/2}\right]$$

$0 < y < \pi/2$, $\mathrm{Re}(a) > 0$. Formulas (10.3) and (10.4) appear on page 468 in [13] as (79.2.10) and (79.2.12), respectively. Furthermore (79.2.13) in [13] is

(10.5)

$$\sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{(2k+1)^2 - b^2}\left[(2k+1)^2 + x^2\right]^{-a/2} C_{2n+1}^{(a)}\left\{x\left[(2k+1)^2 + x^2\right]^{1/2}\right\}$$
$$\cdot J_{2n+a+1}\left\{y\left[(2k+1)^2 + x^2\right]^{1/2}\right\}$$
$$= \frac{\pi(x^2+b^2)^{-a/2}}{4\cos(b\pi/2)} C_{2n+1}^{(a)}\left[x(x^2+b^2)^{-1/2}\right] J_{2n+a+1}\left\{y[x^2+b^2]^{1/2}\right\},$$

$0 < y < \pi/2$, $\mathrm{Re}(a) > 0$. Our (10.4) corrects a misprint in (79.2.12) in [13] where the argument of the Bessel function on the right side is recorded as $y(x^2 + a^2)^{1/2}$ instead of $y(x^2 + b^2)^{1/2}$. Hansen [13] attributes (10.3)–(10.5) to F. Oberhettinger [17].

Observe that (10.3), (10.4) and (10.5) are Mittag-Leffler expansions in the complex variable b. Indeed one can prove the aforementioned formulas by applying the method indicated in §7.4, pages 134–135 in [23]. In fact one can do more. One can replace C_{2n} and C_{2n+1} in (10.3)–(10.5) by any symmetric polynomial, that is a polynomial $p_n(x)$ of degree n satisfying $p_n(-x) = (-1)^n p_n(x)$. The reason is that

$$(x^2+b^2)^{-(n+a)/2} J_{a+n}\left(y(x^2+b^2)^{-1/2}\right)$$

is an entire function of b and

$$(x^2+b^2)^{-n/2} p_n\left(x(x^2+b^2)^{-1/2}\right)$$

is a polynomial in b. Thus a proof of (10.3)–(10.5) via the Mittag-Leffler expansion theorem will work for similar formulas with the C_n's replaced by general p_n's. It is possible to further extend (10.3)–(10.5) by replacing $\sin(\pi b)$ and $\cos(\pi b/2)$ in the aforementioned formulas by a Bessel function $J_\mu(b)$ for $-1/2 \le \mu \le 1/2$. The details are left to the interested reader, as an exercise, or to the user who may need such identities.

Finally, some remarks about this work. One advantage of our approach is that it provides a systematic method to prove identities of the type discussed in this work. Using the same proofs one can further generalize some of the results in this work. For example this work can be generalized to generalized hypergeometric functions or to hypergeometric functions of two variables. We decided not to bother with generalizations of the latter type because the formulas become unwieldy, ugly, and possibly useless in this degree of generality. We also believe no new insight is gained by generalizations of this type. If such a complicated formula is needed for a specific application one can then derive it from the techniques of this work.

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