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ANTISTABLE CLASSES OF THIN SETS IN HARMONIC ANALYSIS

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Introduction

The motivation for this study is a property of the class $\mathscr{N}$ of all sets of absolute convergence (of a trigonometric series whose sum of coefficients is infinite): an increasing countable union of compact (or $\mathscr{K}_\sigma$) $\mathscr{N}$-sets is an $\mathscr{N}$-set (Host-Méla-Parreau [8]). Is it still true for any increasing countable union of $\mathscr{N}$-sets? This problem was posed by J. Arbault in [1]. It led me to study in general the operation of increasing countable union, and to a precise study of the class $\mathscr{N}$ and various related classes of thin sets. My Thèse de Doctorat [11], under the supervision of A. Louveau, contains some of the ideas developed in this paper.

Stability under finite union or countable union of classes of thin sets naturally introduced in harmonic analysis (e.g., sets of uniqueness [2] or Helson sets [18]) are classical problems (most of them are collected in the appendix of [17]). On the other hand, the stability of these classes under increasing countable union, to my knowledge, has never been studied. This paper can be considered as mixing harmonic analysis and descriptive set theory, in the same vein as the work done, in the study of sets of uniqueness, on σ-ideals and the operation of countable union [13]. But contrary to the operation of countable union, the operation of increasing countable union has no good descriptive properties [3]. In particular this operation is not idempotent, and ω_1 iterations are needed in general to obtain the closure of a class under this operation. The general study of the operation of increasing countable union and of related operations is done, from a combinatorial point of view, in [12].

The notion of a set of absolute convergence was introduced by P. Fatou in 1906 [7] and was successively studied by N. Lusin 1912 [19], V. V. Niemytzki 1926 [22], Marcinkiewicz 1938 [21], R. Salem 1941 [23], J. Arbault 1952 [1], J. E. Björk and R. Kaufman 1967 [18] and B. Host, J.-F. Méla and F. Parreau 1991 [8]. In the first section, we present the classical properties of the class $\mathscr{N}$

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of sets of absolute convergence and several classes linked to $\mathscr{N}$: it is a new presentation which uses the operation of increasing countable union to obtain known and new results.

In the second section, we show that the inclusions between classes proved in the first section are strict. In particular, we prove the existence of a set of resolution (even $\mathscr{K}_{\sigma\delta}$) which is not a set of absolute convergence (this should be compared with the fact that all $\mathscr{K}_\sigma$ sets of resolution are of absolute convergence). This last problem was posed by N. Bary more than thirty years ago.

In the third section, we study the properties of non stability of our classes of thin sets. Increasing countable union and its iterates are examples of Hausdorff operations. In [12] we have defined an order on Hausdorff operations which compares their respective power. Among the classes which lack some properties of stability under Hausdorff operations (e.g., increasing countable union or finite union), we have singled out the ones we call the *antistable* classes which have no stability property whatsoever (except those shared by all classes): a class is antistable only if the order of inclusion between its images under the Hausdorff operations is equivalent to the order of the Hausdorff operations. In fact we transform a negative property (not to be stable under finite or increasing countable union) in a positive property (to be antistable) which allows us to build many examples of sets. We prove that the class of sets of absolute convergence and most of the other classes considered in this paper are indeed antistable. This result, which can be considered as very negative by harmonic analysts and which may explain why so many related notions have been introduced to study $\mathscr{N}$, can also be viewed as a transfer theorem which gives a uniform way of building very complicated (or exotic) thin sets in harmonic analysis. In particular it allows us to solve the original problem of J. Arbault (and prove that ω_1 iterations are needed to obtain the closure of $\mathscr{N}$ under increasing countable union) and also allows us to present some similar results for other classes of thin sets in the fourth section (e.g., pseudo Dirichlet sets [2] or asymptotic H-sets [20]). In that final part, we introduce the notion of asymptotic Dirichlet sets, which provides examples of "large" weak Dirichlet sets.

1. Definitions and classical properties

1.1. Notations

Let ω (resp. ω_1) be the first infinite (resp. uncountable) ordinal. We also denoted by ω the set of positive integers. We will identify the set $\mathscr{P}(\omega)$ of subsets of ω with 2^ω via the map $A \mapsto 1_A$. The natural topology on 2^ω is the product topology for which it is a metrizable compact space.

Let $\mathbf{T}$ be the torus $\mathbf{R}/\mathbf{T}$ with its structure of compact topological group. For all $x \in \mathbf{T}$, let $\|x\|$ be the distance from x to 0. Note that

$$\|x\| \leq |\sin \pi x| \leq \pi \|x\|.$$

Every element x of $\mathbf{T}$ can be expressed in the form $x = \sum_{i \geq 1} \varepsilon_i(x) 2^{-i}$ with $\varepsilon_i(x)$ either 0 or 1 ($\varepsilon_i(x)$'s are set all equal to 0 for large enough i if x is rational). This defines an injective map from $\mathbf{T}$ to 2^ω which takes x to $(\varepsilon_i(x))_{i \geq 1}$. Most constructions in this article use this remark.

Let E be a Polish space (i.e., a complete metrizable separable space). For each countable ordinal $\alpha \geq 1$, let Σ^0_α (resp. Π^0_α, resp. Δ^0_α) be the Borel additive (resp. multiplicative, resp. ambiguous) class of rank α:

$$\Sigma^0_1 \equiv \text{open}, \quad \Sigma^0_2 \equiv \mathscr{F}_\sigma, \quad \Sigma^0_3 \equiv \mathscr{G}_{\delta\sigma}, \ldots$$
$$\Pi^0_1 \equiv \text{closed}, \quad \Pi^0_2 \equiv \mathscr{G}_\delta, \quad \Pi^0_3 \equiv \mathscr{F}_{\sigma\delta}, \ldots$$
$$\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha, \quad \Delta^0_1 \equiv \text{clopen}.$$

The class of all compact subsets of E is denoted by $\mathscr{K}(E)$. See [13] for more details about Borel classes.

1.2. Definitions of various classes

Here are the classes of thin sets we are interested in:

A subset X of $\mathbf{T}$ is a *set of absolute convergence* if there is a sequence $(a_n)_{n \in \omega}$ of nonnegative reals such that $\sum_{n \in \omega} a_n = +\infty$ and $\sum_{n \in \omega} a_n \|nx\| < +\infty$ for each $x \in X$. The class of all sets of absolute convergence is denoted by $\mathscr{N}$.

A subset X of $\mathbf{T}$ is called *Dirichlet* if there exists a strictly increasing sequence $(n_k)_{k \in \omega}$ of positive integers such that $\|n_k x\| \to 0$ uniformly in $x \in X$. The class of all compact Dirichlet sets is denoted by D.

A universally measurable subset X of $\mathbf{T}$ is called a *weak Dirichlet set* if

$$\forall \mu \in \mathscr{M}^+(\mathbf{T}), \quad \forall \varepsilon > 0, \quad \exists K \in D, \quad \mu(X \setminus K) < \varepsilon,$$

where $\mathscr{M}^+(\mathbf{T})$ is the set of all positive Borel measures on $\mathbf{T}$. The class of all weak Dirichlet sets is denoted by $\mathscr{W}\mathscr{D}$.

A class $\mathscr{C}$ of subsets of a set E is *hereditary* if, for all subsets A and B of E with $B \in \mathscr{C}$ and $A \subset B$, we have $A \in \mathscr{C}$. If E is a metrizable compact set, and $\mathscr{C}$ is a hereditary subclass of $\mathscr{K}(E)$, one defines the class $\mathscr{W}\mathscr{C}$ as the class of all universally measurable subsets X of E such that

$$\forall \mu \in \mathscr{M}^+(E), \quad \forall \varepsilon > 0, \quad \exists K \in \mathscr{C}, \quad \mu(X \setminus K) < \varepsilon,$$

where $\mathscr{M}^+(E)$ is the set of all positive Borel measures of E.

Many related notions have been introduced to study the class $\mathcal{N}$.

A subset X of $\mathbf{T}$ is a *set of type* $\mathcal{N}_0$ if there exists a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that $\sum_{k\in\omega}\|n_k x\| < +\infty$ for each $x \in X$.

A subset X of $\mathbf{T}$ is a *set of resolution* if there exist a sequence $(c_n)_{n\in\omega}$ of non negative reals such that $\limsup_{n\in\omega} c_n > 0$ and a sequence α_n of elements of $\mathbf{T}$ such that the series

$$\sum c_n \cos(2\pi nx - \alpha_n)$$

converges for each $x \in X$. The class of all sets of resolution is denoted by $\mathcal{R}$.

A subset X of $\mathbf{T}$ is an *Arbault set* if there exists a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that $\|n_k x\| \to 0$ for each $x \in X$. The class of Arbault sets is denoted by $\mathcal{A}$.

We introduce a last series of classes which are related to the other definitions.

A compact subset K of $\mathbf{T}$ is a set of *type H* if there exist a non empty interval I of $\mathbf{T}$ and a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that $n_k K \cap I = \emptyset$ *for each* $k \in \omega$.

A compact subset K of $\mathbf{T}$ is a *set of type L* or a *lacunary set* if there exist a sequence $\varepsilon_n \to 0^+$, a sequence $\alpha_n \to +\infty$ and for each $n \in \omega$ a finite sequence (I_k) of intervals such that $|I_k| \le \varepsilon_n$ for each k, $d(I_k, I_{k'}) \ge \alpha_n \varepsilon_n$ for each $k \ne k'$ and $K \subseteq \bigcup I_k$.

A compact subset K of $\mathbf{T}$ *is a set of type* L_0 if there exist a sequence $\varepsilon_n \to 0^+$, $\alpha > 0$ and for each $n \in \omega$ a finite sequence (I_k) of intervals such that $|I_k| \le \varepsilon_n$ for each k, $d(I_k, I_{k'}) \ge \alpha\varepsilon_n$ for each $k \ne k'$ and $K \subseteq \bigcup I_k$.

As a matter of notation, we denote the classes of compact sets by capital letters and the corresponding classes of general sets by the corresponding calligraphic letters. Thus for example, $N = \mathcal{N} \cap \mathcal{K}(\mathbf{T})$, $N_0 = \mathcal{N}_0 \cap \mathcal{K}(\mathbf{T})$, etc.

We finish this part with a notation which will be used throughout the paper: if $\mathcal{C}$ is a class of subsets of a set E, we denote by $\mathcal{C}^{\uparrow}$ the hereditary class of subsets of E consisting of those sets which can be covered by the union of some increasing sequence of elements of $\mathcal{C}$.

1.3. Properties of sets of absolute convergence and Dirichlet sets

The set $\mathcal{M}_\infty^+(\omega)$ consists of all elements of the form $\Theta = \sum_{n\in\omega} a_n\delta_n$, where $(a_n)_{n\in\omega}$ is a sequence of non negative reals with $\sum_{n\in\omega} a_n = +\infty$ and δ_n is the Dirac measure at the point $n \in \omega$. Let $\Theta(f) = \sum_{n\in\omega} a_n f(n)$ for all sequences f. Let $\Theta(I) = \Theta(1_I) = \sum_{n\in I} a_n$ for all subset I of ω.

Let f_x be the sequence $(\|nx\|)_{n\in\omega}$ for each $x \in \mathbf{T}$. Fix a $\Theta \in \mathcal{M}_\infty^+(\omega)$. So

$$\Theta(f_x) = \sum_{n\in\omega} a_n f_x(n) = \sum_{n\in\omega} a_n \|nx\|.$$

Let $G_\Theta = \{x \in \mathbf{T}; \Theta(f_x) < +\infty\}$, which is clearly in $\mathcal{N}$.

Moreover $\mathcal{N}$ is the hereditary closure of the G_Θ's where Θ varies over $\mathcal{M}_\infty^+(\omega)$. As G_Θ is a $\mathcal{K}_\sigma$ subgroup of $\mathbf{T}$, it follows that *$\mathcal{N}$ is the hereditary class generated by its $\mathcal{K}_\sigma$ elements and $\mathcal{N}$ is closed under the operation of generated subgroup*.

It can be deduced from a theorem of Dirichlet that finite sets are Dirichlet. We are going to give classical examples of uncountable Dirichlet sets [18]. A subset A of ω is called *colacunary* if it contains segments of consecutive integers of unbounded length. It is easy to see that for a colacunary subset A of ω,

$$K_A = \{x \in \mathbf{T}; \forall i \in A, \varepsilon_i(x) = 0\}$$

belongs to D [18].

We can immediately deduce the Marcinkiewicz Theorem [21] which states that *there exists two Dirichlet sets whose union is not an $\mathcal{N}$-set*. Indeed, if A and B are two disjoint colacunary subsets of ω then $K_A \cup K_B \notin \mathcal{N}$ because $K_A + K_B = \mathbf{T}$ and $\mathcal{N}$ is closed under the operation of generated subgroup.

If $\mathcal{C}$ is a hereditary class of compact subsets of some metrizable compact space, note that $\mathcal{W}\mathcal{C}$ is closed under the operation of increasing countable union (in Proposition 3.4, we give a stronger property of $\mathcal{W}\mathcal{C}$) and that each measure concentrated on a $\mathcal{W}\mathcal{C}$-set is in fact concentrated on a $\mathcal{C}^\uparrow$-set.

A measure concentrated on a $\mathcal{W}\mathcal{D}$-set is called a *Dirichlet measure*. For a positive measure $\mu \in \mathcal{M}(\mathbf{T})$, the following conditions are equivalent:

(1) *μ is concentrated on a $D^\uparrow$-set*
(2) $\limsup_{|n|\to\infty} |\hat{\mu}(n)| = \int d\mu$ *(where $\hat{\mu}(n) = \int e^{2\pi i n x}\, d\mu(x)$)*
(3) $\liminf_{|n|\to\infty} \int \|nx\|\, d\mu(x) = 0$.

Note that a subset X of $\mathbf{T}$ is a $\mathcal{W}\mathcal{D}$-set if and only if each measure concentrated on X is a Dirichlet measure.

1.4. Relations between the classes $\mathcal{N}$ and $\mathcal{W}\mathcal{D}$

The most important result is the following.

THEOREM 1.5. (1) *$\mathcal{N}$ is a subset of $\mathcal{W}\mathcal{D}$.*

(2) *For each increasing sequence $(K_n)_{n\in\omega}$ of compact weak Dirichlet sets, $\bigcup_{n\in\Omega} K_n$ belongs to $\mathcal{N}$.*

R. Salem introduced Dirichlet measures and proved (1) for compact sets [23]. The converse was proved by J. E. Björk and R. Kaufman independently [17]. B. Host, J.-F. Méla and F. Parreau stated the theorem in the present form [8] and noticed that most of the classical facts about $\mathcal{N}$ and $\mathcal{W}\mathcal{D}$-sets can be easily deduced from it.

COROLLARY 1.6. (1) *$\mathcal{N} \cap \mathcal{K}_\sigma(\mathbf{T}) = \mathcal{WD} \cap \mathcal{K}_\sigma(\mathbf{T})$.*

(2) *$N = \mathcal{N} \cap \mathcal{K}(\mathbf{T})$ is closed under compact increasing union.*

(3) *$\mathcal{N} = N^{\uparrow}$.*

(4) *$\mathcal{WD} = N^{\text{int}}$, where N^{int} is the interior extension of N, i.e., the class of universally measurable sets all of whose compact subsets belong to N.*

(5) *$\mathcal{N}$ is closed under translations.*

(6) *$\mathcal{N}$ is equal to the class of all sets of absolute convergence of a trigonometrical series*

$$\sum a_n \cos(2\pi nx - \alpha_n) \quad \textit{with } \sum |a_n| = +\infty.$$

(7) *For each $X \in \mathcal{N}$ and Y a countable set, $X \cup Y \in \mathcal{N}$.*

In his thesis, J. Arbault defined a $\mathcal{K}_{\sigma\delta}$ set in $\mathcal{WD}$ which is not an $\mathcal{N}$-set [1]. In particular, (1) cannot be improved. Prior to the Björk-Kaufman result, he also asked the following question.

Question. Is the class $\mathcal{N}$ closed under increasing countable union?

We will answer negatively to this question in part 3.18. In view of result (2), this is a curious phenomenon.

Proof of the corollary. Since $\mathcal{N}$ is the hereditary class generated by its $\mathcal{K}_\sigma$ elements and each $\mathcal{K}_\sigma$ set is an increasing union of compact sets, the five first propositions are straightforward consequences of the previous theorem. Since D is closed under translations, $\mathcal{WD}$ is too, thus (5) holds.

In order to prove (6), let

$$E = \left\{x \in \mathbf{T};\ \sum |a_n \cos(2\pi nx - \alpha_n)| < \infty\right\}.$$

Since

$$\begin{aligned} \|2n(x - x_0)\| &\le |\sin 2\pi n(x - x_0)| \\ &\le |\cos(2\pi nx - \alpha_n)| + |\cos(2\pi nx_0 - \alpha_n)| \end{aligned}$$

for all $(x, x_0) \in \mathbf{T}^2$, $E - x_0 \in \mathcal{N}$ if $x_0 \in E$. Using (5), we obtain $E \in \mathcal{N}$. Conversely,

$$\|nx\| \ge \tfrac{1}{2}\|2nx\| \ge \frac{1}{2\pi}|\sin 2\pi nx|$$

for all n, x; thus every $\mathcal{N}$-set is a subset of a set of the form

$$\left\{x \in \mathbf{T};\ \sum |a_n \sin 2\pi nx| < +\infty\right\},$$

with $\Sigma |a_n| = +\infty$.

To prove (7), fix $X \in \mathscr{N}$ and $x_0 \in \mathbf{T}$. Using (5), $X - x_0 \in \mathscr{N}$; thus

$$(X - x_0) \cup \{0\} \in \mathscr{N}$$

and

$$X \cup \{x_0\} = ((X - x_0) \cup \{0\}) + x_0 \in \mathscr{N}.$$

Let $Y = \{x_n;\ n \in \omega\}$, a countable set. As $\mathscr{N} = N^{\uparrow}$, there exists an increasing sequence $(K_n)_{n \in \omega}$ of compact $\mathscr{N}$-sets such that $X \subseteq \bigcup_{n \in \omega} K_n$. Therefore

$$X \cup Y \subseteq \bigcup_{n \in \omega} (K_n \cup \{x_0, \ldots, x_n\}) \in N^{\uparrow} = \mathscr{N}. \quad \square$$

The following property of the class $\mathscr{W}\mathscr{D}$ is due to G. Debs (and published here for the first time).

PROPOSITION 1.7. *If X is an analytic $\mathscr{W}\mathscr{D}$-set, then the group generated by X is also a $\mathscr{W}\mathscr{D}$-set.*

Proof. Define $\Phi : \mathscr{P}(\mathbf{T}) \to \mathscr{P}(\mathbf{T})$ by $\Phi(X) = X - X = \{x - y;\ x, y \in X\}$. The group generated by X is the increasing union over $n \in \omega$ of $\Phi^{(n)}(X)$. So it is enough to prove that if X is an analytic $\mathscr{W}\mathscr{D}$-set, then $X - X$ is also a $\mathscr{W}\mathscr{D}$-set.

Let μ be a positive measure in $\mathscr{M}(\mathbf{T})$ and μ^* be the corresponding outer measure. Define $C : \mathscr{P}(T) \to \mathbf{R}^+$ by $C(X) = \mu^*(X - X)$. One easily checks that C is a capacity. By Choquet's capacitability theorem, for every analytic subset of $\mathbf{T}$,

$$\mu(X - X) = \sup_{K \in \mathscr{K}(X)} \mu(K - K).$$

Thus if μ is concentrated on $X - X$, then for all $\varepsilon > 0$ there exists a $K \in \mathscr{K}(X)$ such that μ is concentrated on $K - K$ within ε. But $K \in N$, thus $K - K \in N$. Therefore μ is a Dirichlet measure. $\square$

1.8. Other classes between D and $\mathscr{W}\mathscr{D}$

R. Salem introduced the class $\mathscr{N}_0$ in order to simplify the definition of the class $\mathscr{N}$. J. Arbault proved that the two classes are distinct and he introduced the class $\mathscr{A}$ in order to prove this fact [1]. The class $\mathscr{R}$ was considered by N. Bary [2].

PROPOSITION 1.9. *We have the following inclusions.*

$$D^{\uparrow} \subseteq \mathcal{N}_0 \subseteq \mathcal{R} \subseteq \mathcal{A} \subseteq \mathcal{WD}.$$

Proof. In order to prove that $\mathcal{R} \subseteq \mathcal{A}$, let $X = \{x \in \mathbf{T};\ \sum c_n \cos(2\pi nx - \alpha_n)$ converges$\}$ where $\limsup_{n\in\omega} c_n > 0$. Thus there exists $\varepsilon > 0$ and a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that $c_{n_k} > \varepsilon$. Fix an $x \in X$. Since $\cos(2\pi n_k x - \alpha_{n_k}) \to 0$, $e^{4i(2\pi n_k x - \alpha_{n_k})} \to 1$. Without loss of generality, we can assume that $n_{k+1} - n_k \to +\infty$ and $\alpha_{n_k} \to \alpha$ by compactness of $\mathbf{T}$. Thus $e^{8\pi i(n_{k+1}-n_k)x} \to 1$; hence $\|8(n_{k+1} - n_k)x\| \to 0$ and $X \in \mathcal{A}$.

Inclusion of $D^{\uparrow}$ in $\mathcal{N}_0$ is an immediate consequence of the following characterization of the $D^{\uparrow}$-sets. The other inclusions are trivial (see [1] or [2]). □

PROPOSITION 1.10. *Let $X \in \mathcal{P}(\mathbf{T})$. The following statements are equivalent.*

(1) $X \in D^{\uparrow}$.

(2) *There exists a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers and a sequence $\varepsilon_k \to 0^+$ such that*

$$X \subseteq \bigcup_{i\in\omega} \bigcap_{k\geq i} \{x \in \mathbf{T};\ \|n_k x\| \leq \varepsilon_k\}.$$

(3) *There exists a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that*

$$X \subseteq \bigcup_{i\in\omega} \bigcap_{k\geq i} \{x \in \mathbf{T};\ \|n_k x\| \leq 2^{-k}\}.$$

The proof is easy. The property (2) is studied and generalized in Section 4. It was introduced by N. Bary in order to give examples of $\mathcal{N}_0$-sets [2].

The connection between $\mathcal{A}$ and H was noted by Rajchman and can be formulated in the following way: $\mathcal{A} \subseteq H^{\uparrow}$.

Indeed, if $(n_k)_{k\in\omega}$ is a strictly increasing sequence of integers, then we have

$$\{x \in \mathbf{T};\ \|n_k x\| \to 0\} \subset \bigcup_{j\in\omega} \bigcap_{k\geq j} \{x \in \mathbf{T};\ n_k x \in]-\tfrac{1}{4}, \tfrac{1}{4}[\}.$$

Note that both H and L are supersets of D and subsets of L_0.

To finish, let us indicate the descriptive complexity of the previous classes of compact sets. In the space $\mathcal{K}(\mathbf{T})$ of compact subsets of $\mathbf{T}$ (which is a metrizable compact space), the classes D, L and N are $\mathcal{G}_\delta$ subsets [10], H and L_0 are $\mathcal{K}_{\sigma\delta}$ subsets, but N_0, R and A are not Borel sets (they are in fact Σ^1_2 or PCA sets, but not better [3]).

2. Noninclusions between classes

2.1. Introduction

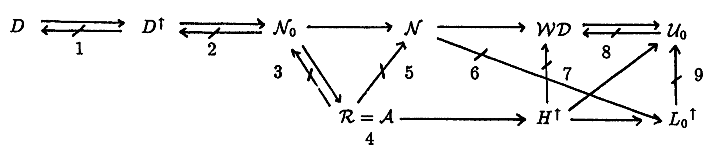
In this section, we will study the converse inclusions between the classes introduced in the first section. The problem each time is to define a set (of the smallest Borel rank possible) of a certain class $\mathscr{C}$ and which is not covered by a countable union of sets in another class $\mathscr{B}$. The schema of the proof is the following. Suppose that such a set X of type $\mathscr{C}$ is proposed; for each sequence $(X_n)_{n \in \omega}$ of sets in $\mathscr{B}$, we will find an $x \in X$ which does not belong to $\bigcup X_n$. We can view x as being a member of 2^ω by using its dyadic decomposition $x = \sum_{i \leq 1} \varepsilon_i 2^{-i}$. We will need a lemma concerning the class $\mathscr{B}$: imposing a limited number of values of ε_i, we can assume that x does not belong to a certain set of type $\mathscr{B}$. Of course, the necessary number of values increases with the size of the class $\mathscr{B}$.

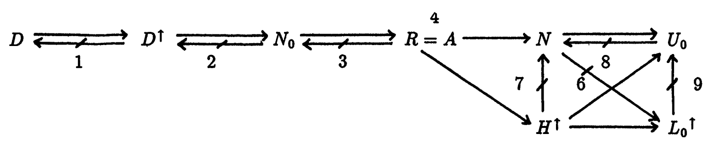
For each class $\mathscr{C}$ of subsets of a set E, let $\mathscr{C}^\sigma$ be the class of all subsets of E which are covered by a countable union of sets in $\mathscr{C}$.

The diagrams show the relationships between the classes and between the corresponding classes of compact sets. An arrow indicates an inclusion and a crossed arrow a non-inclusion.

In the preprint version of this paper, the question whether $\mathscr{R} = \mathscr{A}$ was asked. S. Konyagin has recently proved that $\mathscr{R} = \mathscr{A}$ [15]. Using the result of S. Konyagin, the proofs of Theorems 2.6 and 2.11 can be simplified.

(1) Finite sets are Dirichlet sets, so countable sets are $D^\uparrow$-sets. But there exists a countable compact set which is not a Dirichlet set, for example $\{0\} \cup \{2^{-n}; n \geq 1\}$.





(2) There exists a compact $\mathscr{N}_0$-set which is not a D^σ-set and therefore is not a $D^\uparrow$-set; see Theorem 2.3.

(3) T. W. Körner proved the existence of a compact $\mathscr{R}$-set which is not an $\mathscr{N}_0$-set [16]. We will extend his result in Theorem 2.6.

(4) In Proposition 1.9 we proved the inclusion of $\mathscr{R}$ in $\mathscr{A}$. The converse inclusion was recently proved by S. Konyagin [15].

(5) There exists a compact $\mathscr{R}$-set which is not an $\mathscr{N}^\sigma$-set. This is Theorem 2.11.

(6) J. Arbault proved the existence of an $\mathscr{N}$-set which is not an $\mathscr{A}$-set [1]. We will prove the existence of a compact $\mathscr{N}$-set which is not an L_0^σ set in Theorem 2.8.

(7) J. Arbault proved the existence of an H-set which is not a $\mathscr{W}\mathscr{D}^\sigma$-set [1]. Indeed, the triadic Cantor set K is an H-set and the standard measure on K is not a Dirichlet measure. Moreover each open subset of K has a subset which is homeomorphic to K, so, by the Baire category theorem, K cannot be covered by a countable union of $\mathscr{W}\mathscr{D}$-sets.

(8) $\mathscr{U}_0$ is the class of all sets of extended uniqueness. Since $\mathscr{U}_0$ is closed under finite union [2], $\mathscr{U}_0 \neq \mathscr{W}\mathscr{D}$.

(9) R. Kaufman proved the existence of a lacunary set which is not a $\mathscr{U}_0$-set [14].

2.2. A compact $\mathscr{N}_0$-set which is not a D^σ-set

Recall that there exists a countable compact set which is not a D-set.

THEOREM 2.3. *There exists a compact $\mathscr{N}_0$-set which is not a D^σ-set.*
More precisely, if $(\alpha_k)_{k\in\omega}$ is a sequence of integers such that $\lim_{k\to\infty}(\alpha_{k+1}-\alpha_k) = +\infty$, then the set $K = \{x \in \mathbf{T};\ \sum_{k\in\omega}\|2^{\alpha_k}x\| \le 1\}$ is in $N_0 \setminus D^\sigma$.

In the proof of the theorem, we use the following basic lemma.

LEMMA 2.4. *Let $n, m \in \omega$, $m \ge 2$, $p = \lfloor \log_2 n \rfloor$ and $\varepsilon_1, \dots, \varepsilon_{p+m} \in \{0,1\}$. There exist $\varepsilon_{p+m+1}, \varepsilon_{p+m+2}, \varepsilon_{p+m+3} \in \{0,1\}$ such that for each $x \in \mathbf{T}$,*

$$(\forall i \le p+m+3, \varepsilon_i(x) = \varepsilon_i) \Rightarrow (\|nx\| \ge 2^{-m-2}).$$

Proof. Let $S = \sum_{1\le i\le p+m}\varepsilon_i 2^{-i}$. If $\|nS\| \ge 2^{-m-1}$, then let $\varepsilon_{p+m+1} = \varepsilon_{p+m+2} = \varepsilon_{p+m+3} = 0$. Otherwise let $\varepsilon_{p+m+1} = \varepsilon_{p+m+2} = 1$. □

Proof of Theorem 2.3. It is clear that K is a compact $\mathscr{N}_0$-set. Let $(K_j)_{j\in\omega}$ be a sequence of D-sets. We will find x in $K \setminus \bigcup_{j\in\omega}K_j$ in its dyadic form $x = \sum_{i\ge 1}\varepsilon_i 2^{-i}$ for a suitable choice of ε_i's, either 0 or 1.

Clearly, it is enough to find, by induction on j, an integer k_j and $\varepsilon_i \in \{0,1\}$ for each $\alpha_{k_{j-1}} < i \leq \alpha_{k_j}$ such that:

(1) For each $x \in \mathbf{T}$, if $\varepsilon_i(x) = \varepsilon_i$ for all $i \leq \alpha_{k_j}$ then $\sum_{0 \leq k \leq k_j} \|2^{\alpha_k} x\| \leq 1 - 2^{-j}$ and $x \notin K_i$ for all $i \leq j$.

(2) For all $k \geq k_j$, $\alpha_{k+1} - \alpha_k \geq 2j + 8$.

Let k_{j-1} and ε_i ($i \leq \alpha_{k_{j-1}}$) be given. Since $K_j \in D$, there exists $n_j > 2^{\alpha_{k_j}}$ such that $\|n_j x\| \leq 2^{-j-7}$ for every $x \in K_j$. Let $p_j = \lfloor \log_2 n_j \rfloor$.

Let k_j be the least k such that $\alpha_k \geq p_j + 5$ and $\alpha_{k'+1} - \alpha_{k'} \geq 2j + 8$ for all $k' \geq k$.

Using the previous lemma, we may impose three consecutive values of ε_i (where $p_j + 3 \leq i \leq p_j + j + 8$) to insure $\|n_j x\| \geq 2^{-j-7}$ so that $x \notin K_j$. The other values of ε_i (where $\alpha_{k_{j-1}} < i \leq \alpha_{k_j}$) are set equal to 0.

We consider two cases.

First case. If $\alpha_{k_j - 1} < p_j - j + 2$, we impose ε_i for $i = p_j + 3, p_j + 4, p_j + 5$. So we have

$$\sum_{k_{j-1} \leq k < k_j} \|2^{\alpha_k} x\| \leq \sum_{k_{j-1} \leq k < k_j} 2^{\alpha_k - p_j - 2} \leq 2^{-j}.$$

Second Case. If $p_j - j + 2 \leq \alpha_{k_{j-1}} < p_j + 5$, we impose ε_i for $i = p_j + j + 6, p_j + j + 7, p_j + j + 8$. Note that this is possible because $\alpha_{k_j} \geq \alpha_{k_{j-1}} + 2(j-1) + 8 \geq p_j - j + 2 + 2j + 6 = p_j + j + 8$. So we have

$$\sum_{k_{j-1} \leq k < k_j} \|2^{\alpha_k} x\| \leq \sum_{k_{j-1} \leq k < k_j} 2^{\alpha_k - p_j - 5 - j} \leq 2^{-j}.$$

In both cases we are done. □

2.5. A compact $\mathscr{R}$-set which is not an $\mathscr{N}_0^\sigma$-set

T. W. Körner proved the existence of a compact $\mathscr{R}$-set which is not an $\mathscr{N}_0$-set [16]. Using a different method, we extend his result and give an explicit example.

THEOREM 2.6. *There exists a compact $\mathscr{R}$-set which is not a $\mathscr{N}_0^\sigma$-set.*

More precisely, if $(\alpha_k)_{k \in \omega}$ is a sequence of integers such that $\alpha_{k+1} - \alpha_k \geq k + 9$ for all $k \in \omega$, then the set

$$K = \left\{ x \in \mathbf{T};\ \sum_{k \in \omega} \|2^{\alpha_k} x\|^2 \leq 1 \text{ and } (*)_k \text{ for each } k \in \omega \right\}$$

is in $R \setminus \mathcal{N}_0^\sigma$, *where for each* $k \in \omega$,

$$(*)_k\colon \left(\sum_{i<k} \sin \pi 2^{\alpha_i} x\right) \cdot \sin \pi 2^{\alpha_k} x \leq 0 \textit{ or } \left|\sum_{i<k} \sin \pi 2^{\alpha_i} x\right| \leq 2^{-k}.$$

J. Arbault was the first to consider $\mathcal{N}_0^2$-sets, i.e., subsets of a set of the form

$$\left\{x \in \mathbf{T};\ \sum_{k\in\omega} \|n_k x\|^2 < +\infty\right\}.$$

He proved that $\mathcal{N}_0^2 \not\subset \mathcal{N}_0$ [1]. By straightforward means, we have $\mathcal{N}_0^2 \subset \mathcal{A}$. In order to obtain an $\mathcal{R}$-set and to obtain a compact set, we must add conditions $(*)_k$ which cause extra complications in the definition of K.

Proof of Theorem 2.6. Since all maps considered in the definition of K are continuous or lower semi-continuous, K is compact.

To see that $K \in \mathcal{R}$, fix $x \in K$. We have $u_k = \sin \pi 2^{\alpha_k} x \to 0$ and $(*)_k$ for every k. This allows to prove simply that $\Sigma_{k\in\omega} \sin \pi 2^{\alpha_k} x$ converge. Thus $K \in \mathcal{R}$.

Now we prove that $K \notin \mathcal{N}_0^\sigma$. Let $(\Theta_q)_{q\in\omega}$ be any sequence of elements of $\mathcal{M}_\infty^+(\omega)$ where $\Theta_q = \Sigma_{l\in\omega}\delta_{n_l^q}$ (see part 1.3 for the notations). We will find x in $K \setminus \bigcup_{q\in\omega} G_{\Theta_q}$ in its dyadic decomposition $x = \Sigma_{i\geq 1}\varepsilon_i 2^{-i}$ for a suitable choice of ε_i either 0 or 1.

Note that the condition on the α_k implies that $2^6\Sigma_{k\in\omega}4^{\alpha_k-\alpha_{k+1}} \leq \frac{1}{2}$. Consider a surjection $f\colon \omega \to \omega$ whose fibers are infinite. We will get, by induction on j, an integer k_j and $\varepsilon_i \in \{0, 1\}$ for $\alpha_{k_{j-1}} < i \leq \alpha_{k_j}$ such that for all $x \in \mathbf{T}$ with $\varepsilon_i(x) = \varepsilon_i$ for all $i \leq \alpha_{k_j}$, we have:

(1) $\Sigma_{k_{j-1}<k\leq k_j}\|2^{\alpha_k}x\|^2 \leq 2^{-j} + \Sigma_{k_{j-1}<k\leq k_j}4^{\alpha_k-\alpha_{k+1}}$
(2) $\Theta_q(1_{[2^{\alpha_{k_{j-1}}}, 2^{\alpha_{k_j}}[}\cdot f_x) \geq 1$ for $q = f(j)$
(3) $(*)_k$ for all $k_{j-1} < k \leq k_j$.

Clearly, this is enough to finish the proof: since $f^{-1}\{q\}$ is infinite, $\Theta_q(f_x) = +\infty$ for all $q \in \omega$, thus $x \notin \bigcup_{q\in\omega} G_{\Theta_q}$. Moreover,

$$\sum_{k\in\omega} \|2^{\alpha_k}x\|^2 \leq \sum_{j\geq 2} 2^{-j} + 2^6 \sum_{k\in\omega} 4^{\alpha_k-\alpha_{k+1}} \leq 1;$$

therefore $x \in K$.

Let k_j and ε_i (for $i \leq \alpha_{k_j}$) be given. Let $q = f(j)$ and $p_l = \log_2 n_l^q$ for each $l \geq 2^{\alpha_{k_j}}$.

We consider two cases for $k > k_j$.

First case. There exists l_k with $\alpha_k - 3 < p_{l_k} \le \alpha_{k+1} - 3$. Using Lemma 2.4, we may impose three consecutive values of ε_i where $p_{l_k} + m_k + 1 \le i \le p_{l_k} + m_k + 3$ to get the condition $\|n^q_{l_k} x\| \ge 2^{-m_k-2}$, where

$$m_k = \inf\{j + 9, \alpha_{k+1} - p_{l_k} - 3\}.$$

Note that $m_k \ge 0$, $p_{l_k} + m_k + 3 \le \alpha_{k+1}$ and $\|n^q_{l_k} x\| \ge 2^{-j-11}$.

The other values of ε_i for $\alpha_k - 3 < i \le \alpha_{k+1} - 3$ are set equal to ζ_k, where $\zeta_k = 0$ or 1 and the determination of ζ_k depends on condition $(*)_k$.

Because the values of ε_i for $\alpha_k + 1 \le i \le p_{l_k} + m_k$ are all equal and $\alpha_{k+1} - \alpha_k \le k + 9 \le j + 9$, we have

$$\|2^{\alpha_k} x\| \le 2^{\alpha_k - p_{l_k} - m_k} \le \sup\{2^{3-(j+9)}, 2^{\alpha_k - \alpha_{k+1} + 3}\} \le 2^{-j-6}.$$

Second case. Otherwise all values of ε_i for $\alpha_k - 3 < i \le \alpha_{k+1} - 3$ are set equal to ζ_k where $\zeta_k = 0$ or 1. Thus $\|2^{\alpha_k} x\| \le 2^{\alpha_k - \alpha_{k+1} + 3}$.

Let k_{j+1} be the least value for which the set

$$\{k \in [k_j, k_{j+1}[;\ \exists l, \alpha_k - 3 < p_l \le \alpha_{k+1} - 3\}$$

has cardinality 2^{j+11}. Clearly,

$$\Theta_q(1_{]2^{\alpha_{k_j}}, 2^{\alpha_{k_{j+1}}}]} \cdot f_x) \ge \sum_{k_j < k \le k_{j+1}} \|n^q_{l_k} x\| \ge 1.$$

On the other hand, we have

$$\sum_{k_j < k \le k_{j+1}} \|2^{\alpha_k} x\|^2 \le 2^{j+11}(2^{-j-6})^2 + \sum_{k_j < k \le k_{j+1}} (2^{\alpha_k - \alpha_{k+1} + 3})^2$$
$$\le 2^{-j-1} + \sum_{k_j < k \le k_{j+1}} 4^{\alpha_k - \alpha_{k+1}}.$$

The remaining task is to choose ζ_k for each $k_j < k \le k_{j+1}$ to insure $(*)_k$. Let $S_k = \sum_{i \le \alpha_k - 3} \varepsilon_i 2^{-i}$. We have $|x - S_k| \le 2^{-\alpha_k + 3}$, thus

$$\left|\sum_{i<k} \sin \pi 2^{\alpha_i} x - \sum_{i<k} \sin \pi 2^{\alpha_i} S_k\right| \le \pi \sum_{i<k} 2^{\alpha_i} |x - S_k| \le 2^{-k-1}.$$

Let $\zeta_k = 1$ if $\sum_{i<k} \sin \pi 2^{\alpha_i} S_k > 0$ and otherwise $\zeta_k = 0$. Thus, if $|\sum_{i<k} \sin \pi 2^{\alpha_i} S_k| > 2^{-k-1}$, then $\sum_{i<k} \sin \pi 2^{\alpha_i} x$ has same sign as $\sum_{i<k} \sin \pi 2^{\alpha_i} S_k$ and therefore an opposite sign to $\sin \pi 2^{\alpha_k} x$. Conversely, if $|\sum_{i<k} \sin \pi 2^{\alpha_i} S_k| \le 2^{-k-1}$, then $|\sum_{i<k} \sin \pi 2^{\alpha_i} x| \le 2^{-k-1}$. □

2.7. A compact $\mathcal{N}$-set which is not an L_0^σ-set

J. Arbault proved the existence of an $\mathcal{N}$-set which is not an $\mathcal{A}$-set [1], [2]. We derive an even stronger (yet not more complicated) version of his result.

THEOREM 2.8. *There exists a compact $\mathcal{N}$-set which is not an L_0^σ-set. More precisely,*

$$K = \left\{ x \in \mathbf{T};\ \sum_{n \in \omega} \frac{1}{n} \|2^n x\| \le 1 \right\}$$

is in $N \setminus L_0^\sigma$.

S. Konyagin proved a best result: there exists a compact $\mathcal{N}$-set which is not a σ-porous set ([24] Theorem 5.1) and the L_0-sets are clearly porous. But the present proof is more elementary and the following lemma is fundamental in the next section.

LEMMA 2.9. *Let $\varepsilon \in]0, \frac{1}{8}[$, $m = -\lfloor \log_2 \varepsilon \rfloor$, $\alpha > 0$, $p = \sup\{-\lfloor \log_2 \alpha \rfloor, 0\}$, (I_k) a finite sequence of intervals such that $|I_k| \le \varepsilon$ for each k and $d(I_k, I_{k'}) \ge \alpha\varepsilon$ for each $k \ne k'$. Let $\varepsilon_1, \dots, \varepsilon_{m-2} \in \{0,1\}$.*
There exist $\varepsilon_{m-1}, \varepsilon_m, \dots, \varepsilon_{m+p+1} \in \{0,1\}$ such that for each $x \in \mathbf{T}$,

$$\left(\forall i \le m+p+1, \varepsilon_i(x) = \varepsilon_i\right) \Rightarrow \left(x \notin \bigcup I_k\right).$$

Proof. Let

$$S = \sum_{1 \le i \le m-2} \varepsilon_i 2^{-i} \quad \text{and} \quad A(\tau) = \sum_{m-1 \le i \le m+p+1} \varepsilon_i 2^{-i}$$

where τ is a suitable choice of $(\varepsilon_i)_{m-1 \le i \le m+p+1}$. Let $x \in \mathbf{T}$ with $x = S + A(\tau) + R(x)$, where $R(x) = \sum_{i > m+p+1} \varepsilon_i(x) 2^{-i}$. Observe that $R(x)$ belongs to the interval $[0, 2^{-m-p-1}[$ of $\mathbf{T}$. We must choose τ. Recall that $2^{-m} \le \varepsilon < 2^{-m+1}$ and $2^{-p} \le \alpha$, therefore $\alpha\varepsilon \ge 2^{-m-p}$. Put $\overline{I_k} = [a_k, b_k]$ for each k. Consider two cases.

First case. There exists k such that

$$S \in \left[a_k - 2^{-m-p-1}, b_k + 2^{-m-p-1}\right];$$

then there exists $\tau \in \{0,1\}^{p+2}$ such that

$$S + A(\tau) \in]b_k, b_k + 2^{-m-p-1}],$$

because the possible values of $A(\tau)$ are 2^{-m-p-1} apart and between 0 and $\sum_{m-1\leq i\leq m+p+1}2^{-i} = 2^{-m+2} - 2^{-m-p-1}$, and because $b_k - a_k < 2^{-m+1}$. But $d(I_k, I_{k'}) \geq 2^{-m-p}$ for each $k' \neq k$; thus $x \notin \bigcup I_k$.

Second case. If $S \notin \bigcup[a_k - 2^{-m-p-1}, b_k + 2^{-m-p-1}]$, then let $\tau = 0$, i.e., $A(\tau) = 0$. □

Proof of Theorem 2.8. It is clear that K is a compact $\mathscr{N}$-set. Let $(K_j)_{j\in\omega}$ be a sequence of L_0-sets. We will find x in $K \setminus \bigcup_{j\in\omega}K_j$ in its dyadic form $x = \sum_{i\geq 1}\varepsilon_i 2^{-i}$ for a suitable choice of ε_i, either 0 or 1.

Let $j \in \omega$. Since $K_j \in L_0$, there exist $\varepsilon_j > 0$, $\alpha_j > 0$ and a finite sequence (I_k) of intervals such that $|I_k| \leq \varepsilon_j$ for each k, $d(I_k, I_{k'}) \geq \alpha_j\varepsilon_j$ for each $k \neq k'$ and $K_j \subset \bigcup I_k$. Let $m_j = -\lfloor\log_2 \varepsilon_j\rfloor$ and $p_j = \sup\{-\lfloor\log_2 \alpha_j\rfloor, 0\}$. Using the previous lemma, it is enough to impose values of ε_i for all $i \in [m_j - 1, m_j + p_j + 1]$ to insure $x \notin K_j$. The values of ε_i are set equal to 0 for all $i \notin \bigcup_{j\in\omega}[m_j - 1, m_j + p_j + 1]$. Since ε_j can be chosen as small as desired (see definition of L_0), m_j can be chosen as large as desired. Thus we set $(m_j)_{j\in\omega}$ such that $m_{j+1} \geq m_j + p_j + 2$ for each $j \in \omega$ and

$$\sum_{j\in\omega}\frac{p_j + 5}{2(m_j - 2)} \leq 1.$$

The values of ε_i for every $i \in \bigcup_{j\in\omega}[m_j - 1, m_j + p_j + 1]$ are imposed to insure $x = \sum_{i\geq 1}\varepsilon_i 2^{-i}$ is not in $\bigcup_{j\in\omega}K_j$.

The remaining task is to prove that $x \in K$. But for each $j \in \omega$, we have

$$\sum_{n=m_j-2}^{m_j+p_j}\frac{1}{n}\|2^n x\| \leq \frac{p_j + 3}{2(m_j - 2)}$$

and

$$\sum_{n=m_j+p_j+1}^{m_{j+1}-3}\frac{1}{n}\|2^n x\| \leq \sum_{n=m_j+p_j+1}^{m_{j+1}-3}\frac{2^{n-m_{j+1}+2}}{n} \leq \frac{1}{m_j + p_j + 1} \leq \frac{1}{m_j - 2}.$$

Thus

$$\sum_{n\in\omega}\frac{1}{n}\|2^n x\| \leq \sum_{j\in\omega}\sum_{n=m_j-2}^{m_{j+1}-3}\frac{1}{n}\|2^n x\| \leq \sum_{j\in\omega}\frac{p_j + 5}{2(m_j - 2)} \leq 1$$

and we are done. □

2.10. An $\mathscr{R}$-set which is not an $\mathscr{N}^\sigma$-set

J. Arbault proved the existence of an $\mathscr{A}$-set which is not an $\mathscr{N}$-set [1]. We extend his result in two ways. First, we construct an $\mathscr{R}$-set which is not an $\mathscr{N}$-set (since Konyagin's result it is not more an extension); this solves a problem of N. Bary [2]. Second, we prove that such a set need not be an $\mathscr{N}^\sigma$-set.

THEOREM 2.11. *There exists a $\mathscr{K}_{\sigma\delta}$ set in $\mathscr{R}$ which is not an $\mathscr{N}^\sigma$-set.*
More precisely, if $(\alpha_k)_{k\in\omega}$ is a sequence of integers such that $(\alpha_k - \alpha_{k+1})$ is strictly increasing, then

$$X = \left\{x \in \mathbf{T};\ \sum_{k\in\omega} \sin \pi 2^{\alpha_k} x \text{ converges}\right\}$$

is in $\mathscr{R} \setminus \mathscr{N}^\sigma$.

The following two lemmas will be the first step in all our results about $\mathscr{N}$-sets. For the notation concerning Θ, and f_x, $x \in \mathbf{T}$, see 1.3.

LEMMA 2.12. *Let $\Theta \in \mathscr{M}_\infty^+(\omega)$, $p, m \in \omega$, $m \geq 2$, $I = [2^p, 2^{p+1}[$ and $\varepsilon_1, \dots, \varepsilon_{p+m} \in \{0,1\}$. There exist $\varepsilon_{p+m+1}, \varepsilon_{p+m+2}, \varepsilon_{p+m+3} \in \{0,1\}$ such that for each $x \in \mathbf{T}$,*

$$(\forall i \leq p+m+3, \varepsilon_i(x) = \varepsilon_i) \Rightarrow (\Theta(1_I.f_x) \geq 2^{-m-3}\Theta(I)).$$

Proof. Let $S = \sum_{1\leq i\leq p+m} \varepsilon_i 2^{-i}$ and $x \in \mathbf{T}$, with $x = S + R(x)$ where $R(x) = \sum_{i>p+m} \varepsilon_i(x) 2^{-i}$. Observe that $R(x)$ belongs to the interval $[0, 2^{-p-m}[$ of $\mathbf{T}$.
Consider $J = \{n \in I;\ \|nS\| \geq 2^{-m-1}\}$.

First case. If $\Theta(J) \geq \frac{1}{2}\Theta(I)$, then let $\varepsilon_{p+m+1} = \varepsilon_{p+m+2} = \varepsilon_{p+m+3} = 0$.
Let $n \in J$ and $x \in \mathbf{T}$ with $\varepsilon_i(x) = \varepsilon_i$ for all $i \leq p+m+3$. Thus $R(x) \in [0, 2^{-p-m-3}[$ and $n < 2^{p+1}$; therefore $nR(x) \in [0, 2^{-m-2}]$, whence

$$\|nx\| = \|nS + nR(x)\| \geq \|nS\| - \|nR(x)\| \geq 2^{-m-1} - 2^{-m-2} = 2^{-m-2}.$$

It follows that $\Theta(1_I.f_x) \geq \Theta(1_J.f_x) \geq 2^{-m-2}\Theta(J) \geq 2^{-m-3}\Theta(I)$.

Second case. If $\Theta(J) < \frac{1}{2}\Theta(I)$, then let $\varepsilon_{p+m+1} = \varepsilon_{p+m+2} = 1$.
Let $n \in I \setminus J$ and $x \in \mathbf{T}$ with $\varepsilon_i(x) = \varepsilon_i$ for all $i \leq p+m+2$. Thus

$$(1+\tfrac{1}{2})2^{-p-m-1} \leq R(x) \leq 2^{-p-m} \text{ and } 2^p \leq n < 2^{p+1};$$

therefore $(1+\frac{1}{2})2^{-m-1} \leq nR(x) < 2^{-m+1}$.

Moreover $-2^{-m-1} < nS - z < 2^{-m-1}$ for some $z \in \mathbf{T}$. Because $m \geq 2$, we have

$$2^{-m-2} < nx - z < 2^{-m+1} + 2^{-m-1} \leq 1 - 2^{-m-2}.$$

Thus $\|nx\| \geq 2^{-m-2}$.

It follows that $\Theta(1_I.f_x) \geq \Theta(1_{I\setminus J}.f_x) \geq 2^{-m-2}\Theta(I\setminus J) \geq 2^{-m-3}\Theta(I)$. □

Now let $2^A = \bigcup_{i\in A}[2^i, 2^{i+1}[$ for each subset A of ω. In particular, $2^{[a,b]} = [2^a, 2^{b+1}[$ for all integers $a \leq b$.

LEMMA 2.13. *Let* $\Theta \in \mathscr{M}_\infty^+(\omega)$, $J = 2^{[a,b]}$, $m \geq 2$ *and* $\varepsilon_1, \dots, \varepsilon_{a+m} \in \{0,1\}$. *There exist* $\varepsilon_{a+m+1}, \dots, \varepsilon_{b+m+3} \in \{0,1\}$ *such that for each* $x \in \mathbf{T}$,

$$\big(\forall i \leq b+m+3, \varepsilon_i(x) = \varepsilon_i\big) \Rightarrow \big(\Theta(1_J.f_x) \geq \tfrac{1}{3}2^{-m-3}\Theta(J)\big).$$

Proof. Let

$$J_s = \bigcup_{a\leq j\leq b,\ j=s(\mathrm{mod}\,3)} [2^j, 2^{j+1}[$$

for $s = 0,1,2$. Pick an $s_0 \in \{0,1,2\}$ such that $\Theta(J_{s_0}) \geq \frac{1}{3}\Theta(J)$. Using, by induction on $j \in J_{s_0}$, the previous lemma for $I = [2^j, 2^{j+1}[$, we will get $\varepsilon_{a+m+1}, \dots, \varepsilon_{b+m+3}$ to insure our condition. □

The next lemma is the fundamental step in the proof of the theorem.

LEMMA 2.14. *Let* $(\Theta_q)_{q\in\omega}$ *be a sequence in* $\mathscr{M}_\infty^+(\omega)$ *and* $(\alpha_k)_{k\in\omega}$ *a sequence of integers such that* $\lim_{k\to\infty}(\alpha_{k+1} - \alpha_k) = +\infty$. *There exists a converging to infinity sequence* $(b_k)_{k\in\omega}$ *of integers such that if* $B = \bigcup_{k\in\omega}[\alpha_k + 1, \alpha_k + b_k]$ *and* $(\varepsilon_i)_{i\in B}$ *is any sequence of* 0's *and* 1's, *then there exists a sequence* $(\varepsilon_i)_{i\in B^c}$ *of* 0's *and* 1's *such that*

$$x = \sum_{i\in\omega} \varepsilon_i 2^{-i} \notin \bigcup_{q\in\omega} G_{\Theta_q}.$$

Proof. For $p \in \omega$, let

$$I_p = \bigcup_{k\in\omega} [\alpha_k - 4, \alpha_k + p]$$

and

$$Q = \big\{q \in \omega; \forall p \in \omega, \Theta_q(2^{I_p}) < +\infty\big\}.$$

Two kinds of q's are considered. If $q \notin Q$, then there exists an integer p_q such that $\Theta_q(2^{I_{p_q}}) = +\infty$. We have easily both following results.

CLAIM. *There exists an unbounded non decreasing sequence* $(a_k)_{k\in\omega}$ *of integers such that if*

$$A = \bigcup_{k\in\omega} [\alpha_k - 4, \alpha_k + a_k],$$

then for all $q \in Q$, $\Theta_q(2^A) < +\infty$.

CLAIM. *There exists an unbounded non decreasing sequence* $(m_k)_{k\in\omega}$ *of integers such that for each* $q \notin Q$,

$$\sum_{k\in\omega} 2^{-m_k}\Theta_q(2^{[\alpha_k-4,\alpha_k+p_q]}) = +\infty.$$

Now consider a surjection $f:\omega \to \omega$ whose fibers are infinite. There exists an increasing sequence $(k_j)_{j\in\omega}$ of integers such that for each $j \in \omega$, we have:

(1) if $q = f(j)$ is in Q, then $\sum_{k_j \le k < k_{j+1}} \Theta_q(2^{[\alpha_k+a_k+1,\alpha_{k+1}-5]}) \ge 1$

(2) if $q = f(j)$ is not in Q, then $\sum_{k_j \le k < k_{j+1}} 2^{-m_k}\Theta_q(2^{[\alpha_k-4,\alpha_k+p_q]}) \ge 1$ and $p_q \le m_{k_j}$.

Let $b_k = \inf\{a_k + 3, m_k - 4, \alpha_{k+1} - \alpha_k\}$ for each $k \in \omega$. Plainly, $(b_k)_{k\in\omega}$ converge to infinity. Note that we can assume in the second claim that $2m_k + 3 \le \alpha_{k+1} - \alpha_k$ for each k.

Let $B = \bigcup_{k\in\omega}[\alpha_k + 1, \alpha_k + b_k]$ and let $(\varepsilon_i)_{i\in B}$ be any sequence of 0's and 1's.

By induction on j, using the previous lemma for Θ_q where $q = f(j)$, we get ε_i for all $i \in B^c \cap [\alpha_{k_j} - 4, \alpha_{k_{j+1}} - 5]$ as follows. Consider two cases.

First case. If $q \in Q$, using the previous lemma for $m = 2$ and $J = 2^{[\alpha_k+a_k+1,\alpha_k-5]}$ where k is successively equal to $k_j, \ldots, k_{j+1} - 1$, we impose ε_i for all

$$i \in \bigcup_{k_j \le k < k_{j+1}} [\alpha_k + a_k + 4, \alpha_k + 1] \subseteq B^c \cap \left[\alpha_{k_j} - 4, \alpha_{k_{j+1}} - 5\right]$$

to insure

$$\Theta_q\left(1_{2^{[\alpha_{k_j}-4,\alpha_{k_{j+1}}-5]}} \cdot f_x\right) \ge \frac{1}{3}2^{-5} \sum_{k_j \le k < k_{j+1}} \Theta_q(2^{[\alpha_k+a_k+1,\alpha_{k+1}-5]}) \ge \frac{1}{96}.$$

Second case. If $q \in Q$, using the previous lemma for $J = 2^{[\alpha_k - 4, \alpha_k + p_q]}$ and $m = m_k$, where k is successively equal to $k_j, \dots, k_{j+1} - 1$, we impose ε_i for all

$$\begin{aligned} i \in & \bigcup_{k_j \le k < k_{j+1}} \left[\alpha_k + m_k - 3, \alpha_k + p_q + m_k + 3\right] \\ \subseteq & \bigcup_{k_j \le k < k_{j+1}} [\alpha_k + m_k - 3, \alpha_k + 2m_k + 3] \\ \subseteq & B^c \cap \left[\alpha_{k_j} - 4, \alpha_{k_{j+1}} - 5\right] \end{aligned}$$

to insure

$$\Theta_q\left(1_{2^{[\alpha_{k_j} - 4, \alpha_{k_{j+1}} - 5]}} \cdot f_x\right) \ge \frac{1}{3} \sum_{k_j \le k < k_{j+1}} 2^{-2-m_k} \Theta_q\left(2^{[\alpha_k - 4, \alpha_k + p_q]}\right) \ge \frac{1}{12}.$$

Since $f^{-1}\{q\}$ is infinite for all $q \in \omega$, we have

$$\Theta_q(f_x) \ge \sum_{j \in f^{-1}(q)} \Theta_q\left(1_{2^{[\alpha_{k_j} - 4, \alpha_{k_{j+1}} - 5]}} \cdot f_x\right) = +\infty.$$

Therefore $x = \sum_{i \ge 1} \varepsilon_i 2^{-i} \notin \bigcup_{q \in \omega} G_{\Theta_q}$. $\square$

Proof of Theorem 2.11. It is clear that $X \in \mathscr{R}$.

In order to prove that $X \notin \mathscr{N}^\sigma$, let $(\Theta_q)_{q \in \omega}$ be a sequence of elements of $\mathscr{M}_\infty^+(\omega)$. We will find a x in $X \setminus \bigcup_{q \in \omega} G_{\Theta_q}$ in its dyadic form $x = \sum_{i \ge 1} \varepsilon_i 2^{-i}$ for a suitable choice of ε_i either 0 or 1.

According to the previous lemma, there exists an unbounded non decreasing sequence $(b_k)_{k \in \omega}$ of integers such that if $B = \bigcup_{k \in \omega} [\alpha_k + 1, \alpha_k + b_k]$ and $(\varepsilon_i)_{i \in B}$ is any sequence of 0's and 1's, then there exists a sequence $(\varepsilon_i)_{i \in B^c}$ of 0's and 1's such that

$$x = \sum_{i \ge 1} \varepsilon_i 2^{-i} \notin \bigcup_{q \in \omega} G_{\Theta_q}.$$

Therefore, the only problem is to choose $(\varepsilon_i)_{i \in B}$ to insure that $x \in X$.

We will choose $\zeta_k = 0$ or 1 for $k \in \omega$ and set $\varepsilon_i = \zeta_k$ for $i \in [\alpha_k + 1, \alpha_k + b_k]$. Thus,

$$|\sin \pi 2^{\alpha_k} x| \le \pi \| 2^{\alpha_k} x \| \le \pi 2^{-b_k} \quad \text{for each } k \in \omega.$$

The sequence $(\zeta_k)_{k\in\omega}$ is defined by induction on k. Fix a $k \in \omega$ and consider $S_k = \sum_{i \le \alpha_k} \varepsilon_i 2^{-i}$. We have $|x - S_k| < 2^{-\alpha_k}$; thus

$$\left|\sum_{i<k} \sin \pi 2^{\alpha_i} x - \sum_{i<k} \sin \pi 2^{\alpha_i} S_k\right| \le \pi \sum_{i<k} 2^{\alpha_i} |x - S_k| = O(2^{\alpha_{k-1} - \alpha_k}).$$

Let $\zeta_k = 1$ if $\sum_{i<k} \sin \pi 2^{\alpha_i} S_k > 0$ and $\zeta_k = 0$ otherwise. Thus, if

$$\left|\sum_{i<k} \sin \pi 2^{\alpha_i} S_k\right| > O(2^{\alpha_{k-1} - \alpha_k}),$$

then $\sum_{i<k} \sin \pi 2^{\alpha_i} x$ has same sign as $\sum_{i<k} \sin \pi 2^{\alpha_i} S_k$ and therefore an opposite sign to $\sin \pi 2^{\alpha_k} x$. Thus, for every $p, q \in \omega$,

$$\left|\sum_{p<k<q} \sin \pi 2^{\alpha_k} x\right| \le \sup_{p<k<q} (|\sin \pi 2^{\alpha_k} x|) + \sum_{p<k<q} O(2^{\alpha_{k-1} - \alpha_k}).$$

So $\sum_{k\in\omega} \sin \pi 2^{\alpha_k} x$ converges as desired. □

2.15. Extensions of $\mathcal{N}$

The class $\mathcal{N}$ has three natural extensions closed under increasing countable unions: $\mathcal{WD}$, $\mathcal{N}^\sigma \cap \mathcal{WD}$ and the closure $\mathcal{N}^{\infty\uparrow}$ of $\mathcal{N}$ under increasing countable unions.

THEOREM 2.16. *The class $\mathcal{N}^\sigma \cap \mathcal{WD}$ is not closed under the operation of taking the generated subgroup, i.e., there exists a Borel set $X \in \mathcal{N}^\sigma \cap \mathcal{WD}$ such that the group generated by X does not belong to $\mathcal{N}^\sigma \cap \mathcal{WD}$.*

Since $\mathcal{N}$ is closed under the operation of taking the generated subgroup, $\mathcal{N}^{\infty\uparrow}$ has the same property. Using Proposition 1.7, we then get:

COROLLARY 2.17. *One has*

$$\mathcal{N}^{\infty\uparrow} \subset \mathcal{N}^\sigma \cap \mathcal{WD} \subset \mathcal{WD}$$

and all inclusions are strict.

Proof of Theorem 2.16. Let $(\alpha_k)_{k\in\omega}$ be a sequence of integers such that $\lim_{k\to\infty} \alpha_k - \alpha_{k+1} = +\infty$ *and*

$$Y = \left\{x \in \mathbf{T};\ \lim_{k\to\infty} \sin \pi 2^{\alpha_k} x = 0^+\right\}.$$

Clearly,

$$Y \in \mathscr{A} \subset \mathscr{W}\mathscr{D}.$$

Let (A, C) be a partition of ω in two colacunary sets. Then $X = Y \cap (K_A \cup K_C)$ is in $\mathscr{N}^\sigma \cap \mathscr{W}\mathscr{D}$. Note that Y is a subset of the group generated by X; indeed, $Y \subset X + X$ because $K_A + K_B = \mathbf{T}$.

It only remains to prove that Y does not belong to $\mathscr{N}^\sigma$. Let $(\Theta_q)_{q \in \omega}$ be a sequence of elements of $\mathscr{M}_\infty^+(\omega)$. We will find x in $X \setminus \bigcup_{q \in \omega} G_{\Theta_q}$ with dyadic decomposition $x = \sum_{i \geq 1} \varepsilon_i 2^{-i}$ for a suitable choice of ε_i either 0 or 1. According to Lemma 2.14, there exists a converging to infinity sequence $(b_k)_{k \in \omega}$ of integers such that if $B = \bigcup_{k \in \omega} [\alpha_k + 1, \alpha_k + b_k]$ and $(\varepsilon_i)_{i \in B}$ is any sequence of 0's and 1's, then there exists a sequence $(\varepsilon_i)_{i \in B^c}$ of 0's and 1's such that

$$x = \sum_{i \geq 1} \varepsilon_i 2^{-i} \notin \bigcup_{q \in \omega} G_{\Theta_q}.$$

But the ε_i's are set equal to 0 for every $i \in B$, whence $x \in Y$. □

3. Antistability and further applications

3.1. Hausdorff operations

A subset $\mathscr{F}$ of $2^\omega \setminus \{0\}$ is a *monotone basis on* ω if $\mathscr{F}$ is non empty and is monotone (or cohereditary), i.e., $[A \in \mathscr{F}$ and $A \subset B] \Rightarrow B \in \mathscr{F}$.

To each monotone basis $\mathscr{F}$, we associate its *dual basis*

$$\hat{\mathscr{F}} = \{A \in 2^\omega;\ A^c \in \mathscr{F}^c\},$$

which is also a monotone basis on ω. Note that $\hat{\hat{\mathscr{F}}} = \mathscr{F}$.

If $\mathscr{F}$ is a monotone basis on ω and $(X_n)_{n \in \omega}$ a sequence of subsets of a set E, we define $H_{\mathscr{F}}(X_n)$ as the set of all $x \in E$ such that

$$\{n \in \omega;\ x \in X_n\} \in \mathscr{F}.$$

Note that $x \notin H_{\mathscr{F}}(X_n) \Leftrightarrow \{n \in \omega;\ x \notin X_n\} \in \hat{\mathscr{F}}$. The operation $H_{\mathscr{F}}$ is called the *Hausdorff operation* with basis $\mathscr{F}$.

To each hereditary class $\mathscr{C}$ of subsets of E and monotone basis $\mathscr{F}$, we associate the class $\mathscr{C}^{\mathscr{F}}$ defined by

$$\mathscr{C}^{\mathscr{F}} = \{H_{\mathscr{F}}(X_n);\ (X_n)_{n \in \omega} \subseteq \mathscr{C}\}.$$

We say that $\mathscr{C}$ is an *$\mathscr{F}$-class* (of subsets of E) if $\mathscr{C}^{\mathscr{F}} = \mathscr{C}$.

3.2. Main examples

In this part, $\mathscr{C}$ is always a hereditary class of subsets of a set E.

(1) Let $\sigma = 2^\omega \setminus \{0\}$.

$$\mathscr{C}^\sigma = \left\{X \in \mathscr{P}(E); \exists (X_n)_{n\in\omega} \subseteq \mathscr{C}, X \subseteq \bigcup_{n\in\omega} X_n\right\}$$
$$= \text{the } \sigma\text{-ideal generated by } \mathscr{C}.$$

(2) Let $\mathscr{F}r$ be the Fréchet filter on ω, i.e., the set of all cofinite subsets of ω. For each $(X_n)_{n\in\omega}$, $H_{\mathscr{F}r}(X_n) = \bigcup_{m\in\omega}\bigcap_{n\geq m} X_n$. Thus

$$\mathscr{C}^{\mathscr{F}r} = \mathscr{C}^{\uparrow} = \left\{X \in \mathscr{P}(E); \exists (X_n)_{n\in\omega} \text{ increasing } \subseteq \mathscr{C}, X \subseteq \bigcup_{n\in\omega} X_n\right\}.$$

One has to be careful: in general, $\mathscr{C}^{\uparrow}$ is not the $\uparrow$-class (or $\mathscr{F}r$-class) generated by $\mathscr{C}$.

(3) Let $\mathscr{U}^{(n)} = \{A \in 2^\omega;\ A \cap [0, n-1] \neq \varnothing\}$ for each $n \geq 1$. Let $\mathscr{U} = \mathscr{U}^{(2)}$.

$$\mathscr{C}^{\mathscr{U}^{(n)}} = \left\{X \in \mathscr{P}(E); \exists (X_i)_{0\leq i<n-1} \subseteq \mathscr{C},\quad X \subseteq \bigcup_{i=0}^{n-1} X_i\right\},$$

$$\mathscr{C}^{\mathscr{U}^{(1)}} = \mathscr{C}.$$

The $\mathscr{U}^{(n)}$-classes are the ideals (i.e., the classes closed under finite union) for all $n \geq 2$.

(4) For each $n \geq 1$, let $\mathscr{P}^{(n)}$ be the set of all subsets of ω whose complement has cardinality $\leq n$. Clearly $\mathscr{P}^{(n)} \subset \mathscr{F}r$.

Let $\mathscr{P} = \mathscr{P}^{(1)}$. So $H_{\mathscr{P}}(X_n) = \bigcup_{m\in\omega}\bigcap_{n\neq m} X_n$ for each sequence $(X_n)_{n\in\omega}$.

PROPOSITION 3.3. *Let E be a metrisable compact space. For each sequence $(X_n)_{n\in\omega}$ of compact subsets of E, the set $X = H_{\mathscr{P}}(X_n)$ is compact. For each sequence $(X_n)_{n\in\omega}$ of elements of the multiplicative Borel class Π^0_α, the set $X = H_{\mathscr{P}}(X_n) \in \Pi^0_\alpha$.*

Proof. For each $x \in E$,

$$x \in X \Leftrightarrow \forall m \in \omega\ \left(x \in X_m \text{ or } [\forall n \neq m,\ x \in X_n]\right).$$

Thus X is in the same multiplicative Borel class as the X_n's. $\square$

(5) Let $\mathscr{A}s$ be the set of all subsets of density one of ω, i.e., all subsets A of ω such that

$$d(A) = \lim_{n\to\infty} \frac{\operatorname{card}(A \cap [0, n-1])}{n} = 1.$$

Note that $\mathscr{F}r \subset \mathscr{A}s$ and $\mathscr{A}s$ is a $\mathscr{K}_{\sigma\delta}$ subset of 2^{ω}. For each $(X_n)_{n\in\omega}$,

$$H_{\mathscr{A}s}(X_n) = \left\{ x \in E;\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{X_n}(x) = 1 \right\}.$$

PROPOSITION 3.4. *Let E be a metrizable compact space. For each hereditary class $\mathscr{C}$ of compact subsets of E, $\mathscr{W}\mathscr{C}$ is an $\mathscr{A}s$-class.*

This proposition follows directly from the next lemma, due to R. Lyons [20], and which will be used again in Proposition 4.5.

LEMMA 3.5. *Let E be a metrizable compact space. Let μ be a finite positive measure on E, $(X_n)_{n\in\omega}$ be a sequence of universally measurable subsets of E and $X = H_{\mathscr{A}s}(X_n)$. Then*

$$\mu(X) = \sup_n \mu(X \cap X_n).$$

Proof.

$$X = \left\{ x \in E;\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{X_n}(x) = 1 \right\};$$

thus

$$1_X \le \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{X_n}.$$

Using Lebesgue's Theorem of dominated convergence, we have

$$\begin{aligned} \mu(X) &\le \int_X \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{X_n}(x)\, d\mu(x) \\ &\le \lim_{n\to\infty} \frac{1}{n} \int_X \sum_{k=1}^{n} 1_{X_n}(x)\, d\mu(x) \\ &\le \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mu(X \cap X_n). \end{aligned}$$

For all $\varepsilon > 0$, there exists an integer n such that $\mu(X \cap X_n) > \mu(X) - \varepsilon$. □

3.6. An order relation

PROPOSITION 3.7. *Let $\mathscr{F}$ and $\mathscr{G}$ be two monotone bases on ω. The following conditions are equivalent:*

(1) $\mathscr{C}^{\mathscr{F}} \subseteq \mathscr{C}^{\mathscr{G}}$ *for each hereditary class $\mathscr{C}$ or subsets of each set E.*

(2) $\exists\varphi : \omega \to \omega,\ \varphi^{-1}\mathscr{F} \subseteq \mathscr{G}$.

(3) $\exists\varphi : \omega \to \omega,\ \varphi\hat{\mathscr{G}} \subseteq \hat{\mathscr{F}}$.

If the previous conditions occur, we shall say that $\mathscr{F} \preceq \mathscr{G}$. The proposition is proved in [12, Propositions 1.5 and 1.7].

In particular, $\mathscr{F} \npreceq \mathscr{G} \Leftrightarrow \forall\varphi : \omega \to \omega,\ \exists A \in \hat{\mathscr{G}},\ \varphi(A)^c \in \mathscr{F}$. We will use this equivalence in Part 3.10.

The relation $\preceq$ is a quasi-order (i.e., a reflexive and transitive relation) on the set of all monotone bases on ω. Let $\prec$ be the associated strict quasi-order defined by

$$\mathscr{F} \prec \mathscr{G} \Leftrightarrow \left(\mathscr{F} \preceq \mathscr{G} \text{ and } \mathscr{G} \npreceq \mathscr{F}\right).$$

Let $\equiv$ be the relation of equivalence defined by

$$\mathscr{F} \equiv \mathscr{G} \Leftrightarrow \left(\mathscr{F} \preceq \mathscr{G} \text{ and } \mathscr{G} \preceq \mathscr{F}\right)$$

and let $\perp$ be the relation defined by

$$\mathscr{F} \perp \mathscr{G} \Leftrightarrow \left(\mathscr{F} \npreceq \mathscr{G} \text{ and } \mathscr{G} \preceq \mathscr{F}\right).$$

Remarks. (1) $\mathscr{U}^{(1)}$ is the least element of $\preceq$ modulo $\equiv$ and $\mathscr{U}^{(1)} \equiv \{\omega\}$.
(2) $\mathscr{P}$ is the immediate and single successor of $\mathscr{U}^{(1)}$ for $\preceq$ modulo $\equiv$. Indeed, for all monotone bases $\mathscr{F}$ on ω, we have

$$\begin{aligned} \mathscr{F} \npreceq \mathscr{U}^{(1)} &\Leftrightarrow \forall n \in \omega, \exists A \in \mathscr{F}, n \notin A \\ &\Leftrightarrow \forall n \in \omega, \{n\}^c \in \mathscr{F} \\ &\Leftrightarrow \mathscr{P} \subseteq \mathscr{F}. \end{aligned}$$

(3) σ is the greatest element of $\preceq$ modulo $\equiv$.
(4) $\mathscr{P} \prec \mathscr{F}r \prec \mathscr{A}s$, $\mathscr{F}r \perp \mathscr{U}$ and $\mathscr{A}s \perp \mathscr{U}$.
(5) $\mathscr{P}^{(n-1)} \preceq \mathscr{U}^{(n)}$ and $\mathscr{P}^{(n)} \perp \mathscr{U}^{(n)}$ for each $n \geq 2$.

3.8. Iteration

To obtain the closure of a class under a Hausdorff operation, it is generally necessary to iterate this operation.

Let $\mathscr{F}$ be a monotone basis on ω and $\mathscr{C}$ a hereditary class of subsets of a set E. The class $\mathscr{C}^{\alpha\mathscr{F}}$ is defined by induction on the ordinal α:

$$\mathscr{C}^{0\mathscr{F}} = \mathscr{C}, \quad \mathscr{C}^{<\alpha\mathscr{F}} = \bigcup_{\beta<\alpha} \mathscr{C}^{\beta\mathscr{F}}, \quad \mathscr{C}^{\alpha\mathscr{F}} = \left(\mathscr{C}^{<\alpha\mathscr{F}}\right)^{\mathscr{F}} \quad \text{and} \quad \mathscr{C}^{\infty\mathscr{F}} = \mathscr{C}^{<\omega_1\mathscr{F}}.$$

The height of $\mathscr{F}$, written $\mathrm{ht}(\mathscr{F})$, is the supremum, over all hereditary classes $\mathscr{C}$, of the first α for which $\mathscr{C}^{\alpha\mathscr{F}} = \mathscr{C}^{\infty\mathscr{F}}$. Note that $\mathrm{ht}(\mathscr{U}^{(1)}) = 0$, $\mathrm{ht}(\sigma) = 1$ and $\mathrm{ht}(\mathscr{U}) = \omega$.

In most cases, we will see that $\mathscr{C}^{\alpha\mathscr{F}}$ can be obtained with the single operation of a monotone basis on $\mathscr{C}$.

Fix a one-to-one map $\langle \ \ \rangle : \omega^2 \to \omega$. Let $A_n = \{k \in \omega; \langle n, k\rangle \in A\}$ for each $A \in 2^\omega$ and $n \in \omega$. Let $\mathscr{F}$ and $\mathscr{G}$ be two monotone bases on ω. Consider

$$\mathscr{F} \otimes \mathscr{G} = \left\{A \in 2^\omega; \{n \in \omega; A_n \in \mathscr{G}\} \in \mathscr{F}\right\}$$

which is a monotone basis on ω such that

$$H_{\mathscr{F}\otimes\mathscr{G}}(X_n) = H_{n\to\mathscr{F}}\left[H_{k\to\mathscr{G}}(X_{\langle n,k\rangle})\right]$$

for every sequence $(X_n)_{n\in\omega}$ of sets. In particular

$$\mathscr{C}^{\mathscr{F}\otimes\mathscr{G}} = \left(\mathscr{C}^{\mathscr{G}}\right)^{\mathscr{F}}$$

for each hereditary class $\mathscr{C}$.

Note that

$$\mathscr{C}^{n\mathscr{F}} = \mathscr{C}^{\overbrace{\mathscr{F}\otimes \cdots \otimes \mathscr{F}}^{n \text{ times}}} \quad \text{for all } n \in \omega.$$

But the ideal $\mathscr{C}^{\omega\mathscr{U}}$ generated by $\mathscr{C}$ cannot be obtained with the single operation of a monotone basis on $\mathscr{C}$. Nevertheless for $\mathscr{F} = \mathscr{F}r$ or $\mathscr{A}s$ there exists a monotone basis $\mathscr{F}_\alpha$ such that $\mathscr{C}^{\alpha\mathscr{F}} = \mathscr{C}^{\mathscr{F}\alpha}$ for each countable ordinal α.

Here is one of the possible definitions of $\mathscr{F}_\alpha$. For each countable ordinal α, we can take a sequence $([\alpha]_n)_{n\in\omega}$ of ordinals such that $[\alpha]_n \nearrow \alpha$ if α is a limit ordinal, and $[\alpha]_n = \beta$ if $\alpha = \beta + 1$. Define $\mathscr{F}_0 = \{\omega\}$ and

$$\mathscr{F}_\alpha = \left\{A \in 2^\omega; \left\{n \in \omega; A_n \in \mathscr{F}_{[\alpha]_n}\right\} \in \mathscr{F}\right\}.$$

In particular, $\mathscr{F}_{\alpha+1} = \mathscr{F} \otimes \mathscr{F}_\alpha$.

The definition of $\mathcal{F}_\alpha$ depends a priori on the choice of $([\alpha]_n)_{n\in\omega}$; for example, we do not have unicity of $\mathcal{P}_\omega$ modulo $\equiv$ [12, Proposition 11] but for $\mathcal{F} = \mathcal{F}r$ or $\mathcal{A}s$ we have unicity of $\mathcal{F}_\alpha$ modulo $\equiv$ [12, Propositions 2.8 and 2.9].

Let us recall the main result about these bases [12, Theorem 4.1 and 5.1 and Proposition 5.4].

THEOREM 3.9. *For all ordinals α and β with $1 \le \alpha < \beta < \omega_1$ we have*

$$\mathcal{P}_\beta \not\preceq \mathcal{F}r_\alpha \text{ and } \mathcal{F}r_\beta \not\preceq \mathcal{A}s_\alpha.$$

In particular $\mathrm{ht}(\mathcal{P}) = \mathrm{ht}(\mathcal{F}r) = \mathrm{ht}(\mathcal{A}s) = \omega_1$. *However $\mathcal{P}_\alpha \preceq \mathcal{A}s$ for each ordinal $\alpha < \omega_1$.*

Let $\mathcal{C}^{\alpha\uparrow} = \mathcal{C}^{\alpha\mathcal{F}r} = \mathcal{C}^{\mathcal{F}r_\alpha}$. The basis $\mathcal{F}r_\alpha$ is called the α-iterated Fréchet filter. Observe that $\mathcal{C}^{\infty\uparrow} = \mathcal{C}^{\infty\mathcal{F}r}$ is the closure of $\mathcal{C}$ under increasing countable unions.

3.10. Antistable classes

Consider $I_n = \{A \in 2^\omega;\ n \in A\}$ where $n \in \omega$. Note that I_n is a clopen subset of 2^ω for the product topology. For each monotone basis $\mathcal{F}$, we have $\mathcal{F} = H_{\mathcal{F}}(I_n)$. Let $\mathcal{I} = \bigcup_{n\in\omega}\mathcal{P}(I_n)$. We deduce from Proposition 3.7 that for all monotone bases $\mathcal{F}$ and $\mathcal{G}$ we have

$$\mathcal{F} \not\preceq \mathcal{G} \Rightarrow H_{\mathcal{F}}(I_n) \notin \mathcal{I}^{\mathcal{G}}.$$

So we give the next definition.

DEFINITION 3.11. We say that a hereditary class $\mathcal{C}$ is *antistable* if there exists a sequence $(X_n)_{n\in\omega}$ of elements of $\mathcal{C}$ such that, for all monotone bases $\mathcal{F}$ and $\mathcal{G}$, we have

$$\mathcal{F} \not\preceq \mathcal{G} \Rightarrow H_{\mathcal{F}}(X_n) \notin \mathcal{C}^{\mathcal{G}}.$$

For an antistable class $\mathcal{C}$, one has in particular $\mathcal{F} \preceq \mathcal{G} \Leftrightarrow \mathcal{C}^{\mathcal{F}} \subseteq \mathcal{C}^{\mathcal{G}}$.

Antistability is a very strong property of non-stability, as shown next.

PROPOSITION 3.12. *There exists a class $\mathcal{C}$ such that $\mathcal{C} \neq \mathcal{C}^{\mathcal{F}}$ for every monotone basis $\mathcal{F} \succ \mathcal{U}^{(1)}$ but $\mathcal{C}$ is not antistable.*

To see this, we will use the following lemma [12, Proposition 2.7].

LEMMA 3.13. *Let $\mathscr{F}$ be a monotone basis on ω. For every ordinal $\alpha \leq \mathrm{ht}(\mathscr{F})$, there exists a hereditary class $\mathscr{C} \subset \mathscr{P}(2^\omega)$ such that $\mathscr{C}^{<\alpha\mathscr{F}} \neq \mathscr{C}^{\alpha\mathscr{F}} = \mathscr{C}^{\infty\mathscr{F}}$.*

Proof of Proposition 3.12. By the previous lemma, let $\mathscr{C}$ be a hereditary class such that $\mathscr{C} \neq \mathscr{C}^{\mathscr{P}} = \mathscr{C}^{\infty\mathscr{P}}$. Let $\mathscr{F}$ be a monotone basis $\succ \mathscr{U}^{(1)}$. Since $\mathscr{P}$ is the immediate and single successor of $\mathscr{U}^{(1)}$, we have $\mathscr{C} \neq \mathscr{C}^{\mathscr{P}} \subseteq \mathscr{C}^{\mathscr{F}}$. But $\mathscr{P}_2 \not\preceq \mathscr{P}$ and $\mathscr{C}^{\mathscr{P}} = \mathscr{C}^{\mathscr{P}_2}$, so $\mathscr{C}$ is not antistable. □

We note that if $\mathscr{C}$ is an antistable class, $\mathscr{C}^{\uparrow}$ is antistable too. Since $\mathscr{C}^{\uparrow} = \mathscr{C}^{\mathscr{F}r}$ the next lemma [12, Lemma 4.11] is enough to conclude.

LEMMA 3.14. *Let $\mathscr{F}$ and $\mathscr{G}$ be two monotone bases on ω. We have*

$$\mathscr{F} \not\preceq \mathscr{G} \Rightarrow \mathscr{F} \otimes \mathscr{P} \not\preceq \mathscr{G} \otimes \mathscr{F}r.$$

3.15. Antistability of $\mathscr{N}$

THEOREM 3.16. *Let $(A_n)_{n\in\omega}$ be a sequence of colacunary subsets of ω such that $d(A_n, A_m) \geq 3$ for all $n \neq m$. For each n, let $E_n = \bigcup_{i\in\omega} K_{A_n \cap [i,+\infty[}$, which is a $\mathscr{K}_\sigma$ set in $D^{\uparrow}$. If $\mathscr{F}$ and $\mathscr{G}$ are two monotone bases on ω such that $\mathscr{F} \not\preceq \mathscr{G}$, then $X = H_{\mathscr{F}}(E_n) \notin \mathscr{N}^{\mathscr{G}}$.*

As a consequence, $\mathscr{N}$, $\mathscr{D}^{\uparrow}$ and all hereditary classes $\mathscr{C}$ with $D^{\uparrow} \subseteq \mathscr{C} \subseteq \mathscr{N}$ (like $\mathscr{N}_0$) are antistable.

In the proof of Theorem 3.16, we will use the following lemma, which is a corollary of Lemma 2.13.

LEMMA 3.17. *Let $(\Theta_q)_{q\in\omega}$ be a sequence in $\mathscr{M}_\infty^+(\omega)$ and A a subset of ω such that $\Theta_q(2^A) = +\infty$ for each $q \in \omega$. Let $B = A + [3,5]$ and $(\varepsilon_i)_{i\in B^c}$ be a sequence of 0's and 1's. Then there exists a sequence $(\varepsilon_i)_{i\in B}$ of 0's and 1's such that*

$$x = \sum_{i\geq 1} \varepsilon_i 2^{-i} \notin \bigcup_{q\in\omega} G_{\Theta_q}.$$

Proof. Note that there exist sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ of integers such that

$$B = A + [3,5] = \bigcup_{n\in\omega} [a_n, b_n] \text{ and } b_n + 2 \leq a_{n+1}$$

for all n. Thus $A \subseteq \bigcup_{n\in\omega}[a_n - 3, b_n - 5]$. Consider a surjection $f : \omega \to \omega$ whose fibers are infinite. Since $\Theta_q(2^A) = +\infty$ for each $q \in \omega$, there exists an increasing sequence $(n_k)_{k\in\omega}$ of integers such that for each k,

$$\Theta_q(2^{\bigcup_{n_k \le n < n_{k+1}}[a_n-3,\, b_n-5]}) \ge 1,$$

where $q = f(k)$. Let $(\varepsilon_i)_{i\in B^c}$. Using, by induction on k, Lemma 2.13 for Θ_q, where $q = f(k)$, $J = 2^{[a_n-3,\, b_n-5]}$, $m = 2$ and successively for $n = n_k, \dots, n_{k+1}-1$, we may impose the values ε_i for all $i \in B \cap [a_{n_k}, a_{n_{k+1}}[$ to insure

$$\Theta_q\left(1_{2^{\bigcup_{nk \le n < nk+1}[a_n-3,\, b_n-5]}} f_x\right) \ge \tfrac{1}{3}2^{-5}.$$

Since $f^{-1}\{q\}$ is infinite for all $q \in \omega$, we have

$$\Theta_q(f_x) \ge \sum_{k\in f^{-1}\{q\}} \Theta_q\left(1_{[a_{n_k},\, a_{n_{k+1}}[} \cdot f_x\right) = +\infty$$

for $x = \sum_{i\ge 1}\varepsilon_i 2^{-i}$ and for all $q \in \omega$. □

Proof of Theorem 3.16. Let $(\Theta_q)_{q\in\omega}$ be any sequence of elements of $\mathscr{M}_\infty^+(\omega)$. We will find x in $X \setminus H_{\mathscr{G}}(G_{\Theta_q})$ with dyadic decomposition $x = \sum_{i\ge 1}\varepsilon_i 2^{-i}$ for a suitable choice of ε_i's either 0 or 1.

Let $B_n = A_n + [-5, -3]$ for each $n \in \omega$ and

$$Q = \left\{q \in \omega; \forall n \in \omega, \Theta_q(2^{B_n}) < +\infty\right\}.$$

By the usual means, we can find a subset A of ω such that:

(1) $A \cap B_n$ is finite for each $n \in \omega$.
(2) $\Theta_q(2^A) = +\infty$ for each $q \in Q$.

Let $B = A + [3,5]$. Observe that $B \cap A_n$ is finite for each $n \in \omega$.

Now there exists a $n_q \in \omega$ such that $\Theta_q(2^{B_{n_q}}) = +\infty$ for each $q \notin Q$. Consider a map $\varphi : \omega \to \omega$ which associates n_q to $q \notin Q$ and anything to $q \in Q$. Since $\mathscr{F} \not\leq \mathscr{G}$, there exists a $P \in \hat{\mathscr{G}}$ such that $\varphi(P)^c \in \mathscr{F}$. Let $A' = A \cup \bigcup_{n\in\varphi(P)} B_n$ and $B' = A' + [3,5]$. The values of ε_i are set equal to 0 for all $i \notin B'$. But $B_n + [3,5] = A_n + [-2,2]$, thus

$$(B_n + [3,5]) \cap A_{n'} = \varnothing \quad \text{for all } n \ne n'.$$

Therefore $B' \cap A_n = B \cap A_n$ is finite for each $n \notin \varphi(P)$; thus

$$\{n \in \omega;\, x \in E_n\} \in \mathscr{F} \text{ and } x \in X = H_{\mathscr{F}}(E_n).$$

Furthermore, $\Theta_q(2^{A'}) = +\infty$ for all $q \in P$. According to the previous lemma, there exists a sequence $(\varepsilon_i)_{i \in B'}$ of 0's and 1's such that

$$x = \sum_{i \geq 1} \varepsilon_i 2^{-i} \notin \bigcup_{q \in P} G_{\Theta_q}.$$

Since $P \in \hat{\mathscr{I}}$, then $\{q \in \omega;\ x \notin G_{\Theta_q}\} \in \hat{\mathscr{I}}$ and $x \notin H_{\mathscr{I}}(G_{\Theta_q})$. □

3.18. Solutions of the problem of Arbault and of other questions about $\mathscr{N}$

We can now answer several problems including the question about the stability of $\mathscr{N}$ under increasing countable unions (see Part 1.5). Note that the following results are false if one considers only $\mathscr{K}_\sigma$ sets.

THEOREM 3.19. *There exists an increasing sequence of $\mathscr{K}_{\sigma\delta}$ sets of absolute convergence whose the union is not a set of absolute convergence. More generally, the inclusions*

$$\mathscr{N} \subset \mathscr{N}^{\uparrow} \subset \mathscr{N}^{2\uparrow} \subset \cdots \subset \mathscr{N}^{\alpha\uparrow} \subset \cdots \subset \mathscr{N}^{\infty\uparrow}$$

are strict, and in fact, for each countable ordinal α, there exists a $\mathscr{K}_{\sigma\delta}$ set in $D^{\alpha\uparrow}$ which is not in $N^{\beta\uparrow}$ for any $\beta < \alpha$.

Proof. Let $(E_n)_{n \in \omega}$ be as in Theorem 3.16 and α a countable ordinal. Since $\mathscr{P}_\alpha \not\preceq \mathscr{F}r_\beta$ for each ordinals $\beta < \alpha$, $H_{\mathscr{P}_\alpha}(E_n) \notin \mathscr{N}^{<\alpha\uparrow}$. But $H_{\mathscr{P}_\alpha}(E_n) \in D^{(1+\alpha)\uparrow}$ because $\mathscr{P} \preceq \mathscr{F}r$. Using Proposition 3.3, we deduce that $H_{\mathscr{P}_\alpha}(E_n)$ is $\mathscr{K}_{\sigma\delta}$ because the E_n's are $\mathscr{K}_{\sigma\delta}$. Observe that if we use $\mathscr{F}r_\alpha$ rather than $\mathscr{P}_\alpha$ we obtain a $\Sigma^0_{1+2\alpha+1}$ set. □

THEOREM 3.20. *For all $n \geq 1$, there exists a $\mathscr{K}_{\sigma\delta}$ set in $\mathscr{W}\mathscr{D}$ (even in $D^{2\uparrow}$ or in $\mathscr{N}^{\uparrow}$) which is a union of $n + 1$ $\mathscr{N}$-sets and not of n.*

Proof. Let $(E_k)_{k \in \omega}$ be as in Theorem 3.16 and $n \geq 1$. Since $\mathscr{P}^{(n)} \not\preceq \mathscr{U}^{(n)}$, $H_{\mathscr{P}^{(n)}}(E_k) \notin \mathscr{N}^{\mathscr{U}^{(n)}}$, we know $H_{\mathscr{P}^{(n)}}(E_k)$ is not a union of n $\mathscr{N}$-sets. But $\mathscr{P}^{(n)} \preceq \mathscr{F}r$ and $\mathscr{P}^{(n)} \preceq \mathscr{U}^{(n+1)}$, so

$$H_{\mathscr{P}^{(n)}}(E_k) \in D^{2\uparrow} \subset \mathscr{N}^{\uparrow} \subset \mathscr{W}\mathscr{D}$$

and $H_{\mathscr{P}^{(n)}}(E_k)$ is a union of $n + 1$ $\mathscr{N}$-sets. □

THEOREM 3.21. *There exists a Borel set in $\mathscr{N}^{\uparrow}$ which is not a finite union of $\mathscr{N}$-sets.*

Proof. Let $(E_k)_{k\in\omega}$ be as in Theorem 3.16. Since $\mathscr{F}r \not\preceq \mathscr{U}^{(n)}$ for all $n \geq 1$, $H_{\mathscr{F}_r}(E_k) \notin \bigcup_{n\geq 1}\mathscr{N}^{\mathscr{U}^{(n)}}$; thus $H_{\mathscr{F}_r}(E_k)$ is not a finite union of $\mathscr{N}$-sets.

□

THEOREM 3.22. *There exists a Borel set in $D^\sigma \cap \mathscr{W}\mathscr{D}$ which is not in $\mathscr{N}^{\infty\uparrow}$.*

Proof. Let $(E_n)_{n\in\omega}$ be as in Theorem 3.16. Since $\mathscr{A}s \not\preceq \mathscr{F}r_\alpha$ for all countable ordinals α, $H_{\mathscr{A}s}(E_n) \notin \mathscr{N}^{\infty\uparrow}$. But $\mathscr{W}\mathscr{D}$ is an $\mathscr{A}s$-class, so $H_{\mathscr{A}s}(E_n) \in \mathscr{W}\mathscr{D}$. And clearly $H_{\mathscr{A}s}(E_n) \in D^\sigma$. □

In the previous theorem, note that $H_{\mathscr{A}s}(E_n)$ is a Σ^0_6 set. We will see that a $\mathscr{K}_{\sigma\delta}$ set with the same property can be found. The next result is not a consequence of Theorem 3.16, but its proof follows the same way.

THEOREM 3.23. *Let $(A_n)_{n\in\omega}$ be a sequence of colacunary infinite subsets of ω such that $d(A_n, A_m) \geq 3$ for all $n \neq m$. Then $X = H_{\mathscr{A}s}(K_{A_n}) \notin \mathscr{N}^{\infty\uparrow}$.*

In particular, there exists a $\mathscr{K}_{\sigma\delta}$ set in $D^\sigma \cap \mathscr{W}\mathscr{D}$ which is not in $\mathscr{N}^{\infty\uparrow}$.

Proof. Let $(\Theta_q)_{q\in\omega}$ be any sequence of elements of $\mathscr{M}^+_\infty(\omega)$ and let α be a countable ordinal. We will find x in $X \setminus H_{\mathscr{F}r_\alpha}(G_{\Theta_q})$ with dyadic decomposition $x = \sum_{i\geq 1}\varepsilon_i 2^{-i}$ for a suitable choice of ε_i's, either 0 or 1.

Let $B_n = A_n + [-5, -3]$ for each $n \in \omega$ and

$$Q = \left\{q \in \omega; \forall n \in \omega, \Theta_q(2^{B_n}) < +\infty\right\}.$$

It is easy to prove the following fact: let $(u^q_n)_{n\in\omega}$ be sequences of non negative reals such that $\sum_{n\in\omega} u^q_n = +\infty$ for each $q \in \omega$; there exists a set C of density zero (i.e., $C^c \in \mathscr{A}s$) such that $\sum_{n\in C} u^q_n = +\infty$ for each $q \in \omega$. Using this fact with $u^q_n = \Theta_q(2^{B_n})$, we take a subset A of ω such that $\Theta_q(1^A) = +\infty$ for each $q \in Q$ and $A \cap B_n = \varnothing$ for each $n \in C^c$, where $C^c \in \mathscr{A}s$. let $B = A + [3,5]$. Observe that $B \cap A_n = \varnothing$ for each $n \in C^c$.

There exists a $n_q \in \omega$ such that $\Theta_q(2^{B_{n_q}}) = +\infty$ for each $q \notin Q$. Consider a map $\varphi : \omega \to \omega$ which associates n_q to $q \notin Q$ and anything to $q \in Q$. Since $\mathscr{A}s \not\preceq \mathscr{F}r_\alpha$, there exists a $P \in \widehat{\mathscr{F}r_\alpha}$ such that $\varphi(P)^c \in \mathscr{A}s$. Let $A' = A \cup \bigcup_{n\in\varphi(P)} B_n$ and $B' = A' + [3,5]$. The ε_i's are set equal to 0 for all $i \notin B'$. But $B_n + [3,5] = A_n + [-2,2]$; thus

$$(B_n + [3,5]) \cap A_{n'} = \varnothing \quad \text{for all } n \neq n'.$$

Therefore $B' \cap A_n = B \cap A_n = \varnothing$ for each $n \notin \varphi(P) \cup C$. Since $\mathscr{A}s$ is a filter, $\varphi(P)^c \cap C^c \in \mathscr{A}s$, then

$$\{n \in \omega;\ x \in E_n\} \in \mathscr{A}s \text{ and } x \in X = H_{\mathscr{A}s}(K_{A_n}).$$

Furthermore, $\Theta_q(2^{A'}) = +\infty$ for all $q \in P$. According to Lemma 3.17, there exists a sequence $\widetilde{(\varepsilon_i)_{i \in B'}}$ of 0's and 1's such that $x = \sum_{i \geq 1} \varepsilon_i 2^{-i} \notin \bigcup_{q \in P} G_{\Theta_q}$. Since $P \in \widetilde{\mathscr{F}r_\alpha}$, we have

$$\{q \in \omega;\ x \notin G_{\Theta_q}\} \in \widetilde{\mathscr{F}r_\alpha} \text{ and } x \notin H_{\mathscr{F}r_\alpha}(G_{\Theta_q}). \quad \square$$

3.24. Antistability of various classes of compact sets

THEOREM 3.25. *Let $(A_n)_{n \in \omega}$ be a sequence of colacunary subsets of ω such that as $k \to +\infty$,*

$$d(A_n \cap [k, +\infty[, A_m \cap [k, +\infty[) \to +\infty$$

uniformly for all $n \neq m$. For each n, let $E_n = K_{A_n}$ which is a compact Dirichlet set. If $\mathscr{F}$ and $\mathscr{G}$ are two monotone bases on ω such that $\mathscr{F} \not\leq \mathscr{G}$, then $X = H_{\mathscr{F}}(E_n) \notin L_0^{\mathscr{G}}$.

So D, L_0, and all hereditary classes $\mathscr{C}$ with $D \subseteq \mathscr{C} \subseteq L_0$, (like H and L) are antistable.

COROLLARY 3.26. *There exists a sequence $(F_n)_{n \in \omega}$ of compact $D^\uparrow$-sets such that if $\mathscr{F}$ and $\mathscr{G}$ are two monotone bases on ω such that $\mathscr{F} \not\leq \mathscr{G}$, then $H_{\mathscr{F}}(F_n) \notin (L_0^\uparrow)^{\mathscr{G}}$.*

So $\mathscr{N}_0$, $\mathscr{R}$, $\mathscr{A}$, and their corresponding classes of compact sets, N_0, R and A, are antistable.

Proof. For each $n \in \omega$, let $F_n = H_{\mathscr{P}}(E_{\langle n,k \rangle})$ which is a compact $D^\uparrow$-set. Let $\mathscr{F}$ and $\mathscr{G}$ be two monotone bases on ω such that $\mathscr{F} \not\leq \mathscr{G}$. According to Lemma 3.14, we have $\mathscr{F} \otimes \mathscr{P} \not\leq \mathscr{G} \otimes \mathscr{F}r$. Then $H_{\mathscr{F}}(F_n) = H_{\mathscr{F} \otimes \mathscr{P}}(E_n)$ does not belong to $(L_0^\uparrow)^{\mathscr{G}} = L_0^{\mathscr{G} \otimes \mathscr{F}r}$. $\square$

Proof of Theorem 3.25. Let $(K_j)_{j \in \omega}$ be any sequence of L_0-sets. We will find x in $X \setminus H_{\mathscr{G}}(K_j)$ with dyadic decomposition $x = \sum_{i \geq 1} \varepsilon_i 2^{-i}$ for a suitable choice of ε_i's, either 0 or 1.

Let $j \in \omega$. Since $K_j \in L_0$, there exist a sequence $\varepsilon_j^k \to 0^+$, $\alpha_j > 0$ and for each k a finite sequence (I_l^k) of intervals such that $|I_l^k| \leq \varepsilon_j^k$ for each l, $d(I_l^k, I_{l'}^k) \geq \alpha_j \varepsilon_j^k$ for each $l \neq l'$ and $K_j \subset \bigcup_l I_l^k$. Let

$$p_j = \sup\{-\lfloor \log_2 \alpha_j \rfloor, 0\}, \quad m_j^k = -\lfloor \log_2 \varepsilon_j^k \rfloor$$

and

$$J_j^k = \left[m_j^k - 1, m_j^k + p_j + 1\right]$$

for each $k \in \omega$. As $k \to +\infty$, $m_j^k \to +\infty$ and $d(A_n \cap [k, +\infty[, A_m \cap [k, +\infty[) \to +\infty$ uniformly for all $n \neq m$, J_j^k meets at most one A_n for large enough k. Thus for each $j \in \omega$, there exists an integer k_j such that $J_j = J_j^{k_j}$ meet at most A_{n_j} and the J_j's are pairwise disjoint.

Consider the map $\varphi : \omega \to \omega$ which takes j to n_j. Since $\mathscr{F} \not\preceq \mathscr{G}$, there exists a $P \in \hat{\mathscr{G}}$ such that $\varphi(P)^c \in \mathscr{F}$. Using Lemma 2.9, by induction on $j \in P$, we choose ε_i for $i \in J_j$ to insure

$$x = \sum_{i \geq 1} \varepsilon_i 2^{-i} \notin K_j.$$

The other ε_i's are set equal to 0. Since $P \in \hat{\mathscr{G}}$, $\{j \in \omega;\ x \notin K_j\} \in \hat{\mathscr{G}}$ and thus $x \notin H_{\mathscr{G}}(K_j)$. Furthermore $\varphi(P)^c \in \mathscr{F}$; then $\{n \in \omega;\ x \in K_{A_n}\} \in \mathscr{F}$, and so $x \in X = H_{\mathscr{F}}(E_n)$. □

3.27. Increasing countable compact unions of compact thin sets

We study the stability of various classes of compacts under increasing countable compact union.

Let $\mathscr{C}$ be a hereditary class of compact subsets of a space E. The class $\mathscr{C}_{\mathscr{K}}^{\alpha\uparrow}$ is defined by induction on the ordinal α:

$$\mathscr{C}_{\mathscr{K}}^{\uparrow} = \mathscr{C}^{\uparrow} \cap \mathscr{K}(E), \mathscr{C}_{\mathscr{K}}^{<\alpha\uparrow} = \bigcup_{\beta<\alpha} \mathscr{C}_{\mathscr{K}}^{\beta\uparrow},$$

$$\mathscr{C}_{\mathscr{K}}^{\alpha\uparrow} = \left(\mathscr{C}_{\mathscr{K}}^{<\alpha\uparrow}\right)_{\mathscr{K}}^{\uparrow} \text{ and } \mathscr{C}_{\mathscr{K}}^{\infty\uparrow} = \mathscr{C}_{\mathscr{K}}^{<\omega_1\uparrow}.$$

Note that $\mathscr{C}_{\mathscr{K}}^{\infty\uparrow}$ is the closure of $\mathscr{C}$ under increasing countable compact union.

According to Proposition 3.5, $\mathscr{C}^{\mathscr{P}} \subseteq \mathscr{C}_{\mathscr{K}}^{\uparrow}$, whence we deduce the following results.

THEOREM 3.28. *There exists a compact set in $D_{\mathscr{K}}^{\alpha\uparrow}$ which is not in $L_0^{<\alpha\uparrow}$ for each countable ordinal α.*

Consequently, the classes of compact sets D, H, L, L_0, N_0, R and A are not closed under increasing countable compact union.

Proof. Let $(E_n)_{n\in\omega}$ be as in Theorem 3.25 and α a countable ordinal. Since $\mathscr{P}_\alpha \not\preceq \mathscr{F}r_\beta$ for all ordinal $\beta < \alpha$, $H_{\mathscr{P}_\alpha}(E_n) \notin L_0^{<\alpha\uparrow}$. But $H_{\mathscr{P}_\alpha}(E_n) \in D_{\mathscr{K}}^{\alpha\uparrow}$ because $\mathscr{P} \preceq \mathscr{F}r$. □

COROLLARY 3.29. *There exists a compact $\mathscr{N}$-set which is not an $\mathscr{A}$-set. More precisely, there exists a $D^{2\uparrow}_{\mathscr{K}}$-set, which is both an $\mathscr{N}$-set and a set of uniqueness, but is not an $\mathscr{A}$-set.*

Note that the H^{σ}-sets (and so the $D^{2\uparrow}_{\mathscr{K}}$-sets) are sets of uniqueness [9], [2]. T. W. Körner proved the existence of a compact $\mathscr{N}$-set which is not a set of uniqueness (and so which is not an $\mathscr{A}$-set) [17].

Let us recall that $\mathscr{N} = N^{\uparrow}$.

COROLLARY 3.30. (a) $\mathscr{N}_0 \subset N_0^{\uparrow}$ *but* $N_0^{\uparrow} \not\subset \mathscr{N}_0$.
(b) $\mathscr{A} \not\subset A^{\uparrow}$ *and* $A^{\uparrow} \not\subset \mathscr{A}$.

Proof. Each element of $\mathscr{N}_0$ is a subset of a $\mathscr{K}_\sigma$ set in $\mathscr{N}_0$ which is an $N_0^{\uparrow}$-set. Conversely, $D^{2\uparrow}_{\mathscr{K}} \subset N_0^{\uparrow} \subset A^{\uparrow}$ and $D^{2\uparrow}_{\mathscr{K}} \not\subset \mathscr{A}$, thus $N_0^{\uparrow} \not\subset \mathscr{N}_0$ and $A^{\uparrow} \not\subset \mathscr{A}$. Moreover $A \subset N$, thus $A^{\uparrow} \subset N^{\uparrow} = \mathscr{N}$. In view of Theorem 2.11, $\mathscr{A} \not\subset \mathscr{N}$, thus $\mathscr{A} \not\subset A^{\uparrow}$. □

THEOREM 3.31. *For all $n \geq 1$, there exists a compact set in $\mathscr{N}_0$ (even in $D^{\uparrow}$) which is the union of $n+1$ Dirichlet sets and is not a union of n L_0-sets.*

Proof. Let $(E_k)_{k\in\omega}$ be as in Theorem 3.25 and $n \geq 1$. Since $\mathscr{P}^{(n)} \not\preceq \mathscr{U}^{(n)}$, $H_{\mathscr{P}^{(n)}}(E_k) \notin L_0^{\mathscr{U}^{(n)}}$; thus $H_{\mathscr{P}^{(n)}}(E_k)$ is not a union of n L_0-sets. But $\mathscr{P}^{(n)} \preceq \mathscr{F}r$ and $\mathscr{P}^{(n)} \preceq \mathscr{U}^{(n+1)}$, whence $H_{\mathscr{P}^{(n)}}(E_k) \in D^{\uparrow} \subset \mathscr{N}_0$ and $H_{\mathscr{P}^{(n)}}(E_k)$ is a union of $n+1$ Dirichlet sets. □

THEOREM 3.32. *For all $n \geq 1$, there exists a compact set which is the union of $n+1$ Dirichlet sets and is not a union of n $L_0^{\infty\uparrow}$-sets.*

Proof. Let $(E_k)_{k\in\omega}$ be as in Theorem 3.25, let $n \geq 1$ and let α be a countable ordinal. Since $\mathscr{U}^{(n+1)} \not\preceq \mathscr{U}^{(n)} \otimes \mathscr{F}r_\alpha$, $H_{\mathscr{U}^{(n+1)}}(E_k) \notin (L_0^{\mathscr{F}r_\alpha})^{\mathscr{U}^{(n)}}$, thus $H_{\mathscr{U}^{(n+1)}}(E_k)$ is not a union of n $L_0^{\alpha\uparrow}$-sets for each $\alpha < \omega_1$, therefore $H_{\mathscr{U}^{(n+1)}}(E_k)$ is not a union of n $L_0^{\infty\uparrow}$-sets. □

THEOREM 3.33. *There exists a $\mathscr{K}_{\sigma\delta}$ set in $D^{\sigma} \cap \mathscr{W}\mathscr{D}$ (more precisely in $D^{\mathscr{A}s}$) which is not in $(\mathscr{N} \cup L_0)^{\infty\uparrow}$.*

Proof. Let $(E_k)_{k\in\omega}$ be as in Theorem 3.25. Since $\mathscr{A}s \not\preceq \mathscr{F}r_\alpha$ for each countable ordinal α, $H_{\mathscr{A}s}(E_k) \notin L_0^{\infty\uparrow}$. According to Theorem 3.23, $H_{\mathscr{A}s}(E_k) \notin \mathscr{N}^{\infty\uparrow}$. Moreover,

$$(\mathscr{N} \cup L_0)^{\infty\uparrow} = \mathscr{N}^{\infty\uparrow} \cup L_0^{\infty\uparrow}$$

and $H_{\mathscr{A}s}(E_k)$ is a $\mathscr{K}_{\sigma\delta}$ set. □

4. Pseudo and asymptotic classes

4.1. $\mathscr{F}$-uniform convergence

Let $\mathscr{F}$ be a free filter on ω, i.e., a monotone basis $\mathscr{F}$ containing $\mathscr{F}r$ and closed under finite intersection. Let $(f_n)_{n\in\omega}$ be a sequence of real valued maps on a set E. We shall say that $(f_n)_{n\in\omega}$ *converges $\mathscr{F}$-uniformly* to f if there exists a sequence $\varepsilon_n \to 0^+$ such that for each $x \in E$,

$$\{n \in \omega;\, |f_n(x) - f(x)| < \varepsilon_n\} \in \mathscr{F}.$$

With $\mathscr{F} = \{\omega\}$, we obtain the usual uniform convergence. The $\mathscr{F}r$-uniform convergence was introduced by A. Denjoy [6, p. 183] under the name of *pseudo-uniform convergence* and was also studied under the name of quasi-normal convergence or equal convergence [5]. The pseudo-uniform convergence was considered in the present context by N. Bary [2], J. Arbault [1] and Z. Bukovská [4].

A subset X of $\mathbf{T}$ is a *set of type $D - \mathscr{F}$* if there exists a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that $\|n_k\cdot\|$ converge $\mathscr{F}$-uniformly to 0, i.e., if there exists a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers and a sequence $\varepsilon_k \to 0^+$ such that for each $x \in X$,

$$\{k \in \omega;\, \|n_k x\| < \varepsilon_k\} \in \mathscr{F}.$$

A subset X of $\mathbf{T}$ is a *set of type $H - \mathscr{F}$* if there exist an interval I of $\mathbf{T}$ and a strictly increasing sequence $(n_k)_{k\in\omega}$ of integers such that for each $x \in X$,

$$\{k \in \omega;\, n_k x \notin I\} \in \mathscr{F}.$$

A subset X of $\mathbf{T}$ is a *set of type $L_0 - \mathscr{F}$* if there exist a sequence $\varepsilon_n \to 0^+$, $\alpha > 0$ and for each $n \in \omega$, a finite sequence (I_k^n) of intervals such that $|I_k^n| \le \varepsilon_n$ for each k and $d(I_k^n, I_{k'}^n) \ge \alpha\varepsilon_n$ for each $k \ne k'$, such that for each $x \in X$,

$$\left\{n \in \omega;\, x \in \bigcup_k I_k^n\right\} \in \mathscr{F}.$$

Note the inclusions between these classes.

PROPOSITION 4.2. *We have $D^{\mathscr{F}} \subseteq D - \mathscr{F} \subseteq H - \mathscr{F} \subseteq L_0 - \mathscr{F}$ for each free filter $\mathscr{F}$ on ω.*

For each class $\mathscr{C} = D$, H or L_0, a set of type $\mathscr{C} - \mathscr{F}r$ will be called pseudo $\mathscr{C}$-set and a set of type $\mathscr{C} - \mathscr{A}s$ an asymptotic $\mathscr{C}$-set. The pseudo Dirichlet sets were considered by N. Bary who proved they are in $\mathscr{N}_0$. Z.

Bukovská noted that the pseudo Dirichlet sets are exactly the $D^{\uparrow}$-sets. We can complete this result.

PROPOSITION 4.3. *We have $D - \mathscr{F}r_{\alpha} = D^{\alpha\uparrow}$ for each countable ordinal α.*

Proof. We have just to prove that $D - \mathscr{F}r_{\alpha} \subseteq D^{\alpha\uparrow}$ for each countable ordinal α. In the case $\alpha = 1$, consider the set X of all $x \in \mathbf{T}$ such that $\{k \in \omega;\ \|n_k x\| < \varepsilon_k\} \in \mathscr{F}r$. Then $X \subseteq \bigcup_{i \in \omega}\{k \in \omega;\ \|n_k x\| < \varepsilon_k$ *for each* $k \geq i\}$, which is an increasing union of Dirichlet sets. Let α be a countable ordinal > 1. Consider the set X of all $x \in \mathbf{T}$ such that $\{k \in \omega;\ \|n_k x\| < \varepsilon_k\} \in \mathscr{F}r_{\alpha}$. Using the definition of $\mathscr{F}r_{\alpha}$ in Part 3.8, we have that

$$\left\{j \in \omega;\ \left\{k \in \omega;\ \|n_{\langle j,k\rangle} x\| < \varepsilon_{\langle j,k\rangle}\right\} \in \mathscr{F}r_{[\alpha]_j}\right\} \in \mathscr{F}r$$

for each $x \in X$. Therefore $X = H_{\mathscr{F}r}(X_j)$ where X_j is the set of all $x \in \mathbf{T}$ such that

$$\left\{k \in \omega;\ \|n_{\langle j,k\rangle} x\| < \varepsilon_{\langle j,k\rangle}\right\} \in \mathscr{F}r_{[\alpha]_j}$$

which is clearly a set of type $D - \mathscr{F}r_{[\alpha]_j}$. So we conclude the proof by induction on α. □

4.4. Asymptotic Dirichlet sets

The asymptotic H-sets were considered by R. Lyons because they are annihilated by the same measures as the H-sets [20]. We are interested in the class of all asymptotic Dirichlet sets which provides examples of "large" weak Dirichlet sets.

PROPOSITION 4.5. *All asymptotic Dirichlet sets are weak Dirichlet sets.*

Proof. Let X be an asymptotic Dirichlet set and let $(n_k)_{k \in \omega}$ and $(\varepsilon_k)_{k \in \omega}$ be sequences witnessing that. Then $X \subset H_{\mathscr{A}s}(X_k)$ with $X_k = \{x \in \mathbf{T};\ \|n_k x\| < \varepsilon_k\}$. By Lemma 3.5 we have $\mu(X) = \sup_k(X \cap X_k) = \sup_{(k_i)} \mu(X \cap \bigcap_i X_{k_i})$ and $\bigcap_i X_{k_i} \in D$ for each sequence $(k_i)_{i \in \omega}$. □

THEOREM 4.6. *There exists an asymptotic Dirichlet set which is not covered by a countable union of $\mathscr{N}$ and L_0-sets.*

More precisely, if $(a_k)_{k \in \omega}$ and $(b_k)_{k \in \omega}$ are two sequences of integers such that

$$b_k - a_k \to +\infty \text{ and } a_{k+1} - b_k \to +\infty$$

as $k \to +\infty$, then $X = H_{\mathscr{A}s}(K_{[a_k, b_k]})$ is in $(D - \mathscr{A}s)$ and not in $(\mathscr{N} \cup L_0)^{\sigma}$.

Proof. Since $b_k - a_k \to +\infty$ as k, X clearly belongs to $D - \mathscr{A}s$. Let $(\Theta_q)_{q\in\omega}$ be a sequence of elements of $\mathscr{M}_\infty^+(\omega)$ and let $(K_j)_{j\in\omega}$ be a sequence of elements of L_0. We will find x in

$$X \setminus \left(\bigcup_{q\in\omega} G_{\Theta_q} \cup \bigcup_{j\in\omega} K_j \right)$$

with dyadic decomposition $x = \Sigma_{i\geq 1}\varepsilon_i 2^{-i}$ for a suitable choice of ε_i's either 0 or 1.

Let

$$A = \bigcup_{k\in\omega} [a_k - 5, b_k - 3] \quad \text{and} \quad Q = \left\{ q \in \omega;\, \Theta_q(2^A) = +\infty \right\}.$$

Let $u_k^q = \Theta_q(2^{[a_k - 5, b_k - 3]})$ for each $q \in Q$ and $k \in \omega$. Recall an argument used in the proof of Theorem 3.23: let $(u_n^q)_{n\in\omega}$ be sequences of non negative reals such that $\Sigma_{n\in\omega} u_n^q = +\infty$ for each $q \in Q$; there exists a set C of density zero (i.e., $C^c \in \mathscr{A}s$) such that $\Sigma_{n\in C} u_n^q = +\infty$ for each $q \in Q$. Let

$$A' = \left(\bigcup_{k\in C^c} [a_k - 5, b_k - 3] \right)^c \quad \text{and} \quad B' = A' + [3,5].$$

Note that $\Theta_q(2^{A'}) = +\infty$ for each $q \in Q$, and $B'^c = \bigcup_{k\in C}[a_k, b_k]$.

Let $j \in \omega$. Since $K_j \in L_0$, there exist a sequence $\varepsilon_j^n \to 0^+$, $\alpha_j > 0$ and for each n a finite sequence (I_l^n) of intervals such that $|I_l^n| \leq \varepsilon_j^n$ for each l, $d(I_l^n, I_{l'}^n) \geq \alpha_j \varepsilon_j^n$ for each $l \neq l'$ and $K_j \subset \bigcup_l I_l^n$. Let $p_j = \sup\{-\lfloor \log_2 \alpha_j \rfloor, 0\}$, $m_j^n = -\lfloor \log_2 \varepsilon_j^n \rfloor$ and

$$J_j^n = \left[m_j^n - 1, m_j^n + p_j + 1 \right] \quad \text{for each } n \in \omega.$$

Since $a_{k+1} - b_k \to +\infty$ as k, J_j^n meets at most one interval $[a_k, b_k]$ for large enough n. Therefore, there exists an integer n_j for each $j \in \omega$ such that the J_j's, where $J_j = J_j^{n_j}$, are pairwise disjoint and the set C' of all k such that $[a_k, b_k]$ meets at most one interval, J_j, is a set of density zero.

Since $\mathscr{A}s$ is a filter, $C'' = C \cup C'$ is a set of density zero. The values of ε_i are set equal to 0 for all $i \in \bigcup_{k\in C''^c}[a_k, b_k] = B''$. Thus $x \in X = H_{\mathscr{A}s}(K_{[a_k, b_k]})$. Furthermore, simultaneously using Lemma 3.17 and 2.9, we can choose a sequence $(\varepsilon_i)_{i\in B''^c}$ of 0's and 1's such that $x = \Sigma_{i\geq 1}\varepsilon_i 2^{-i} \notin \bigcup_{q\in\omega} G_{\Theta_q} \cup \bigcup_{j\in\omega} K_j$. □

4.7. Another result of antistability

We prove that classes of type $\mathscr{C} - \mathscr{F}$ do not have properties of stability. In particular, there exists a $D^{2\uparrow}$-set which is not an asymptotic Dirichlet set.

THEOREM 4.8. *Let $\mathscr{F}$ be a monotone basis on ω and let $\mathscr{G}$ be a free filter on ω with $\mathscr{F} \nsubseteq \mathscr{G}$. There exists a set in $D^{\mathscr{F}}$ which is not in $L_0 - \mathscr{G}$.*

More precisely, if $(A_n)_{n \in \omega}$ is a sequence of colacunary subsets of ω such that as $k \to +\infty$,

$$d(A_n \cap [k, +\infty[, A_m \cap [k, +\infty[) \to +\infty$$

uniformly for all $n \neq m$, then

$$X = H_{\mathscr{F}}(K_{A_n}) \in D^{\mathscr{F}} \setminus (L_0 - \mathscr{G}).$$

Proof. Consider a sequence $\varepsilon_k \to 0^+$, $\alpha > 0$ and for each $k \in \omega$, a finite sequence (I_l^k) of intervals such that $|I_l^k| \leq \varepsilon_k$ for each l and $d(I_l^k, I_{l'}^k) \geq \alpha\varepsilon_k$ for each $l \neq l'$. We will find $x \in X$ with dyadic decomposition $x = \sum_{i \geq 1} \varepsilon_i 2^{-i}$ for a suitable choice of ε_i's either 0 or 1 and such that $\{k \in \omega;\ x \in \bigcup_l I_l^k\} \notin \mathscr{G}$.

Let $p = \sup\{-\lfloor \log_2 \alpha \rfloor, 0\}$, $m_k = -\lfloor \log_2 \varepsilon_k \rfloor$ and $J_k = [m_k - 1, m_k + p + 1]$ for each $k \in \omega$. Since as $k \to +\infty$, $m_k \to +\infty$ and $d(A_n \cap [k, +\infty[, A_m \cap [k, +\infty[) \to +\infty$ uniformly for all $n \neq m$, J_k meet at most one A_n for large enough k, i.e., $k \geq k_0$. Thus for each $k \geq k_0$, there exists an integer n_k such that J_k meets at most A_{n_k} and the J_k's are pairwise disjoint.

Consider the map $\varphi : \omega \to \omega$ which takes k to n_k. Since $\mathscr{F} \nsubseteq \mathscr{G}$, there exists a $P \in \hat{\mathscr{G}}$ such that $\varphi(P)^c \in \mathscr{F}$. Since $\mathscr{G}$ is a free filter, $P' = P \cap [k_0, +\infty[$ belongs to $\hat{\mathscr{G}}$. Using Lemma 2.9, by induction on $k \in P'$, we pick ε_i for $i \in J_k$ to insure that $x = \sum_{i \geq 1} \varepsilon_i 2^{-i}$ is not in $\bigcup_l I_l^k$. The other ε_i's are set equal to 0. Since $P' \in \hat{\mathscr{G}}$, $\{k \in \omega;\ x \notin \bigcup_l I_l^k\} \in \hat{\mathscr{G}}$, whence

$$\left\{k \in \omega;\ x \in \bigcup_l I_l^k\right\} \notin \mathscr{G}.$$

Furthermore, $\varphi(P')^c \in \mathscr{F}$, whence $\{n \in \omega;\ x \in K_{A_n}\} \in \mathscr{F}$, so that $x \in X = H_{\mathscr{F}}(K_{A_n})$. □

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