ILLINOIS JOURNAL OF MATHEMATICS
Volume 37, Number 3, Fall 1993

# TOTAL CURVATURE OF FOLIATIONS

R. LANGEVIN AND C. POSSANI

The total curvature of foliations will be studied with the extensive use of integral geometry. More precisely there is a dictionary between curvature integrals and counting of contact points. Each correspondence of the dictionary will be called an "exchange theorem".

It is not a surprise that lower bounds of curvature integrals of a foliation given on the boundary of a domain depend on the integral geometry of the boundary data.

## I. Exchange theorem

Throughout this paper we'll use the so called "Exchange Theorem". In this paragraph we will state it and make some remarks. For the proof we refer the reader to [B.L.R], or [L].

Let $W \subset \mathbf{R}^{n+1}$ be an open set with $\overline{W}$ compact, $U \subset W$ an open subset of $W$, and $\mathscr{F}$ a codimension one foliation of $W$ with isolated singularities. If $u \in S^n \subset \mathbf{R}^{n+1}$, let $L(u)$ be the oriented line through the origin of $\mathbf{R}^{n+1}$. For each $t \in L(u)$ let $T_t$ be the hyperplane orthogonal to $L(u)$ passing through $t$. We define $|\mu|(\mathbf{F}, T_t) \in \mathbf{N} \cup \infty$ to be the number of points of $U$ where the hyperplane $T_t$ is tangent to a leaf of $\mathbf{F}$. This number is finite for almost all pairs $(L(u); t)$.

Let $K(x)$ be the Lipschitz-Killing curvature of the leaf through the point $x$.

With these notations we have:

I.1. EXCHANGE THEOREM.

$$\int_{P^n}\int_{L(u)} |\mu|(\mathscr{F}; T_t)\, dt\, du = \int_U |K|(x)\, dx \tag{I.1}$$

(*Both sides of the equality may be infinite*).

Received September 10, 1991.
1991 Mathematics Subject Classification. Primary 57R30; Secondary 51M99.

This inequality may, of course, be written as

$$(\mathrm{I}.2)\qquad 1/2\int_{S^n}\int_{L(u)}|\mu|(\mathcal{F},T_t)\,dt\,du = \int_U |K|(x)\,dx$$

When $n = 1$, that is when $W$ is an open set in $R^2$ with compact closure, the left integral can be understood as an integral over the space $\mathcal{G}$ of non-oriented lines of the plane. This set is endowed with the density $m$ associated to the form $dr\,d\theta$ where $r$ is the distance of the line $T$ to the origin, and $\theta$ the angle between the axis $0x$ and the line $T$. For more details see [S].

The exchange theorem becomes

$$(\mathrm{I}.3)\qquad \int_U |K|(x)\,dx = \int_{\mathcal{G}} |\mu|(\mathcal{F},L)\,dm(L).$$

Instead of considering contact with lines, we can count contacts between the leaves of $F$ and line segments of fixed length. Let $\mathcal{G}_d$ be the set of non-oriented line segments in $R^2$ of length $d$. We parametrize a segment $I$ of $\mathcal{G}_d$ by coordinates $(r,\theta,s)$ where $(r,\theta)$ are the coordinates of the line $L$ supporting $I$ and $s$ the coordinate of the center of $I$ in $L$. This gives a density $m = |dr \wedge d\theta \wedge ds|$ on $\mathcal{G}_d$ which is invariant by the action of the isometries, and therefore another version of the exchange theorem is

$$(\mathrm{I}.4)\qquad \int_U |K|(x)\,dx = \frac{1}{d}\int_{\mathcal{G}_d} |\mu|(\mathcal{F},I)\,dm(I)$$

From now on we will denote by $K(\mathcal{F})$ the integral $\int_U |K|(x)\,dx$.

If we consider the analogous situation in a hyperbolic surface $M$, then, although the set of geodesics of $M$ cannot be given a measure invariant by the isometries, the measure $m = |\mathrm{coshr.dr} \wedge d\theta \wedge ds|$ on the set $\mathcal{G}_d$ of geodesic segments of length $d$ is invariant by the action of "local isometries" and I.4 still holds, see [L.L1].

## II. Total curvature of foliations of bounded plane domains

In this paragraph we will consider foliations of a given domain $D \subset \mathbf{R}^2$ homeomorphic to the unit disc $D^2 = \{(x,y) \in \mathbf{R}^2 | x^2 + y^2 \leq 1\}$, and we suppose that its boundary $\partial D$ is piecewise $\mathcal{C}^2$.

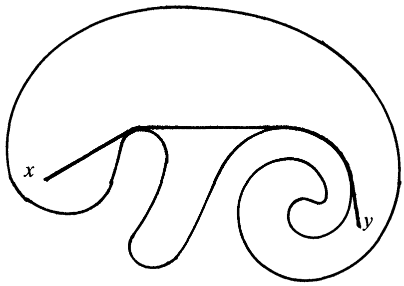


FIG. 1

II.1 DEFINITION. The distance between two points $x$ and $y$ in $D$ is defined by

$$d(x, y) = \inf\{l(\gamma)|\gamma : [a, b] \to D \text{ is a regular curve},$$
$$\gamma(a) = x, \gamma(b) = y \text{ and } l(\gamma) \text{ is the length of } \gamma\}$$

It is clear that $d$ is a metric in $D$. In fact, as $D \approx D^2$ there always exists exactly one curve $\gamma$ joining $x$ and $y$ which satisfies $l(\gamma) = d(x, y)$. Such a $\gamma$ is called a geodesic of $D$.

II.2. DEFINITION. The diameter of $D$ is defined as

$$d = \sup\{d(x, y)|x, y \in D\}.$$

(see Fig. 1)

Let $\mathscr{F}$ be a foliation of curves of $D$, tangent to $\partial D$, with isolated singularities of positive index. $\mathscr{F}$ need not be orientable.

II.3. THEOREM. *With the above notations we have*

$$K(\mathscr{F}) \geq l(\partial D) - 2d.$$

This theorem is analogous to one of $R$. Langevin and G. Levitt that asserts if $\mathscr{F}$ is an orientable foliation of the disc $D^2$, tangent to the boundary, then

$$K(\mathscr{F}) \geq 2\pi - 4.$$

*Proof.* First of all, let us define the index of a singularity of a non-orientable foliation. The index of a singularity $P$ is an integer which measures the variation over $\mathbf{RP}^1$ of the field of directions of the leaves of $\mathscr{F}$ around a neighbourhood of $P$. The index we consider is the above integer index divided by 2. If the singularity is orientable, then this index is the usual one.

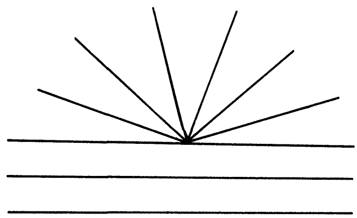

FIG. 2

We will suppose in this paragraph that $\mathscr{F}$ has only singularities of positive index.

We are going to show that we can suppose $\mathscr{F}$ has 2 singularities of sunset type, which are of index $1/2$ (Fig. 2).

All singularities with the property that the oscillation of its field of directions is uniformly bounded in any circle centered at the singularity, can be substituted by a sunset or a source increasing the total curvature of $\mathscr{F}$ as little as desired.

This can be done by considering in the boundary of the disc $D_r$, with center at the singular point $P$ and radius $r$, a homotopy between the angle function determined by $\mathscr{F}$ and the angle function of one of the foliations in Fig. 3.

A source can be replaced by two sunsets by the modification in Fig. 4.

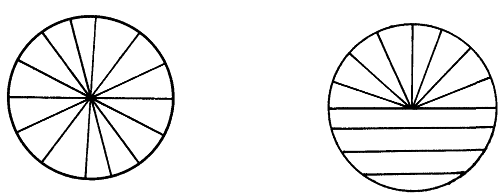

FIG. 3

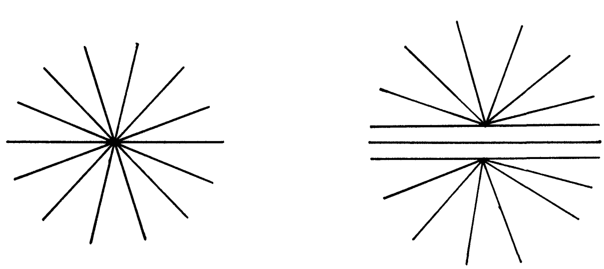

FIG. 4

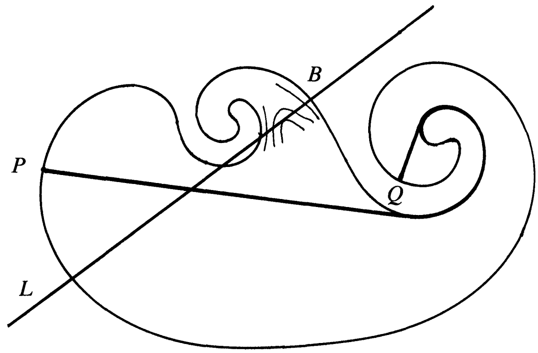


FIG. 5

If the oscillation of the field of directions of $\mathscr{F}$ in the neighbourhood of $P$ is not uniformly bounded then $K(\mathscr{F}) \to \infty$ in a neighbourhood of $P$ and we are done.

Let $P$ and $Q$ be the two sunsets of $\mathscr{F}$, and $\gamma$ a geodesic of $D$ joining them. We are going to estimate the number of points of contact of leaves of $\mathscr{F}$ and a line $L$ of the plane. Except for a set of lines of measure zero, each line $L$ meets $D$ in a finite number of segments.

If $\overline{AB}$ is a segment of $L \cap D$ and $\overline{AB} \cap \gamma = \varnothing$ then $\overline{AB}$ divides $D$ into two discs, one of them containing $\{P, Q\}$. In the other disc, $\mathscr{F}$ is orientable and certainly there's at least one point of contact between $\overline{AB}$ and leaves of $\mathscr{F}$ (see Fig. 5).

Let $n(L)$ be the number of segments of $L \cap D$ in which $L$ meets $\gamma$, and $c(L)$ the number of segments of $L \cap D$ in which $L$ doesn't meet $\gamma$. Then we have

$$|\mu|(\mathscr{F}; L) \geq c(L). \tag{II.4}$$

Cauchy's formula [S] gives us

$$l(c) = \int_{\mathscr{G}} \sharp\{\text{segments } L \cap D\} = \frac{1}{2}\int_{\mathscr{G}} \sharp\{L \cap c\}$$

if $c$ is a piecewise regular closed curve boundary of a disc $D$,

$$l(c) = \frac{1}{2}\int_{\mathscr{G}} \sharp\{L \cap c\} \quad \text{if } c \text{ is a piecewise regular arc.}$$

Then we have

$$l(\partial D) = \int_{\mathscr{G}} (n(L) + c(L)) = \int_{\mathscr{G}} n(L) + \int_{\mathscr{G}} c(L) = \int_{\mathscr{G}} \sharp\{L \cap \gamma\} + \int_{\mathscr{G}} c(L)$$

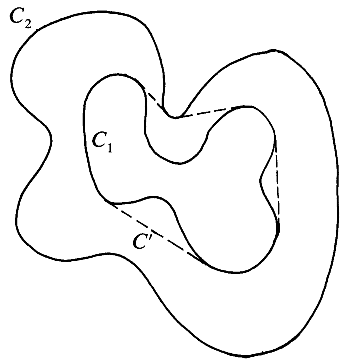


FIG. 6

Applying II.4 and the exchange theorem,

$$l(\partial D) \leq 2l(\gamma) + \int_{\mathscr{G}} |\mu|(\mathscr{F}; L) = 2l(\gamma) + K(\mathscr{F}),$$

and then

$$K(\mathscr{F}) \geq l(\partial D) - 2l(\gamma) \geq l(\partial D) - 2d.$$

With the same techniques we can analyse the case where $D$ is a region of $\mathbf{R}^2$ homeomorphic to the annulus $A = \{(x, y) \in \mathbf{R}^2 | 0 < r_1 \leq x^2 + y^2 \leq r_2\}$, with boundary $\partial D = C_1 \cup C_2$ where $C_1$ and $C_2$ are piecewise regular closed curves (Fig. 4). Suppose $C_1$ is the interior curve of $\partial D$.

II.5. DEFINITION. If $D$ is a region in $\mathbf{R}^2$ homeomorphic to the annulus $A$, with piecewise regular boundary $\partial D = C_1 U C_2$, $C_1$ being the interior curve, we call interior convex envelope of $C_1$ the curve $C_1'$ which is the boundary of the convex-hull of $C_1$ in $D$, that is the boundary of the intersection of all sets of $\mathbf{R}^2$ that are contained in the interior of $C_2$ and convex in $D$ (Fig. 6).

We can then prove in an analogous way:

II.6. THEOREM. *If $D \subset \mathbf{R}^2$ is a region homeomorphic to the annulus $A$ as above, and if $\mathscr{F}$ is a foliation of $D$, without singularities, tangent to the boundary then*

$$K(\mathscr{F}) \geq l(C_2) + l(C_1) - 2l(C_1')$$

If, in theorem II.3, the curve $\partial D$ is convex then there exists a tight foliation of $D$. By a tight foliation we mean a foliation $\mathscr{F}$ that realizes the minimum

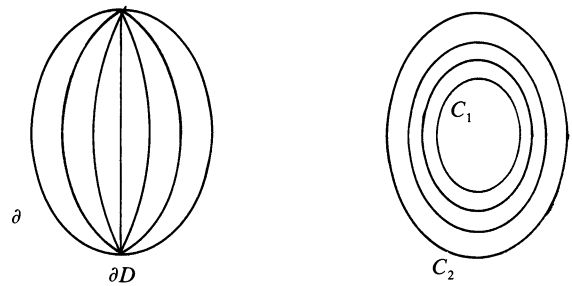


FIG. 7

of $K(\mathscr{F})$. Analogously, in Theorem II.6, if $C_1$ and $C_2$ are convex, $C_1' = C_1$, $K(\mathscr{F}) \geq l(C_1) - l(C_1)$, then there exist tight foliations (see Fig. 7).

If $\partial D$ in Theorem II.3, or $C_i$, $i = 1, 2$, in Theorem II.6 are not convex then we can show that there do not exist tight foliations of $D$.

The non-existence of tight foliations when the boundary of the domain in not a convex curve follows from the following fact: if $P \in \partial D$ is a point of inflexion of $\partial D$ and a regular point for $\mathscr{F}$, then there will be an open set of lines in the plane with the property that each line in this set has more than one point of contact with the leaves of $\mathscr{F}$ in a neighbourhood of $x$.

But in this case we can exhibit a sequence of foliations of $D$, $(\mathscr{F}_n)$ with

$$K(\mathscr{F}_n) \to l(\partial D) - 2d.$$

We can think of the limit of this sequence of foliations as a foliation having $\partial D$ as critical set (In general, this critical set could be a union of curves), with the "curvature concentrated in $\partial D$" (see Fig. 8). (More details in [P].)

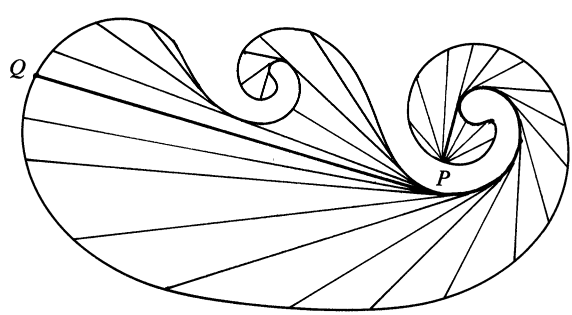


FIG. 8

## III. Foliations in a hyperbolic surface

So far the only known theorem concerning total curvature of foliations in a hyperbolic surface is due to R. Langevin and G. Levitt [L.L.1] and asserts that if $M$ is a hyperbolic compact surface, without boundary and $\mathscr{F}$ is a foliation of $M$ whose singularities are all isolated and of saddle type, then

$$K(\mathscr{F}) \geq (12\log 2 - 6\log 3)|\chi(M)|.$$

In this section we will prove a theorem for two orthogonal foliations in a compact hyperbolic surface $M$, without boundary. Our approach will be quite similar to the one Langevin and Levitt used in [L.L.2] to establish an analogous theorem for two orthogonal foliations in the sphere $S^2 \subset \mathbf{R}^3$.

First of all let us recall that the universal covering of a compact hyperbolic surface without boundary is the hyperbolic plane $H^2$, and we will think of it as the Poincaré disc.

In hyperbolic geometry the interior angle of a regular $n$-gon is not determined by $n$, a priori. Given $n$, there exist a regular polygon of $n$ sides in $H^2$, with interior angle $\alpha$, for each value of $\alpha$ satisfying

$$0 < \alpha < \frac{n-2}{n}.\pi.$$

For technical reasons, that will be clear later, we need $n$ odd and $\alpha$ equal to $\pi/2$. The smallest $n$ we can get is 5. Given $n$ and $\alpha$ then the length, $l$, of the side of the regular $n$-gon in $H^2$ becomes determined. For $n = 5$ and $\alpha = \pi/2$ we have $l \approx 4{,}275$.

We can now state our theorem.

III.1. Theorem. *Let $M$ be a compact, without boundary, hyperbolic (that is, with a metric of constant curvature $-1$) surface and let $\mathscr{F}$ and $\mathscr{F}^{\perp}$ be orthogonal foliations of $M$ with isolated singularities. Then*

$$K(\mathscr{F}) + K(\mathscr{F}^{\perp}) \geq \frac{2\pi^2}{5l}|\chi(M)|,$$

*where $l$ is the length of the side of the regular pentagon in $H^2$, whose interior angles are $\pi/2$.*

Before proving this we make some definitions and remarks.

III.2. Definition. We let $G$ be space of all orientation preserving isometries of $H^2$. We define an equivalence relation in $G$ by

$$g_1 \approx g_2 \Leftrightarrow p \circ g_1 \circ g_2^{-1} = p.$$

It's easy to check that $\approx$ is actually an equivalence relation. We will denote $G/\approx$ by $\overline{G}$.

III.3. Definition. As in Section I:
$\mathscr{G}$ is the space of all geodesics of $H^2$, with the canonical measure given by the density $m = \cosh r\, dr\, d\theta$.
$\mathscr{G}_r$ is the space of all geodesic segments of length $r > 0$, with the density $m = \cosh r\, dr\, d\theta\, ds$.
We identify, in $H^2$, geodesic segments which project by $p$ over the same geodesic arc in $M$.

III.4. Definition. In $\mathscr{G}_r$ we define an equivalence relation by $C_1 \approx C_2 \Leftrightarrow (g \in G,\ g(C_1) = C_2 \Rightarrow [g] = [Id]$ or $[g] = [\theta_A]$, where $A$ is the middle point of $C_2$ and $\theta_A$ is the geodesic symmetry through $A$).

It's easy to see that $\approx$ is an equivalence relation, and we indicate $\mathscr{G}_{r/\approx}$ by $\overline{\mathscr{G}}_r$. It's not hard to check that the measure in $\mathscr{G}_r$ induces a measure in $\overline{\mathscr{G}}_r$.
We define now a measure in $\overline{G}$. Consider a fixed segment $C_0$ in $H^2$, of length $r$. For each $C \in \mathscr{G}_r$ we have two isometries $g_i \in G$, $i = 1, 2$ with the property that $g_i(C_0) = C$. (They satisfy $g_1 = \theta_A \circ g_2$). So $G$ is a two-fold covering of $\mathscr{G}_r$ and we can consider in $G$ a measure induced by $\mathscr{G}_r$. This measure induces, through the canonical projection, a measure in $\overline{G}$.

*Remark*. The measures in $\overline{\mathscr{G}}_r$ and in $\overline{G}$ could have been obtained by parametrizing the geodesic segments by the position in $M^2$ of its middle point, and by the direction of the tangent line at this point. The above construction allows us to think of an equivalence class $[g]$ as a geodesic segment in $M$. We return now to our theorem.

*Proof of Theorem III.*1. Let $\mathscr{F}$ be a foliation of $M$ and $\mathscr{F}^{\perp}$ be the orthogonal foliation. $\mathscr{F}$ and $\mathscr{F}^{\perp}$ induce by $p: H^2 \to M$, foliations in $H^2$, and we will also denote them by $\mathscr{F}$ and $\mathscr{F}^{\perp}$. As $p$ is locally an isometry, and considering the identifications we made above, we have

$$K(\mathscr{F}) = \frac{1}{r}\int_{\overline{\mathscr{G}}_r} |\mu|(\mathscr{F};[C])$$

where $[C]$ is the equivalence class of a segment $C$ of length $r$, and $\mu(\mathscr{F};[C])$ is the number of points of contact between leaves of $\mathscr{F}$ in $H^2$ and a representative of the class $[C]$.

We can also consider $K(\mathscr{F})$ as an integral on the space $\overline{G}$:

$$K(\mathscr{F}) = \frac{1}{2l}\int_{\overline{G}} |\mu|(\mathscr{F};[g](C))$$

where $C$ is fixed, with $l(C) = l$, $|\mu|(\mathscr{F};[g](C))$ is the number of contact points between $\mathscr{F}$ in $H^2$ and $g(C)$, $g \in [g]$.

The factor $\frac{1}{2}$ appears in this equality since each point of contact between leaves of $\mathscr{F}$, and a segment $C$ will be counted twice in $\overline{G}$.

We can fix $C$, and let $\overline{G}$ act on $\mathscr{F}$ to obtain

$$K(\mathscr{F}) = \frac{1}{2l}\int_{\overline{G}} |\mu|([g]\mathscr{F};C),$$

where $|\mu|(g,\mathscr{F};C)$ is the number of points of contact between leaves of $g(\mathscr{F})$, $g \in [g]$ and $C$.

From now on we will fix $P$, a regular pentagon in $H^2$ whose interior angles are $\pi/2$ and count contacts between its sides and leaves of $g(\mathscr{F})$. We have

$$K(\mathscr{F}) = \frac{1}{10l}\int_{\overline{G}} |\mu|([g]\mathscr{F};P).$$

III.5. Lemma. $|\mu|([g]\mathscr{F};P) + |\mu|([g]\mathscr{F}^{\perp};P) \geq 1$.

*Proof.* Here we need an odd number of sides and interior angles $\pi/2$.

This idea is due to R. Conelly, after [L.L.2]. Except for a set of isometries of zero measure, at the vertices $v_i$ of $P$ we have two cases to consider.

(i) The leaf through $v_i$, $L_i$ is locally contained in the exterior of $P$. We call it an exterior leaf.

(ii) $v_i$ divides $L_i$ into two half-leaves, one of them locally contained in the interior of $P$, the other contained in the exterior of $P$. We call it an interior leaf (see Fig. 9).

As we have 5 vertices there are two consecutives vertices in which the leaves of $\mathscr{F}$ are both exterior or both interior (see Fig. 9).

If we have two consecutive vertices, say $v_1$, $v_2$, for which $\mathscr{F}$ has interior leaves, or for which $\mathscr{F}^{\perp}$ has exterior leaves, then there exists a point $v$ on the edge $v_1v_2$, through which the leaf $L_v$ of $\mathscr{F}$ is either tangent or perpendicular to the side $v_1v_2$. If the leaf is tangent then $|\mu|(\mathscr{F},P) \geq 1$; if it is orthogonal then $\mu(\mathscr{F}^{\perp},P) \geq 1$ (see Fig. 10).

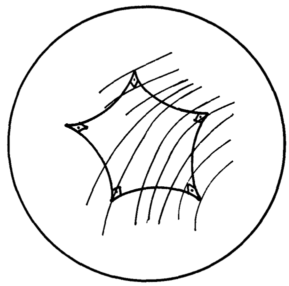

FIG. 9

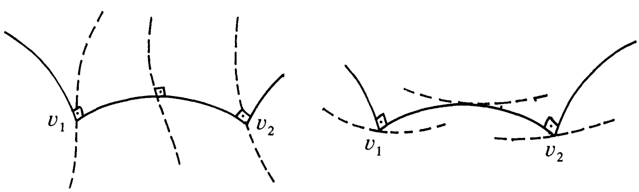


FIG. 10

This concludes the proof of Lemma III.5 and we return to the proof of the theorem. We have

$$K(\mathscr{F}) + K(\mathscr{F}^{\perp}) = \frac{1}{10l}\int_{\overline{G}} |\mu|([g]\mathscr{F}; P) + \frac{1}{10l}\int_{\overline{G}} |\mu|([g]\mathscr{F}^{\perp}; P)$$

$$= \frac{1}{10l}\int_{\overline{G}} \left[|\mu|([g]\mathscr{F}; P) + |\mu|([g]\mathscr{F}^{\perp}; P)\right]$$

and then

$$K(\mathscr{F}) + K(\mathscr{F}^{\perp}) \geq \frac{1}{10l}\int_{\overline{G}} 1 = \frac{1}{10l} m(\overline{G}).$$

The theorem then follows from the two following lemmas.

II.6. LEMMA. *The area of a fundamental domain of $M$ is* $2\pi\,|\chi(M)|$.

*Proof.* This is an immediate consequence of the Gauss-Bonnet theorem applied either to $M$ or to a fundamental domain $F_g$.

II.7. LEMMA. $m(\overline{G}) = 4\pi^2|\chi(M)|$.

According to the above remark, the measure in $\overline{G}$ can be viewed as the measure over segments of fixed length in $H^2$ (or in $M$). The middle point of a segment varies in $F_g$ (or in $M$) whose area is $2\pi|\chi(M)|$. The direction of a segment varies in a segment $[0; \pi[$. Then the measure of all segments of fixed length is $2\pi^2|\chi(M)|$, and so $m(\overline{G}) = 4\pi^2|\chi(M)|$.

## IV. Foliations and envelopes

In this paragraph we turn our attention to the following question. Suppose we are given in $S^n$ an $n$-plane field, $\mathscr{P}: S^n \to \mathscr{A}_{n,n+1}$. We want to calculate a lower bound for $K(\mathscr{F})$, where $\mathscr{F}$ is a orientable codimension one foliation of the unit ball $B^n$, with isolated singularities, which extends $\mathscr{P}$, that is, if $x \in S^n$ such that

$$\mathscr{P}(x) = T_x\mathscr{F}_x.$$

First of all note that such a foliation may not exist. As it will be clear in the following, even in this case our theorem makes sense, as a result on $n$-plane fields that extend $\mathscr{P}$.

We will suppose that the envelope determined by $\mathscr{P}$ is a herisson $H$ with support function $h: S^n \to \mathbf{R}$. By this we mean that $H$ is the solution of the system:

$$\begin{cases} \langle x|z\rangle = h(z) \\ \langle x|dz\rangle = dh(z), \quad z \in S^n \end{cases}$$

We would like to remark that the notion of a herisson (hedgehog) was introduced by R. Langevin, G. Levitt, and H. Rosenberg in [L.L.R], and this paper is the reference for the reader who wants to have more details on it.

The symmetric part of the support function $h$, is

$$h^s(u) = \frac{1}{2}(h(u) + h(-u)), \quad u \in S^n.$$

Notice that, $h^s$ doesn't depend of the origin in $\mathbf{R}^n$.

We are now able to state our theorem.

IV.1. THEOREM. *Let $\mathscr{P}: S^n \to \mathscr{A}_{n;n+1}$ be an n-plane field given along $S^n$, and $\mathscr{F}$ an orientable codimension one foliation of the unit ball $B^n \subset \mathbf{R}^{n+1}$ which has isolated singularities and extends $\mathscr{P}$. Suppose the envelope*

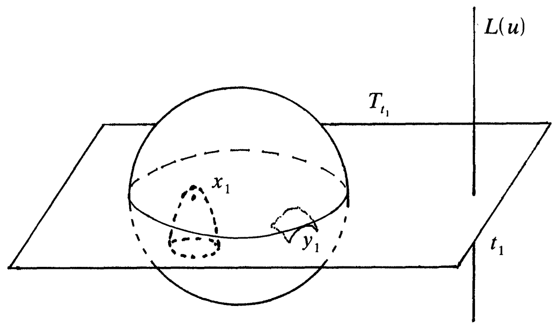


FIG. 11

*determined by $\mathscr{P}$ is a herisson with support function $h$. Then*

$$K(\mathscr{F}) \geq \int_{S^n} |h^s(u)|\, du.$$

*Proof.* With the notations of the first paragraph, we have

$$K(\mathscr{F}) = \frac{1}{2}\int_{S^n}\int_{L(u)} |\mu|(\mathscr{F}; T_t)\, dt\, du.$$

We will prove that

$$\text{IV.2.} \qquad \frac{1}{2}\int_{L(u)} |\mu|(\mathscr{F}; T_t)\, dt \geq |h^s(u)|, \quad \forall u \in S^n,$$

and this will give us the inequality of Theorem IV.1.

As $H$ is a herisson, the orthogonal projection $P_{L(u)}|_H : H \to L(u)$ has, for almost all $u \in S^n$, exactly two critical points $x_1, x_2$. Let's suppose $h(u) = P_{L(u)}(x_1)$, and say that $t_i = P_{L(u)}(x_i)$, $i = 1, 2$, and let $y_i \in S^n$ be the points in $S^n$ where $\mathscr{P}(y_i) = T_{t_i}$, $i = 1, 2$ (see Fig. 11).

We are going to show that for almost all $t \in [t_2, t_1]$ there exists at least one point of contact between $T_t$ and leaves of $\mathscr{F}$. This proves IV.2 above.

As $\mathscr{F}$ is orientable we can define in $B^n$ a kind of "Gauss map" $\gamma_{\mathscr{F}}$ which assigns to each regular point, $x$, of $\mathscr{F}$, the unitary vector $n(x)$ normal, in $x$, to $\mathscr{F}_x$. The vector $n(x)$ is determined by the orientation of $\mathscr{F}$.

This function $\gamma_{\mathscr{F}}$ is continuous in B-Sing($\mathscr{F}$). We indicate by $\tilde{\gamma}_{\mathscr{F}}$, it's restriction to $S^n$, and due to the fact that $H$ is a herisson $\tilde{\gamma}_{\mathscr{F}}$ is injective and hence is a homeomorphism over its image in $S^n$.

With $u$ fixed in $S^n$, let $D_t = B^n \cap T_t$ and $S_t = \partial D_t$. If $t \in [t_2, t_1]$ and $T_t \cap \operatorname{Sing}(\mathscr{F}) = \emptyset$ then either $u$ or $-u$ belongs to $\gamma_{\mathscr{F}}(D_t)$, as we shall now show.

In fact, either $u \in \tilde{\gamma}_{\mathscr{F}}(S_{t_1})$ or $-u \in \tilde{\gamma}_{\mathscr{F}}(S_{t_2})$.

$P_{L(u)}^{-1}([t_2; t_1]) \cap S^n$ is a cylinder whose image by $\tilde{\gamma}_{\mathscr{F}}$ is a cylinder in $S^n$ with boundary

$$\tilde{\gamma}_{\mathscr{F}}(S_{t_1}) \cup \tilde{\gamma}_{\mathscr{F}}(S_{t_2}).$$

$\tilde{\gamma}_{\mathscr{F}}(S_t)$ is an hypersurface in $S^n$ that separates $S^n$ into two hemispheres, one of them containing $u$, the other containing $-u$.

$\tilde{\gamma}_{\mathscr{F}}(D_t)$ must contain one of the two hemispheres determined by $\tilde{\gamma}_{\mathscr{F}}(S_t)$, then $\gamma_{\mathscr{F}}(D_t)$ must contains $u$ or $-u$.

But this means that there is $x \in$ such that $n(x) = u$ or $n(x) = -u$, and then $|\mu|(\mathscr{F}; T_t) \geq 1$.

This gives us inequality IV.2., and we are finished with Theorem IV.1.

*Remark* 1. When $n = 1$, the integral $\int_{S^1} |h^s(u)|\, du$ is called the absolute length of $H$ [L.L.R].

*Remark* 2. If $\mathscr{F}$ is a foliation of $B^{n+1}$ such that $S^n$ is a leaf, then we have

$$H = S^n, \qquad h^s(u) = 2, \forall u \in S^n,$$

and

$$K(\mathscr{F}) \geq \frac{1}{2}\int_{S^n} 2\, du = \operatorname{area}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)},$$

where $\Gamma$ is the gamma function.

## V. Exchange theorem with plane curves

In this paragraph we generalize the Exchange Theorem by "counting contacts" with a regular curve of $\mathbf{R}^2$, instead of line segments. This seems to be very natural, after the constructions in Section III, where we used a pentagon to estimate $K(\mathscr{F})$.

We believe it will be possible to improve these ideas in higher dimensions. For example, if $\mathscr{F}$ is a foliation by surfaces of an open set of $\mathbf{R}^3$, we can "count contacts" of leaves of $\mathscr{F}$, with a fixed surface of $\mathbf{R}^3$ which will roll over the leaves by the action of the isometries of $\mathbf{R}^3$, and obtain integral formulas relating the total curvature of $\mathscr{F}$, the principal curvatures of its leaves and the principal curvatures of the surface.

In this section, $W$ will be an open set in $\mathbf{R}^2$, with $\overline{W}$ compact, $\mathscr{F}$ will be a foliation of $W$, with isolated singularities and $C$ will be a regular curve fixed in $\mathbf{R}^2$, parametrized by arc length.

We define $\varphi : W \times C \to G$ by $\varphi(x; y; s) = g$ if and only if $g$ is an isometry of $\mathbf{R}^2$ which sends $s \in C$ to $(x, y) \in W$ in such a way that $\mathscr{F}_{(x,y)}$ and $g(C)$ are tangent in $(x; y)$ and have the same orientation.

LEMMA. *The Jacobian of the map $\varphi$ satisfies*

$$|\operatorname{Jac} \varphi_{(x,y,s)}| = |K_{\mathscr{F}}(x, y) - K_g(s)|$$

*where $K_{\mathscr{F}}(x, y)$ is the curvature of the leaf $\mathscr{F}_{(x,y)}$ at the point $(x, y)$ and $K_g(s)$ is the curvature of $g(C)$ at $s$, both considered with sign.*

We recall that the space $G$ of all orientation preserving isometries of $\mathbf{R}^2$ is the cylinder $\mathbf{R}^2 \times S^1$, and that in $G$ we have a canonical measure given by the density

$$m = dx\, dy\, d\theta$$

Given $(x, y, s) \in W \dashv C$ there are exactly two isometries $g_i$, $i = 1, 2$ which send $s$ to $(x, y)$ in such a way that the leaf $\mathscr{F}_{(x,y)}$ is tangent in $(x, y)$ to the curve $g_i(C)$, $i = 1, 2$. These isometries differ by a rotation of angle $\pi$ (see Fig. 12).

*Proof of the lemma.* We consider two orthogonal bases:

$[e_1, e_2, C'(s)]$ in the tangent space to $W \times C$ in $(x, y, s)$ where $[e_1, e_2]$ is the canonical basis of $\mathbf{R}^2$;

$[e_1, e_2, f]$ in the tangent space to $\overline{G} = \mathbf{R}^2 \times S^1$ where $[e_1, e_2]$ is the canonical basis of $\mathbf{R}^2$, and $f$ is the unitary vector tangent to the family of isometry classes in $\overline{G}$ which roll $C$ over $\mathscr{F}_{(x,y)}$ without sliding in the positive sense in $\mathscr{F}_{(x,y)}$.

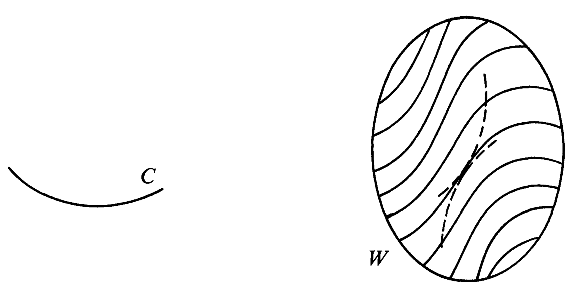


FIG. 12

When $C$ rolls over $\mathscr{F}_{(x,y)}$ without sliding then the point of contact between $g(C)$ and $\mathscr{F}$ has, at each moment, velocity of translation equal to zero and component of rotation equal to $K_{\mathscr{F}} - K_g$. Then

$$|\mathrm{Jac}\, \varphi_{(x;y;s)}| = \left| \det \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & K_{\mathscr{F}}(x,y) - K_g(s) \end{bmatrix} \right| = |K_{\mathscr{F}}(x,y) - K_g(s)|$$

The exchange theorem with curves follows directly from this lemma.

THEOREM. *If $W$ is an open set in $\mathbf{R}^2$, with $\overline{W}$ compact, $\mathscr{F}$ is a foliation of $W$ by curves with isolated singularities, and $C$ is a regular $C^2$ curve in $\mathbf{R}^2$, parametrized by arc length, then*

$$\frac{1}{2}\int_G |\mu|(\mathscr{F}; g(C))\, dm(g) = \int_{W\times C} |K_{\mathscr{F}}(x;y) - K_g(s)|\, dx\, dy\, ds,$$

*where $|\mu|(\mathscr{F}; g(C))$ is the number of contact points between leaves of $\mathscr{F}$ and $g(C)$, $K_{\mathscr{F}}(x,y)$ denotes the curvature of the leaf of $\mathscr{F}$ through $(x,y)$ at $(x,y)$, $K_g(s)$ denotes the curvature of $g(C)$ at $s$ (both curvatures take orientations into account).*

*Proof.*

$$\frac{1}{2}\int_G |\mu|(\mathscr{F}; g(C))\, dm(g) = \int_{W\times C} |\mathrm{Jac}\, \varphi_{(x;y;s)}|\, dx\, dy\, ds.$$

*Remark.* An analogous result is valid in any surface of constant curvature.

*Example.* We consider $W = ]0,1[\times]0,1[ \subset \mathbf{R}^2$, $\mathscr{F}$ the foliation of $W$ by horizontal lines and $C = S(r) = \{(x,y) \in \mathbf{R}^2 | x^2 + y^2 = r^2\}$ the circle of radius $r$ in $\mathbf{R}^2$, $0 < r < \frac{1}{2}$, parametrized by arc length.

Then $K_{\mathscr{F}_{(x,y)}} = 0$ and $|K_g(s)| = 1/r$, and

$$\int_{W\times C} |K_{\mathscr{F}_{(x,y)}} - K_g(s)| = \mathrm{area}(W)\cdot \int_C \frac{1}{r} = 1 \cdot \int_0^{2\pi} \frac{1}{r} r\, d\theta = 2\pi.$$

Each isometry $g$ of $\mathbf{R}^2$ is determined by the image of the origin, or in other words by the center of $g(C)$, and by an angle $\theta \in S^1$ (its component of rotation).

If the center of $g(C)$ lies in the rectangle $]0,1[\times[r, 1-r]$, then $|\mu|(\mathscr{F}; g(C)) = 2$, if the center lies in $]0,1[\times[-r,r] \cup ]0,1[\times[1-r; 1+r]$ then $|\mu|(\mathscr{F}; g(C)) = 1$, and $|\mu|(\mathscr{F}; g(C)) = 0$ otherwise.

It follows that

$$\int_G |\mu|(\mathscr{F}; g(C)) = 2\pi[2(1 - 2r) + 2r + 2r] = 4\pi.$$

## References


[B.L.R] F. Brito, R. Langevin and H. Rosenberg, *Intégales de courbures sur des variétés feuilletées*. J. Diff. Geom. **16** (1981), 19–50.

[L] R. Langevin, *Feuilletages Tendus*, Bull. Soc. Math. France **107** (1979), 271–281.

[L.L.1] R. Langevin and G. Levitt, *Courbure totale des feuilletages des surfaces*, Comment. Math. Helvetici **57** (1982), 175–195.

[L.L.2] ______, *Sur la courbure totale des feuilletages des surfaces à bord*. Bol. Soc. Bras. Mat. **16** (1985), 1–13.

[L.L.R] R. Langevin, G. Levitt and H. Rosenberg, *Hérissons et multihérissons (enveloppes paramétrées par leur application de Gauss)*, Banach Center Publ. 20 (1988), 245–253.

[P] C. Possani, *Curvatura total de Folheações*, Thesis, Instituto de Matematica e Estastistica Universidade de Sao Paulo, 1989.

[S] L.A. Santalo, *Integral geometry and geometric probability*, Number 1, Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, Mass., 1976.



Université de Bourgogne
Dijon, France

Universidade de Sao Paulo
Sao Paulo, Brazil