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PLURIHARMONIC SYMBOLS OF COMMUTING TOEPLITZ OPERATORS

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1. Introduction and Results

Our setting throughout the paper is the unit ball B_n of the complex n-space $\mathbf{C}^n$; dimension n is fixed and thus we usually write $B = B_n$ unless otherwise specified. The Bergman space $A^2(B)$ is the closed subspace of $L^2(B) = L^2(B, V)$ consisting of holomorphic functions where V denotes the volume measure on B normalized to have total mass 1. For $u \in L^\infty(B)$, the *Toeplitz operator* T_u *with symbol* u is the bounded linear operator on $A^2(B)$ defined by $T_u(f) = P(uf)$ where P denotes the orthogonal projection of $L^2(B)$ onto $A^2(B)$. The projection P is the well-known Bergman projection which can be explicitly written as follows:

$$P(\psi)(z) = \int_B \frac{\psi(w)}{(1 - \langle z, w \rangle)^{n+1}} \, dV(w) \quad (z \in B)$$

for functions $\psi \in L^2(B)$. Here $\langle\, , \rangle$ is the ordinary Hermitian inner product on $\mathbf{C}^n$. See [7, Chapters 3 and 7] for more information on the projection P.

In one dimensional case, Axler and Čučković [3] has recently obtained a complete description of harmonic symbols of commuting Toeplitz operators: if two Toeplitz operators with harmonic symbols commute, then either both symbols are holomorphic, or both symbols are antiholomorphic, or a nontrivial linear combination of symbols is constant (the converse is also true and trivial). Trying to generalize this characterization to the ball, one may naturally think of pluriharmonic symbols. A function $u \in C^2(B)$ is said to be *pluriharmonic* if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. As is well known, a real-valued function on B is pluriharmonic if and only if it is the real part of a holomorphic function on B. Hence every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function.

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In the present paper we consider the same problem of characterizing pluriharmonic symbols of commuting Toeplitz operators on the ball. Our first result is a necessary condition in terms of $\mathscr{M}$-harmonicity (see Section 2 for relevant definitions) for such symbols.

THEOREM 1. *Let f, g, h, and k be holomorphic functions on B such that $f+\bar{g}$ and $h+\bar{k}$ are pluriharmonic symbols of two commuting Toeplitz operators on $A^2(B)$. Then $f\bar{k}-h\bar{g}$ is $\mathscr{M}$-harmonic on B.*

The proof in [3] shows that the converse of Theorem 1 is also true in one dimensional case. Unfortunately, we were not able to prove or disprove the converse of Theorem 1 on the ball in general. However, Theorem 1 is enough to produce a simple characterization in case one of symbols is holomorphic (or antiholomorphic which amounts to considering adjoint operators). Its proof will make use of a recent characterization (see Proposition 7) of Ahern and Rudin [2] on $\mathscr{M}$-harmonic products.

THEOREM 2. *Suppose that u and v are pluriharmonic symbols of two commuting Toeplitz operators on $A^2(B)$. If u is nonconstant and holomorphic, then v must be holomorphic.*

Recall that a bounded linear operator on a Hilbert space is called *normal* if it commutes with its adjoint operator. Since the adjoint operator of the Toeplitz operator with symbol u is the Toeplitz operator with symbol $\bar{u}$, the following is an immediate consequence of Theorem 2 whose proof is therefore omitted.

COROLLARY 3. *The Toeplitz operator with holomorphic symbol u is normal on $A^2(B)$ if and only if u is constant.* □

In Section 2 we collect some facts about $\mathscr{M}$-harmonic functions which are needed in Section 3 where we prove Theorems 1 and 2. In Section 4 we conclude the paper with some remarks and discussions related to the converse of Theorem 1 and a possible pluriharmonic version of Corollary 3.

2. $\mathscr{M}$-Harmonic Functions

For $z, w \in B$, $z \neq 0$, define

$$\varphi_z(w) = \frac{z - |z|^{-2}\langle w, z\rangle z - \sqrt{1-|z|^2}\,(w - |z|^{-2}\langle w, z\rangle z)}{1 - \langle w, z\rangle}$$

and $\varphi_0(w) = -w$. Then $\varphi_z \in \mathscr{M}$, the group of all automorphisms

(= biholomorphic self-maps) of B. Furthermore, each $\varphi \in \mathscr{M}$ has a unique representation $\varphi = U \circ \varphi_z$ for some $z \in B$ and unitary operator U on $\mathbf{C}^n$. For $u \in C^2(B)$ and $z \in B$, we define

$$(\tilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0)$$

where Δ denotes the ordinary Laplacian. The operator $\tilde{\Delta}$ is called the *invariant* Laplacian because it commutes with automorphisms of B in the sense that $\tilde{\Delta}(u \circ \varphi) = (\tilde{\Delta} u) \circ \varphi$ for $\varphi \in \mathscr{M}$. We say that a function $u \in C^2(B)$ is *$\mathscr{M}$-harmonic* on B if it is annihilated on B by $\tilde{\Delta}$. One can easily see that $\mathscr{M}$-harmonic functions are precisely harmonic ones in one dimensional case. As is the case for harmonic functions, $\mathscr{M}$-harmonic functions are characterized by a certain mean value property (see [7, Chapter 4]): a function $u \in C(B)$ is $\mathscr{M}$-harmonic on B if and only if

$$(u \circ \varphi)(0) = \int_S (u \circ \varphi)(r\zeta)\, d\sigma(\zeta) \quad (0 \le r < 1)$$

for every $\varphi \in \mathscr{M}$. Here σ denotes the rotation invariant probability measure on the unit sphere S, the boundary of B. This is the so-called *invariant* mean value property. The following *area version* of this invariant mean value property also gives a characterization of $\mathscr{M}$-harmonicity of functions continuous up to the boundary (see [7, Proposition 13.4.4]): a function $u \in C(\overline{B})$ is $\mathscr{M}$-harmonic on B if and only if

$$(u \circ \varphi)(0) = \int_B (u \circ \varphi)\, dV$$

for every $\varphi \in \mathscr{M}$.

The key step to our proof of Theorem 1 is adapted from that of [3]. That is, we will use a slight variant of the characterization of $\mathscr{M}$-harmonicity given by the area version of invariant mean value property. To state it, let us introduce some notations. We associate with each $v \in C(B)$ its so-called radialization $\mathscr{A}(v)$ defined by the formula

$$\mathscr{A}(v)(z) = \int_{\mathscr{U}} (v \circ U)(z)\, dU \quad (z \in B)$$

where dU denotes the Haar measure on the group $\mathscr{U}$ of all unitary operators on $\mathbf{C}^n$. Using Proposition 1.4.7 of [7], one can easily verify that

$$\mathscr{A}(v)(z) = \int_S v(|z|\zeta)\, d\sigma(\zeta) \quad (z \in B)$$

and hence $\mathscr{A}(v)$ is indeed a radial function on B. We write $\mathscr{A}(v) \in C(\overline{B})$ if $\mathscr{A}(v)$ has a continuous extension up to the boundary.

PROPOSITION 4. *Suppose that $u \in C(B) \cap L^1(B)$. Then u is $\mathscr{M}$-harmonic on B if and only if*

$$\int_B (u \circ \varphi)\, dV = (u \circ \varphi)(0) \tag{1}$$

and

$$\mathscr{A}(u \circ \varphi) \in C(\overline{B}) \tag{2}$$

for every $\varphi \in \mathscr{M}$.

Proof. We first prove the easy direction. Suppose that u is $\mathscr{M}$-harmonic on B. Let $\varphi \in \mathscr{M}$. By the invariant mean value property, we have

$$(u \circ \varphi)(0) = \int_S (u \circ \varphi)(r\zeta)\, d\sigma(\zeta)$$

for every $r \in [0, 1)$. Integrating in polar coordinates, we have (1). The above also shows that $\mathscr{A}(u \circ \varphi)$ is constant on B, with value $(u \circ \varphi)(0)$, and therefore (2) holds.

To prove the other direction (which we need for the proof of Theorem 1), suppose that (1) and (2) hold. Let $\varphi \in \mathscr{M}$ and put $v = \mathscr{A}(u \circ \varphi)$. We first show that v is $\mathscr{M}$-harmonic on B. Since $v \in C(\overline{B})$ by (2), it is sufficient to show the area version of invariant mean value property of v. To do this, fix $\psi \in \mathscr{M}$. Then

$$\int_B (v \circ \psi)\, dV = \int_B \int_{\mathscr{U}} (u \circ F_U)(z)\, dU\, dV(z) \tag{3}$$

where $F_U = \varphi \circ U \circ \psi \in \mathscr{M}$.

For a fixed unitary operator U on $\mathbf{C}^n$, consider the inverse mapping $G_U \in \mathscr{M}$ of F_U and put $a = F_U(0) = (\varphi \circ U \circ \psi)(0)$. Then, since $|\varphi^{-1}(0)| = |\varphi(0)|$, we have ([7, Theorem 2.2.5])

$$1 - |a|^2 = \frac{\left(1 - |\varphi(0)|^2\right)\left(1 - |\psi(0)|^2\right)}{\left|1 - \langle \varphi^{-1}(0), (U \circ \psi)(0) \rangle\right|^2} \geq \left(1 - |\varphi(0)|^2\right)\left(1 - |\psi(0)|^2\right).$$

On the other hand, we have [7, Theorem 2.2.6]

$$J_R G_U(w) = \left(\frac{1-|a|^2}{|1-\langle w, a\rangle|^2} \right)^{n+1} \le \left(\frac{4}{1-|a|^2} \right)^{n+1} \quad (w \in B)$$

where $J_R G_U(w)$ denotes the real Jacobian of G_U at $w \in B$. It follows that the function $J_R G_U$ is bounded on B uniformly in U. Therefore, since $u \in L^1(B)$ by assumption, a change of variables shows that

$$\int_{\mathscr{U}} \int_B |u \circ F_U| \, dV dU = \int_{\mathscr{U}} \int_B |u| J_R G_U \, dV dU < \infty.$$

Now one can interchange the order of integrations on the right side of (3) to obtain

$$\begin{aligned} \int_B (v \circ \psi) \, dV &= \int_{\mathscr{U}} \int_B (u \circ F_U) \, dV dU \\ &= \int_{\mathscr{U}} (u \circ F_U)(0) \, dU \\ &= \int_{\mathscr{U}} (u \circ \varphi \circ U)(\psi(0)) \, dU \\ &= \mathscr{A}(u \circ \varphi)(\psi(0)) \\ &= (v \circ \psi)(0) \end{aligned}$$

where the second equality holds by (1). Hence v is $\mathscr{M}$-harmonic on B. Since v is radial, the invariant mean value property shows that v is constant. Consequently,

$$(u \circ \varphi)(0) = v(0) = v(z) = \int_S (u \circ \varphi)(|z|\zeta) \, d\sigma(\zeta) \quad (z \in B).$$

Since $\varphi \in \mathscr{M}$ is arbitrary, the above shows that u has the invariant mean value property and hence that u is $\mathscr{M}$-harmonic on B as desired. □

3. Proofs

First, we recall some well known facts on the Hardy space $H^2(B)$ consisting of holomorphic functions f on B for which

$$\sup_{0<r<1} \int_S |f(r\zeta)|^2 \, d\sigma(\zeta) < \infty.$$

Note that $H^2(B) \subset A^2(B)$ by an integration in polar coordinates. To each $f \in H^2(B)$ corresponds its boundary function f^* on S defined by $f^*(\zeta) = \lim_{r \nearrow 1} f(r\zeta)$ for σ-almost every $\zeta \in S$. In addition, we have $f^* \in L^2(\sigma)$ and

$$\lim_{r \nearrow 1} \int_S |f(r\zeta) - f^*(\zeta)|^2 \, d\sigma(\zeta) = 0.$$

See [7, Chapter 5] for details. One can easily verify by using the above that if $f, g \in H^2(B)$, then

$$\lim_{r \nearrow 1} \int_S f(r\zeta)\bar{g}(r\zeta) \, d\sigma(\zeta) = \int_S f^* \bar{g}^* \, d\sigma$$

and hence $\mathscr{A}(f\bar{g}) \in C(\overline{B})$.

Next, before turning to the proof of Theorem 1, we prove a couple of lemmas. For $\varphi \in \mathscr{M}$, let U_φ denote the linear operator on $A^2(B)$ defined by $U_\varphi f = (f \circ \varphi)J_\varphi$ where J_φ is the complex Jacobian of φ and write U_φ^* for its adjoint operator.

Lemma 5. *Let $\varphi \in \mathscr{M}$. Then $U_\varphi U_\varphi^* = U_\varphi^* U_\varphi$ is the identity operator on $A^2(B)$.*

In other words, the conclusion of the lemma is that U_φ is unitary on $A^2(B)$.

Proof. Since $|J\varphi|^2$ is the real Jacobian of φ, a change of variables yields

$$\int_B |(f \circ \varphi)|^2 |J\varphi|^2 \, dV = \int_B |f|^2 \, dV$$

for every $f \in A^2(B)$, and hence U_φ is an isometry of $A^2(B)$ into $A^2(B)$. Clearly $U_{\varphi^{-1}}$ is the inverse operator for U_φ. An invertible linear isometry on a Hilbert space is a unitary operator (see for example [5, Theorem 12.13]). The proof is complete.

Lemma 6. *Let $\varphi \in \mathscr{M}$ and let $u \in L^\infty(B)$. Then*

$$U_\varphi T_u U_\varphi^* = T_{u \circ \varphi}.$$

Recall that P denotes the Bergman projection of $L^2(B)$ onto $A^2(B)$.

Proof. Define V_φ: $L^2(B) \to L^2(B)$ by $V_\varphi f = (f \circ \varphi)J\varphi$. As in the proof of Lemma 5, V_φ is unitary on $L^2(B)$. Since $V_\varphi = U_\varphi$ when restricted to $A^2(B)$, we see that V_φ takes $A^2(B)$ onto $A^2(B)$ and hence

$$PV_\varphi = V_\varphi P. \tag{4}$$

If $f \in A^2(B)$, then we see from (4) that

$$\begin{aligned} T_{u\circ\varphi}U_\varphi f &= T_{u\circ\varphi}((f\circ h)J\varphi) = P((u\circ\varphi)(f\circ\varphi)J\varphi) \\ &= P\big(V_\varphi(uf)\big) = V_\varphi(P(uf)) = U_\varphi T_u f. \end{aligned}$$

Thus $T_{u\circ\varphi}U_\varphi = U_\varphi T_u$, and since U_φ is unitary by Lemma 5, we have $T_{u\circ\varphi} = U_\varphi T_u U_\varphi^*$. The proof is complete. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $u = f + \bar{g}$ and $v = h + \bar{k}$. Since u and v are bounded on B, functions f, g, h, and k must be in $H^2(B)$ by an application of the Korányi-Vagi theorem (see [7, Theorem 6.3.1]). In particular, functions f, g, h, and k are all in $A^2(B)$. Let 1 denote the constant function 1 on B. Then we have

$$T_uT_v1 = T_u(Pv) = T_u\big(h + \bar{k}(0)\big) = P\big(fh + \bar{k}(0)f + h\bar{g} + \bar{g}(0)\bar{k}(0)\big).$$

Note that $\int_B F\,dV = F(0)$ for holomorphic functions $F \in L^1(B)$. Since the projection P is orthogonal, it follows that

$$\begin{aligned} \int_B T_uT_v1\,dV &= \int_B fh + \bar{k}(0)f + h\bar{g} + \bar{g}(0)\bar{k}(0)\,dV \\ &= f(0)h(0) + f(0)\bar{k}(0) + \bar{g}(0)\bar{k}(0) + \int_B h\bar{g}\,dV. \end{aligned} \tag{5}$$

Similarly,

$$\int_B T_vT_u1\,dV = f(0)h(0) + h(0)\bar{g}(0) + \bar{g}(0)\bar{k}(0) + \int_B f\bar{k}\,dV. \tag{6}$$

Since $T_uT_v = T_vT_u$ by assumption, letting $\alpha = f\bar{k} - h\bar{g}$, we have by (5) and (6) that

$$\int_B \alpha\,dV = \alpha(0). \tag{7}$$

We also have (by a remark at the beginning of this section) that

$$\mathscr{A}(\alpha) \in C(\overline{B}). \tag{8}$$

Let $\varphi \in \mathscr{M}$. Multiplying both sides of the equation $T_u T_v = T_v T_u$ by U_φ on the left and by U_φ^* on the right, we obtain by Lemma 5 that

$$U_\varphi T_u U_\varphi^* U_\varphi T_v U_\varphi^* = U_\varphi T_v U_\varphi^* U_\varphi T_u U_\varphi^*$$

and therefore by Lemma 6

$$T_{u\circ\varphi} T_{v\circ\varphi} = T_{v\circ\varphi} T_{u\circ\varphi}. \tag{9}$$

Equations (7) and (8) were derived under the assumption that $T_u T_v = T_v T_u$. Thus (9) says that (7) and (8) remain valid with $\alpha \circ \varphi$ in place of α. That is,

$$\int_B (\alpha \circ \varphi)\, dV = (\alpha \circ \varphi)(0)$$

and $\mathscr{A}(\alpha \circ \varphi) \in C(\overline{B})$ for any $\varphi \in \mathscr{M}$. It follows from Proposition 4 that α is $\mathscr{M}$-harmonic on B. This completes the proof.

Having proved Theorem 1, we now turn to the proof of Theorem 2 which states that if one of symbols of two commuting Toeplitz operators is nonconstant and holomorphic, then the other one must be also holomorphic. In the proof we apply a consequence of the following recent theorem of Ahern and Rudin [2] on $\mathscr{M}$-harmonic products.

PROPOSITION 7. *Let f and g be holomorphic functions such that $f\bar{g}$ is $\mathscr{M}$-harmonic on B.*

(a) *If $n \leq 2$, then either f or g is constant.*

(b) *If $n \geq 3$, and if both f and g are nonconstant, then there exist an integer $2 \leq m \leq n-1$, a unitary operator U on $\mathbf{C}^n$, and entire functions F on $\mathbf{C}^{m-1}$, and G on $\mathbf{C}^{n-m}$, such that*

$$f(Uz) = F\left(\frac{z_2}{1-z_1}, \ldots, \frac{z_m}{1-z_1}\right), \quad g(Uz) = G\left(\frac{z_{m+1}}{1-z_1}, \ldots, \frac{z_n}{1-z_1}\right).$$

Moreover, $f(B) = F(\mathbf{C}^{m-1})$, $g(B) = G(\mathbf{C}^{n-m})$, and $(f\bar{g})(B) = \mathbf{C}$ or $\mathbf{C} \setminus \{0\}$.

Combining Proposition 7 with Liouville's theorem, we have the following:

LEMMA 8. *Let f and g be holomorphic functions such that $f\bar{g}$ is $\mathscr{M}$-harmonic on B. If one of them is bounded on B, then either f or g is constant.*

Proof of Theorem 2. Write $v = h + \bar{k}$ where h, k are holomorphic on B. Then, by Theorem 1, uk is $\mathscr{M}$-harmonic on B. Since u is bounded and nonconstant on B by assumption, we see from Lemma 8 that k must be constant and hence v is holomorphic on B. Conversely, since Toeplitz operators with holomorphic symbols are simply multiplication operators, it is straightforward that two Toeplitz operators with holomorphic symbols commute on $A^2(B)$. □

4. Some related remarks

Throughout the section f, g, h, and k denote holomorphic functions on B, normalized so that $f(0) = g(0) = h(0) = k(0) = 0$ for simplicity. In view of Theorem 1 one may ask (under additional boundedness hypothesis as in Lemma 8 if desired) whether there is any further description of such functions for which

$$\tilde{\Delta}(f\bar{k}) = \tilde{\Delta}(h\bar{g}). \tag{10}$$

Both sides of the above are assumed to be not identically zero; otherwise we are back to Proposition 7. In one dimensional case, it is elementary to verify that condition (10) implies $f = \lambda h$ and $g = \bar{\lambda} k$ for some constant λ. In higher dimensional cases, we do not know whether the same is true in general. This question can be rephrased as follows: does it follow from (10) that $f\bar{k} - h\bar{g}$ is pluriharmonic? The answer is known to be *yes* if an additional smoothness condition of certain order, depending on dimension n, is satisfied up to the boundary: if a function $u \in C^n(\bar{B})$ is $\mathscr{M}$-harmonic on B, then u is pluriharmonic on B. See [1] or [4]. We also remark in passing that there is in fact a more precise version of this fact ([6]): if u is $\mathscr{M}$-harmonic on B and if the nth radial derivative $\mathscr{D}^n u$ satisfies the L^2-growth condition

$$\left\{\int_S |(\mathscr{D}^n u)(r\zeta)|^2 \, d\sigma(\zeta)\right\}^{1/2} = o\left(\log\frac{1}{1-r}\right) \quad (r \nearrow 1),$$

then u is pluriharmonic on B. Note that $T_{f+\bar{g}}T_{h+\bar{k}} = T_{h+\bar{k}}T_{f+\bar{g}}$ if and only if $T_f T_{\bar{k}} - T_{\bar{k}} T_f = T_h T_{\bar{g}} - T_{\bar{g}} T_h$ for functions f, g, h, and k bounded on B. Thus, for example, we have the following:

If f, g, h, and k are of class C^n on $\bar{B}$, and if $T_f T_{\bar{k}} - T_{\bar{k}} T_f = T_h T_{\bar{g}} - T_{\bar{g}} T_h$ on $A^2(B)$, then $f = \lambda h$ and $g = \bar{\lambda} k$ for some constant λ.

Trying to obtain a pluriharmonic version of Corollary 3, one is led to a special case of (10) which may be of some independent interest. That is, the question is now whether the condition

$$\tilde{\Delta}|f|^2 = \tilde{\Delta}|g|^2 \tag{11}$$

implies $f = \lambda g$ for some unimodular constant λ. We could prove only in some special cases that the answer is yes. Those are included in the rest of the paper with hope that they may serve as a motivation for someone to settle the question in the affirmative or negative direction. We first prove a couple of lemmas.

LEMMA 9. *Let Ω be a given connected open subset of $\mathbf{C}^n$. If F_j and G_j $(1 \leq j \leq m)$ are holomorphic functions such that $\sum_{j=1}^m F_j \overline{G}_j = 0$ on Ω, then $\sum_{j=1}^m F_j(z)\overline{G}_j(w) = 0$ for all $z, w \in \Omega$.*

Proof. Assume, without loss of generality, that an open ball β with center at the origin is contained in Ω. Define

$$H(z,w) = \sum_{j=1}^{m} F_j(z)\overline{G}_j(\overline{w}) \quad (z, \overline{w} \in \Omega).$$

It is sufficient to show that $H(z, \overline{w}) = 0$ for all $z, w \in \beta$ by real analyticity. Let L be the invertible linear operator on $\mathbf{C}^n \times \mathbf{C}^n$ defined by

$$L(z,w) = (z + iw, z - iw).$$

Then, since $H(z, \bar{z}) = 0$ for all $z \in \beta$ by hypothesis, we have $H \circ L = 0$ on $V \cap (\mathbf{R}^n \times \mathbf{R}^n)$ where $V = L^{-1}(\beta \times \beta)$. Note that the function $H \circ L$ is holomorphic on V. A consideration of Taylor coefficients therefore shows that $H \circ L$ vanishes on V. In other words, $H = 0$ on $\beta \times \beta$, completing the proof.

LEMMA 10. *Let Ω be a given connected open subset of $\mathbf{C}^n$. If F_j and G_j $(1 \leq j \leq m)$ are holomorphic functions such that $\sum_{j=1}^m |F_j|^2 = \sum_{j=1}^m |G_j|^2$ on Ω, then there is a unitary operator U on $\mathbf{C}^m$ such that $(F_1, \ldots, F_m) = U \circ (G_1, \ldots, G_m)$ on Ω.*

Proof. The lemma is trivial if $m = 1$. To proceed by induction on m, let $m > 1$ and suppose that the lemma is proved for $m - 1$. Let $F = (F_1, \ldots, F_m)$ and $G = (G_1, \ldots, G_m)$. We may assume that Ω contains the origin. We may further assume that $|F(0)| = |G(0)| = 1$. Pick unitary operators U_1 and U_2 on $\mathbf{C}^m$ such that $U_1(F(0)) = U_2(G(0)) = (1, 0, \ldots, 0)$. Let

$$U_1 \circ F = (f_1, \ldots, f_m) \text{ and } U_2 \circ G = (g_1, \ldots, g_m).$$

Then we have $\Sigma_{j=1}^{m} f_j \bar{f}_j = \Sigma_{j=1}^{m} g_j \bar{g}_j$ on Ω and hence, by Lemma 9,

$$\sum_{j=1}^{m} f_j(z)\bar{f}_j(w) = \sum_{j=1}^{m} g_j(z)\bar{g}_j(w)$$

for all $z, w \in \Omega$. Taking $w = 0$, we obtain $f_1 = g_1$ on Ω. Thus, by induction hypothesis, there exists some unitary operator U on $\mathbf{C}^{m-1}$ such that $(f_2, \dots, f_m) = U \circ (g_2, \dots, g_m)$ on Ω. Now let

$$U_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & U & \\ 0 & & & \end{pmatrix}.$$

Then U_3 is a unitary operator on $\mathbf{C}^m$ and we have $F = U_1^{-1} \circ U_3 \circ U_2 \circ G$. The proof is complete. □

In what follows, we let $\nabla f = (D_1 f, \dots, D_n f)$ and $\mathscr{R} f = \Sigma_{j=1}^{n} z_j D_j f$ where D_j denotes the differentiation with respect to z_j-variable. With these notations, equation (11) becomes

$$|\nabla f|^2 + |\mathscr{R} g|^2 = |\nabla g|^2 + |\mathscr{R} f|^2. \tag{12}$$

We assert the following:

Suppose that (12) *holds on* B_2. *If* $\nabla f(0) = \nabla g(0) \neq 0$, *then* $f = g$ *on* B_2.

Proof. By (12) and Lemma 10 there is a unitary operator (α_{ij}) on $\mathbf{C}^3$ such that

$$\begin{pmatrix} D_1 f \\ D_2 f \\ \mathscr{R} g \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} D_1 g \\ D_2 g \\ \mathscr{R} f \end{pmatrix}. \tag{13}$$

Assume that $\nabla f(0) = \nabla g(0) = (1, 0)$ without loss of generality. Then, evaluating both sides of (13) at the origin, one can easily find that $\alpha_{11} = 1$ and $\alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{31} = 0$. It follows that $D_1 f = D_1 g$. Hence $D_2 f - D_2 g$ does not depend on z_1-variable. In order to prove $D_2 f = D_2 g$ it is therefore sufficient to show that $D_2 f(0, z_2) = D_2 g(0, z_2)$ for $|z_2| < 1$. Evaluating both sides of (12) at points $(0, z_2)$, we obtain that $|D_2 f(0, z_2)| = |D_2 g(0, z_2)|$ and thus there exists a unimodular constant λ such that

$$D_2 f(0, z_2) = \lambda D_2 g(0, z_2) \tag{14}$$

for $|z_2| < 1$. Assume that both sides of (14) are not identically zero; otherwise we are done. By (13),

$$z_2 D_2 g(0, z_2) = \alpha_{32} D_2 g(0, z_2) + \alpha_{33} z_2 D_2 f(0, z_2).$$

Insert (14) into the above. A little manipulation yields $\alpha_{32} = \alpha_{23} = 0$ and $\alpha_{33} = \bar{\lambda}$. Thus, we have $\mathscr{R}f = \lambda \mathscr{R}g$. Evaluating both sides of this at points $(z_1, 0)$, we obtain $\lambda = 1$. The proof is complete.

We now conclude the paper with another special case:

If $f_{l+1} = 0$ and $f_j \neq 0$ for some $1 \leq j \leq l$ where f_m denotes the mth degree term in the homogeneous expansion of f on B, then (12) *implies $f = \lambda g$ for some unimodular constant λ.*

Thus, if there were counter examples, then there would be no “gap” in their homogeneous expansions.

Proof. First note that the invariant mean value property of $|f|^2 - |g|^2$ yields

$$\int_S |f_m|^2 \, d\sigma = \int_S |g_m|^2 \, d\sigma \quad (m = 1, 2, \ldots)$$

where g_m denotes the mth degree term in the homogeneous expansion of g on B. Hence $g_{l+1} = 0$ and $g_j \neq 0$ by hypothesis. Now, by Lemma 10 as before, there exists a unitary operator U on $\mathbf{C}^{n+1}$ such that $(\nabla f, \mathscr{R}g) = U \circ (\nabla g, \mathscr{R}f)$. In particular, there are some vectors $\alpha, \beta \in \mathbf{C}^n$ and a constant λ with $|\alpha|^2 + |\lambda|^2 = |\beta|^2 + |\lambda|^2 = 1$ such that

$$\mathscr{R}f = \langle \nabla g, \alpha \rangle + \lambda \mathscr{R}g \quad \text{and} \quad \mathscr{R}g = \langle \nabla f, \beta \rangle + \bar{\lambda} \mathscr{R}f. \tag{15}$$

If $|\lambda| = 1$, then $\alpha = \beta = 0$ and (15) shows $\mathscr{R}f = \lambda \mathscr{R}g$, hence $f = \lambda g$. So, we assume $|\lambda| < 1$ and derive a contradiction. Equate terms of the same degree in the homogeneous expansions of both sides of two equations of (15) to obtain

$$m f_m = \langle \nabla g_{m+1} m, \alpha \rangle + \lambda m g_m \quad \text{and} \quad m g_m = \langle \nabla f_{m+1}, \beta \rangle + \bar{\lambda} m f_m,$$

so that

$$m(1 - |\lambda|^2) f_m = \langle \nabla g_{m+1}, \alpha \rangle + \lambda \langle \nabla f_{m+1}, \beta \rangle$$

for $m = 1, 2, \ldots$. Since $f_{l+1} = g_{l+1} = 0$, the above shows that $f_m = g_m = 0$ for all $1 \leq m \leq l$, which is a contradiction. The proof is complete. □

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