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ARITHMETICALLY LONG ORBITS OF SOLVABLE LINEAR GROUPS

Dedicated to Marty Isaacs for his 50th Birthday

OLAF MANZ AND THOMAS R. WOLF

Introduction

Let V be a finite faithful irreducible $\mathcal{F}[G]$-module for a finite solvable group G.

A number of interesting results regarding orbit sizes have connections with the structure and character theory of solvable groups, in part because the chief factors of a solvable group G are finite irreducible G-modules. Huppert (see [HB, Theorem XII.7.3]) proved that if G acts transitively on $V - \{0\}$ (i.e. the orbit sizes are 1 and $|V| - 1$), then G is isomorphic to a subgroup of the semi-linear group $\Gamma(V) = \Gamma(q^n)$, where $q = |\mathcal{F}|$ and $n = \dim(V)$, or $|V| = 3^2, 5^2, 7^2, 11^2, 23^2$, or 3^4. By saying $G \leq \Gamma(V)$, we mean that the elements of V may be identified or labeled by the elements of the field $GF(q^n)$ in such a way that G is a subgroup of

$$\Gamma = \{x \to ax^{\sigma} | 0 \neq a \in GF(q^n), \sigma \in \operatorname{Gal}(GF(q^n)/GF(q))\} \leq GL(V).$$

Observe that Γ is metacyclic of order $n(q^n - 1)$. A consequence of Huppert's result is classification of solvable two-transitive permutation groups. Saeger [Sa] generalized this by showing that if V is a primitive G-module with relatively few orbits, then $G \leq \Gamma(V)$ or q^n is one of a handful of values. Passman [Pa 1, 2] classified those G that act half-transitively on $V - \{0\}$, i.e., the G-orbits of $V - \{0\}$ are of equal size.

Our concern here is the existence of large orbits, specifically an orbit divisible by many prime divisors of $|G|$. Our main result is:

THEOREM A. *Suppose G is a solvable group and V is a finite faithful irreducible G-module. Choose $H \leq G$ and W a primitive H-module such that $V \cong W^G$. If $H/\mathbf{C}_H(W) \not\leq \Gamma(W)$, then there exists $v \in V$ such that $|G: \mathbf{C}_G(V)|$ is divisible by every prime divisor $p \geq 5$ of $|G|$.*

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Of course regular orbits would be nice, particularly for applications. This is not always possible, particularly with imprimitive modules. And, in the primitive case, there is always the semi-linear group $\Gamma(q^n)$ which has order $n(q^n - 1)$ and orbit sizes 1 and $q^n - 1$. Espuelas [Es] showed if V is a primitive G-module and $|G||V|$ is odd, then G has a regular orbit or $G \leq \Gamma(V)$. If, in Theorem A, we assume that $|G||V|$ is odd, then Espuelas' result can be used there exists $v \in V$ such that $|G\colon \mathbf{C}_G(v)|$ is divisible by all prime divisors of $|G|$ (unless, of course $H/\mathbf{C}_H(W) \leq \Gamma(W)$). In proving Theorem A, one may assume that each $v \in V$ is centralized by a Sylow-p-subgroup of G for some $p \geq 5$ (dependent upon v). The case where p is not dependent upon v ($p \geq 5$) can only occur when $G \leq \Gamma(V)$. This result [Wo1] provides an important step for our results. We mention other papers [Be, Ca, Ha] that deal with existence of regular orbits.

Our main theorem will be proved in Section 2. But first we apply the theorem to a conjecture of Huppert, which roughly states that a group G must have an irreducible character whose degree is divisible by many primes. We give the best results known for solvable G.

1. Huppert's $\rho - \sigma$ conjecture

We let $\pi(n)$ denote the set of prime divisors of an integer n and $\pi(G:H) = \pi(|G:H|)$. For a group G, we let

$$\rho(G) = \{p \text{ prime} \mid p \mid \chi(1) \text{ for some } \chi \in \mathrm{Irr}(G)\}$$

and

$$\sigma(G) = \max\{\pi(\chi(1)) \mid \chi \in \mathrm{Irr}(G)\}.$$

Of course, $\rho(G)$ is a set, while $\sigma(G) \in \mathbf{N}$. Huppert has conjectured the following:

(a) There is a function $f\colon \mathbf{N} \to \mathbf{N}$ such that $|\rho(G)| \leq f(\sigma(G))$ for all group G.

(b) For solvable G, $|\rho(G)| \leq 2 \cdot \sigma(G)$.

Given primes p_i and q_i, one may construct a group H_i, a semi-direct product of an extra-special p_i-group and cyclic group of order q_i, such that $\rho(H_i) = \{p_i, q_i\}$ and $\sigma(H_i) = 1$. If $p_1, \dots, p_n, q_1, \dots, q_n$ are chosen to be distinct, then the group $G_n := H_1 \times \cdots \times H_n$ satisfies $|\rho(G_n)| = 2n$ and $\sigma(G_n) = n$. Consequently the bound in (b) would be best possible.

Isaacs [Is2] was first to give a bound (exponential) for solvable groups. Gluck [Gl2] gave a quadratic bound and Gluck and Manz [GM] give the linear bound $|\rho(G)| \leq 3\sigma(G) + 32$. We show the additive constant can be lowered so that $|\rho(G)| \leq 3\sigma(G) + 2$. The additive constant refers specifically to the set $\{2, 3\}$. Part of the difficulties with this set arise in the next lemma, which we use in both theorem A and (directly) in Lemma 1.2 below. This next lemma is a consequence of a theorem of Gluck [Gl1]. We let $\pi_0(G:H) = \pi(G:H) \setminus \{2,3\}$.

1.1 LEMMA. *Suppose G is a solvable permutation group on Ω (not necessarily transitive). Then we may choose $\Delta \leq \Omega$ such that*
(a) $\text{stab}_G(\Delta)$ *is a $\{2,3\}$-group, and*
(b) $\text{stab}_G(\Delta) = 1$ *provided $|G|$ is odd.*

Proof. See [GM, Lemma 7] for (a) and [Gl1, Corollary 1] for (b). □

1.2 LEMMA. *Suppose that M is a normal elementary abelian subgroup of the solvable group G. Assume that $M = \mathbf{C}_G(M)$ is a completely reducible G-module (possibly of mixed characteristic). Set $V = \text{Irr}(M)$ and write $V = V_1 \oplus \cdots \oplus V_m$ for irreducible G-modules V_i. For each i, write $V_i = Y_i^G$ for primitive modules Y_i. Assume that $\mathbf{N}_G(Y_i)/\mathbf{C}_G(Y_i)$ is nilpotent-by-nilpotent for each i. If $M \leq N \trianglelefteq G$, there exists $\theta \in \text{Irr}(N)$ whose degree is divisible by at least half the primes of $\pi_0(N/M)$.*

Proof. We may write each V_i as a direct sum of the G-conjugates of Y_i, $i = 1, \ldots, m$. Consequently, $V = X_1 \oplus \cdots \oplus X_n$ for subspaces X_i of V permuted by G (not necessarily transitively) with $\{Y_1, \ldots, Y_m\} \subseteq \{X_1, \ldots, X_n\}$. Furthermore, if $N_i = \mathbf{N}_G(X_i)$, $C_i = \mathbf{C}_G(X_i)$ and $F_i/C_i = \mathbf{F}(N_i/C_i)$, then X_i is a primitive, faithful N_i/C_i-module and N_i/F_i is nilpotent.

Let $K = \bigcap_i N_i \trianglelefteq G$ be the kernel of the permutation representation of G on $\{X_1, \ldots, X_n\}$. Since $\bigcap_i C_i = M$, we have $\bigcap_i F_i/M = \mathbf{F}(K/M) \trianglelefteq G/M$. Let $H = \bigcap_i F_i$, so that $H/M = \mathbf{F}(K/M)$. Observe that K/H is nilpotent. Set $C = K \cap N$ and $F = H \cap N = H \cap C$. Observe that $F/M = \mathbf{F}(C/M)$ and that C/F is nilpotent because K/H is. Because $C/M/\mathbf{F}(C/M)$ is nilpotent, a fairly standard argument yields the existence of $\mu \in \text{Irr}(C/M)$ such that $\pi(\mu(1)) = \pi(C/F)$ (e.g., see Lemma 1.1 of [HM]).

By Lemma 1.1 (a), we may choose $\Delta \subseteq \{X_1, \ldots, X_n\}$ such that $\text{stab}_N(\Delta)/(N \cap K) = \text{stab}_N(\Delta)/C$ is a $\{2,3\}$-group. Furthermore, we can assume that Δ intersects each N-orbit non-trivially. Without loss of generality, $\Delta = \{X_1, \ldots, X_l\}$ for some $l \in \{1, \ldots, n\}$. Set $\lambda = \lambda_1 \ldots \lambda_l \in V$ for non-principal $\lambda_i \in X_i$. Finally suppose that $Q \in \text{Syl}_q(N)$ for a prime $q \geq 5$, and Q centralizes λ. Thus $Q \leq \text{stab}_N(\Delta)$. But $\text{stab}_N(\Delta)/C$ is a $\{2,3\}$-group. Thus $Q \leq C$. For each i, $F_i \cap C/C_i \cap C$ is isomorphic to a normal nilpotent subgroup of N_i/C_i, and N_i/C_i acts irreducibly on X_i. Thus, for $i = 1, \ldots, l$, λ_i is not centralized by a non-trivial Sylow-subgroup of $F_i \cap C/C_i \cap C$. Since $Q \cap F_i \in \text{Syl}_q(F_i \cap C)$, we have that $q \nmid |F_i \cap C/C_i \cap C|$ for $i = 1, \ldots, l$. By our choice of Δ, each F_j/C_j $(j = 1, \ldots, n)$ is conjugate to some F_i/C_i

with $i \in \{1, \dots, l\}$. Hence

$$q \nmid |F_j \cap C / C_j \cap C|$$

for all $j = 1, \dots, n$. Since $\bigcap_i C_i = M$ and $\bigcap_i (F_i \cap C) = F$, we have that $q \nmid |F/M|$. We have seen above that $Q \leq C$ and so $q \nmid |N/C|$. Thus $|N : \mathbf{C}_N(\lambda)|$ is divisible by every prime in $\pi_0(N/C) \cup \pi_0(F/M)$.

Now let

$$\beta \in \mathrm{Irr}(N|\mu) \quad \text{and} \quad \chi \in \mathrm{Irr}(N|\lambda).$$

By the last two paragraphs, $\beta(1)$ is divisible by every prime divisor of $|C/F|$ and $\chi(1)$ is divisible by every prime in $\pi_0(N/C) \cup \pi_0(F/M)$. The conclusion of the lemma is met with $\theta = \beta$ or $\theta = \chi$. □

1.3 Lemma. *Suppose that $M = \mathbf{C}_G(M)$ is a normal elementary abelian subgroup of a solvable group G and a completely reducible G-module (possibly of mixed characteristic). Assume that G splits over M. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is divisible by at least half of the primes in $\pi_0(G/M)$.*

Proof. We proceed by induction on $|M|$. Write $M = M_1 \oplus \cdots \oplus M_n$ for $n \geq 1$ irreducible G-modules M_i. Set $V_i = \mathrm{Irr}(M_i)$ so that each V_i is an irreducible G-module and $V = V_1 \oplus \cdots \oplus V_n$ is a faithful G/M-module. For each i, choose $H_i \leq G$ and X_i an irreducible primitive H_i-module with $X_i^G = V_i$. If $H_i/\mathbf{C}_{H_i}(X_i) \leq \Gamma(X_i)$ for each i, this lemma follows from Lemma 1.2. We assume without loss of generality that $H_1/\mathbf{C}_{H_1}(X_1) \nleq \Gamma(X_1)$.

Let $K = \mathbf{C}_G(M_1) \trianglelefteq G$. Let H be a complement for M in G and let $J = NH$ where $N = M_2 \oplus \cdots \oplus M_n$. Now $J \cap K = N(H \cap K)$ acts on N and $\mathbf{C}_{J \cap K}(N) = N$. By induction, there exists $\tau \in \mathrm{Irr}(J \cap K)$ such that $\tau(1)$ is divisible by at least half the primes in $\pi_0((J \cap K)/N) = \pi_0(K/M)$, as $(J \cap K)/N \cong K/M$. Now $J \cap K \trianglelefteq J$ and centralizes $M/N \cong M_1$. Thus $J \cap K \trianglelefteq KJ = G$ and $K/N = M/N \times (J \cap K)/N$.

By the choice of M_1, Theorem A implies that there exists $\lambda \in V_1$ such that

$$\pi_0(G/K) = \pi_0(G : I_G(\lambda)).$$

Set $\beta = \lambda \cdot \tau \in \mathrm{Irr}(K)$. Now $I_G(\beta) \subseteq I_G(\lambda)$. Thus $\pi_0(G : I_G(\beta)) \supseteq \pi_0(G/K)$. If $\chi \in \mathrm{Irr}(G|\beta)$, then as $K \trianglelefteq G$, $\pi_0(\chi(1)) \supseteq \pi_0(G/K) \cup \pi_0(\tau(1))$. Since $\tau(1)$ is divisible by at least half the primes in $\pi_0(K/M)$, certainly $\chi(1)$ is divisible by at least half the primes in $\pi_0(G/M)$. □

1.4 THEOREM. *Let G be solvable. Then*
(a) $|\rho(G)| \leq 3 \cdot \sigma(G) + 2$.
(b) $|\rho(G)| \leq 2 \cdot \sigma(G) + 2$ *if* $r \mid |G/\mathbf{O}_r(G)|$ *whenever* $\mathbf{O}_r(G)$ *is non-abelian.*

Proof. Let $\mathscr{R} = \{r \text{ prime} \mid \mathbf{O}_r(G) \in \mathrm{Syl}_r(G) \text{ and } \mathbf{O}_r(G) \text{ is non-abelian}\}$ and $F = \mathbf{F}(G)$. Certainly $\rho(G) \subseteq \pi(G/F) \cup \mathscr{R}$ and by Ito's Theorem [Is, 12.33], equality holds.

By a theorem of Gaschütz (see [Hu, III.4.2, III.4.4, and III.4.5]), $\mathbf{F}(G)/\Phi(G)$ is a faithful completely reducible G/F-module and $G/\Phi(G)$ splits over $\mathbf{F}(G)/\Phi(G)$. Applying Lemma 1.3, there exists $\chi \in \mathrm{Irr}(G)$ with $\pi_0(\chi(1)) \geq \pi_0(G/F)/2$. Hence

$$\sigma(G) \geq \pi_0(G/F)/2.$$

Under hypothesis (b), $\mathscr{R} \subseteq \pi(G/F)$ and thus

$$\rho(G) = \pi(G/F) \subseteq \pi_0(G/F) \cup \{2,3\}.$$

In this case, $2\sigma(G) \geq |\rho(G)| - 2$, as desired.

Now $\prod_{r \in \mathscr{R}} \mathbf{O}_r(G) \trianglelefteq G$ and each $\mathbf{O}_r(G)$ is non-abelian. Thus there exists $\eta \in \mathrm{Irr}(G)$ such that $\mathscr{R} \subseteq \pi(\eta(1))$. Since

$$\sigma(G) \geq \max\{|\mathscr{R}|, \pi_0(G/F)/2\}$$

and since

$$\rho(G) = \pi(G/F) \cup \mathscr{R} \subseteq \pi_0(G/F) \cup \mathscr{R} \cup \{2,3\},$$

part (a) follows. □

Suppose $|G|$ is odd. If we employ Lemma 1.1 (b) and Theorem 2.7 instead of Lemma 1.1(a) and Theorem A, then the conclusions of Lemmas 1.2 and 1.3 remain valid with π replacing π_0. Consequently, we get:

1.5 THEOREM. *If $|G|$ is odd, then*:
(a) $|\rho(G)| \leq 3 \cdot \sigma(G)$.
(b) (*Espuelas* [Es]) $\rho(G) \leq 2\sigma(G)$ *if* $r \mid |G/\mathbf{F}(G)|$ *whenever* $\mathbf{O}_r(G)$ *is non-abelian.*

Gluck [Gl3] has verified Huppert's conjecture $|\rho(G)| \leq 2\sigma(G)$ for solvable G in a number of special cases. This bound is not correct for arbitrary G, but appears to be of the correct order of magnitude. If L is A_5 or $PSL(2,8)$, then $|\rho(L)| = 3$ and $\sigma(L) = 1$. It has been verified that $|\rho(S)| \leq 3\sigma(S)$ for simple S by Alvis and Barry [AB] and Manz, Staszewski, and Willems [MSW]. Altering the construction at the beginning of this section by letting $H_1 = A_5$ instead, then $|\rho(G_n)| = 2n + 1$ and $\sigma(G_n) = n$. Possibly $\sigma(G) \leq 2|\rho(G)| + 1$ is valid for all G.

2. Orbits

Here we prove Theorem A. Recall that a G-module V is quasi-primitive if V_N is homogeneous for all $N \trianglelefteq G$.

2.1 THEOREM. *Suppose that V is a faithful quasi-primitive G-module, G solvable. Then there exist normal subgroups Z, U, T, A, C, and $F = \mathbf{F}(G)$ of G satisfying*:

(a) *U is cyclic and Z =* socle(U);
(b) *$U \leq T$, $U = \mathbf{C}_T(U)$ and $|T:U| \leq 2$;*
(c) *$F/T = F_1/T \times \cdots \times F_l/T$ where each F_i/T is an irreducible G/F-module of order e_i^2 for a prime power e_i. We let $e = \prod_{i=1}^{l} e_i = |F:T|^{1/2}$;*
(d) *$A = \mathbf{C}_G(Z)$ and A/F acts faithfully on F/T;*
(e) *$C = \mathbf{C}_G(F/T)$, $C \cap A = F$, and $C/F \leq \mathbf{Z}(G/F)$;*
(f) *Each prime divisor of e divides $|Z|$;*
(g) *If W is an irreducible U-submodule of V, then* $\dim(V) = te \dim(W)$ *for an integer t.*

Proof. Parts (a)–(d), (f) follow from Lemma 2.3 and Corollary 2.4 of [Wo2], because every normal abelian subgroup of G is cyclic. Define $C = \mathbf{C}_G(F/T) \geq F$. Part (d) implies that $C \cap A = F$. Now Aut(Z) is abelian and G/A is G-isomorphic to a subgroup of Aut(Z). Since $C \cap A = F$, it follows that $C/F \leq \mathbf{Z}(G/F)$. This proves (e). Part (g) follows from [Wo2, Lemma 2.5]. □

2.2 LEMMA. *Suppose V is a finite faithful irreducible G-module. Assume that one of the following occurs*:

(i) *$A \trianglelefteq G$, A is abelian, and V_A is irreducible;*
(ii) *$A = \mathbf{C}_G(A) \trianglelefteq G$ and V_A is homogeneous; or*
(iii) *G is solvable, V is quasi-primitive, and $e = 1$ (as in Theorem 2.1).*

Then $G \leq \Gamma(V)$.

Proof. Part (i) is [Hu; II, 3.11]. Under hypothesis (ii) and finiteness of V, V_A is irreducible by Theorem 4.2 of [Pk]. So (i) applies.

Assume that V is quasi-primitive and adopt the notation of Theorem 2.1. If $e = 1$, Theorem 2.1 (c, d) imply that $F = T = \mathbf{C}_G(Z)$. Since $Z \leq U = \mathbf{C}_T(U) \trianglelefteq G$, indeed $U = \mathbf{C}_G(U) \trianglelefteq G$ and hypothesis, (ii) is met. □

Observe that condition (ii) is met when G is solvable, when $F = \mathbf{F}(G)$ is abelian, and V_F is homogeneous.

2.3 Corollary. *Suppose G is a solvable irreducible subgroup of $GL(n, p)$.*
(i) *If $p = 2$ and n is prime then $G \lesssim \Gamma(2^n)$.*
(ii) *If G is quasi-primitive and $n = p^m$ for some m, then $G \lesssim \Gamma(p^{p^m})$.*

Proof. Let V be the corresponding G-module, let $F = \mathbf{F}(G)$, let $1 \neq Q \in \mathrm{Syl}_q(F)$ for a prime q, and let $Z = \mathbf{Z}(Q)$. Note that $q \neq p$.

If Q is non-abelian, then $q | \dim(V)$ because $Q \trianglelefteq G$ and every faithful absolutely irreducible representation of Q has degree divisible by q. Under hypotheses (ii), it thus follows that Q and F are abelian. By Lemma 2.2, $G \leq \Gamma(V)$.

Now assume that $p^n = 2^n$ with n prime. If U is an irreducible Z-submodule of V, then $\dim(U) > 1$ because $Z \neq 1$. Thus V_Z is irreducible. By Lemma 2.2, $G \leq \Gamma(V)$. □

2.4 Lemma. *Suppose that G is a solvable irreducible subgroup of $GL(n, p)$, p prime.*
(a) *If $p^n = p^2$, then $\pi_0(G) \subseteq \pi_0(p^2 - 1)$ and G has a normal Sylow-q-subgroup for each $q \in \pi_0(G)$.*
(b) *If $p^n \in \{2^4, 2^6, 2^8, 3^6\}$, then $|\pi_0(G)| \leq 2$ and G has a normal Sylow-q-subgroup for each $q \in \pi_0(G)$.*
(c) *If $p^n = 3^4$, then $\pi_0(G) \subseteq \{5\}$.*
(d) *If $p^n = 2^{10}$ and $|\pi_0(G)| > 1$, then $G \leq \Gamma(2^5) \mathrm{wr}\, Z_2$ or $G \leq \Gamma(2^{10})$.*

Proof. Let V be the corresponding G-module. If V is not quasi-primitive, then $G \precsim H \mathrm{wr}\, S$, where $S \leq S_m$ is a solvable primitive permutation group on m letters and $1 \neq H$ is a solvable irreducible subgroup of $GL(n/m, p)$. Should $p = 2$, $n > m$ and so $m \leq 5$ in all cases. If $m = 5$, then $p^n = 2^{10}$, $|S| | 20$ and $G \leq S_3 \mathrm{wr}\, H$, whence conclusion (d) holds. Then we may assume that $2 \leq m \leq 4$ and $\pi_0(H) = \emptyset$. With help of Corollary 2.3, it is easy to see all conclusions of the lemma hold. Thus we assume that V is quasi-primitive.

Should $G \leq \Gamma(V)$, the conclusions of the lemma are satisfied. Theorem 2.1 applies and we adopt the notation in Theorem 2.1. In particular, $|F : T| = e^2$ for an integer e, $\dim(V) = te \dim(W)$ where W is an irreducible U-submodule of V, $t \in \mathbf{N}$. By Lemma 2.2, we may assume $e > 1$. Each prime divisor of e divides $|U|$. Furthermore $U | |W| - 1$ as U is cyclic. Also $p \nmid e$, because

$\mathbf{O}_p(G) = 1$. Thus the only possibilities are:

e	p^n	$\|W\|$	$\|U\|$
4	3^4	3	2
3	2^6	4	3
2	3^6	3 or 3^3	divides 26
2	3^4	3 or 3^2	divides 8
2	p^n	p	divides $p - 1$

In the last case, $V_U = V_1 \oplus V_2$ for isomorphic 1-dimensional U-modules V_i, whence $U \leq \mathbf{Z}(GL(V))$. By Theorem 2.1 (d, e), it follows that $|C/F| \mid 12$ in all cases. Thus $\pi_0(C/U) = \emptyset$. The conclusion of the theorem is met unless $\pi_0(G:C) \neq \emptyset$. But F/T is a faithful G/C-module of order $e^2 = 2^2$, 3^2, or 4^2, and F/T is an irreducible G/C-module or the direct sum of two G/C-modules of order 4. Since we may assume that $\pi_0(G/C) \neq \emptyset$, indeed F/T is a faithful irreducible G/C-module of order 2^4. Conclusion (c) now holds (see Corollary 2.3). □

2.5 Proposition. *Let G be solvable. Then the number $|\mathrm{Syl}(G)|$ of distinct Sylow-subgroups of G (for all primes) is at most $|G|$.*

Proof. By induction on $|G|$. We note that equality holds when $|G| \leq 2$. We may choose a maximal normal subgroup M of G and set $q = |G/M|$, a prime. By the inductive hypothesis, $|\mathrm{Syl}(M)| \leq M$. If $P \in \mathrm{Syl}_p(G)$ for $p \neq q$, then $P \in \mathrm{Syl}_p(M)$, and so the number of Sylow subgroups of G for all primes other than q is at most $|M|$. But $|\mathrm{Syl}_q(G)| \leq |G|/q = |M|$. Hence $|\mathrm{Syl}(G)| \leq 2|M| \leq |G|$. □

Next is Theorem A.

2.6 Theorem. *Suppose V is a finite faithful irreducible G-module for a solvable group G. Write $V = W^G$ where W is a primitive H-module, $H \leq G$. If $H/\mathbf{C}_H(W) \nleq \Gamma(W)$, then there exists $v \in V$ such that $\pi_0(G:\mathbf{C}_G(v)) = \pi_0(G)$.*

Proof. By induction on $|G|$. For each $v \in V$, we may assume that $\mathbf{C}_G(v)$ contains a Sylow-p-subgroup of G for some $p \geq 5$, since otherwise the conclusion of the theorem is satisfied.

Step 1. $H = G$ and V is a primitive G-module.

Proof. For $H \leq J \leq G$, W^J is irreducible and thus $H = \mathbf{N}_G(W)$. Since $V = W^G$, we may write $V = W_1 \oplus \cdots \oplus W_m$ for subspaces W_i of V that are transitively permuted by G with $W = W_1$. Set $H_i = \mathbf{N}_G(W_i)$, so that H_i is conjugate to H and

$$H_i/\mathbf{C}_{H_i}(W_i) \cong H/\mathbf{C}_H(W).$$

If $H < G$, we may apply the inductive hypothesis to conclude there exists $y \in W$ such that $\pi_0(H:\mathbf{C}_H(y)) = \pi_0(H:\mathbf{C}_H(W))$.

Let $N = \bigcap_{i=1}^m H_i$, so that G/N faithfully and transitively permutes $\{W_1,\dots,W_m\}$. By Lemma 1.3, we may assume that $\mathrm{stab}_{G/N}\{W_1,\dots,W_l\}$ is a $\{2,3\}$-group for some $l \le m$. Assuming H is proper in G, set

$$x = y + x_2 + \cdots + x_l$$

where $0 \neq x_i \in W_i$ $(2 \le i \le l)$. Then

$$\mathbf{C}_G(x)/\mathbf{C}_N(x) \cong N\mathbf{C}_G(x)/N \le \mathrm{stab}_{G/N}(W_1,\dots,W_l)$$

is a $\{2,3\}$-group. If $q \ge 5$ is prime and $Q \in \mathrm{Syl}_q(G)$ centralizes x, then

$$Q \le \mathbf{C}_N(x) \le \mathbf{C}_N(y) \le \mathbf{C}_H(y).$$

By choice of y, $Q \le \mathbf{C}_H(W) \cap N = \mathbf{C}_N(W)$. Thus $N/\mathbf{C}_N(W) \cong N/\mathbf{C}_N(W_i)$ is a q'-group for all i. Since $\bigcap_i \mathbf{C}_N(W_i) = 1$, indeed $Q = 1$. Thus $\pi_0(G:\mathbf{C}_G(x)) = \pi_0(G)$, as desired. So we may assume that $H = G$.

Step 2. Let π be the set of those prime divisors $p \ge 5$ of G for which $\mathbf{C}_V(P) \neq 0$, $P \in \mathrm{Syl}_p(G)$. Then

(a) $\sum_{p\in\pi}\sum_{p\in \mathrm{Syl}_p(G)}|\mathbf{C}_V(P)| \ge |V|$;
(b) $\mathbf{O}_\pi(G) = 1$; and
(c) $|\pi| \ge 2$.

Proof. By the first paragraph of the proof, each $v \in V$ is centralized by some Sylow-p-subgroup for some $p \in \pi$. Part (a) is a consequence thereof. To prove (b), we may, by the solvability of G, assume that $\mathbf{O}_q(G) \neq 1$ for some $q \in \pi$. By definition of π, $\mathbf{C}_V(\mathbf{O}_q(G)) \neq 0$. This is a contradiction because V is a faithful and homogeneous $\mathbf{O}_q(G)$-module.

If $\pi = \{p\}$, every $v \in V$ is centralized by a Sylow-p-subgroup. By [MW1, Theorem 1.8], $L := \mathbf{O}^{p'p}(G)$ is a cyclic p'-group and V is an irreducible $\mathbf{O}^{p'}(G)$-module. Let Y be an irreducible L-submodule of V, let $0 \neq y \in Y$ and choose $P \in \mathrm{Syl}_p(G)$ such that $P \le \mathbf{C}_G(y)$. Then Y is invariant under $LP = \mathbf{O}^{p'}(G)$. So $Y = V$ is an irreducible L-module. By Lemma 2.2, $G \le \Gamma(V)$. Part (c) follows.

Step 3. Theorem 2.1 applies and we adopt that notation. In particular

(a) $F/T = F_1/T \times \cdots F_l/T$ for irreducible G/F-modules F_i/T of order e_i^2, $e_i \in \mathbf{Z}$;

(b) $e = e_1 \cdots e_l > 1$;

(c) If W is an irreducible U-submodule of V, then $|V| = |W|^{te}$ for an integer t;

(d) $|U| \,|\, |W| - 1$ and each prime divisor of e divides $|U|$.

Proof. Parts (a) and (c) follows form Theorem 2.1, as does the fact that each prime divisor of $e = e_1 \cdots e_l$ divides $|U|$. Since V_U is homogeneous and U is cyclic, then $|U| \,|\, |W| - 1$. That $e > 1$ follows from Lemma 2.2.

Step 4. Some $p \in \pi$ does not divide $|D/U|$.

Proof. Assume each $p \in \pi$ does divide $|D/U|$. If $P \in \mathrm{Syl}_p(G)$, then $P \cap D \in \mathrm{Syl}_p(D)$. Thus each $v \in V$ is centralized by a non-trivial Sylow-q-subgroup of D for some $q \in \pi$. Choose $\pi_1 \subseteq \pi$ minimal such that each $v \in V$ is centralized by a Sylow-q-subgroup of D for some $q \in \pi_1$. Next let $D_1/U \in \mathrm{Hall}_{\pi_1}(D/U)$ so that $D_1 \trianglelefteq G$ and each $v \in V$ is centralized by a non-trivial Sylow-q-subgroup of D_1 for some $q \in \pi_1$.

Since $U = F \cap D_1$, certainly $U = \mathbf{F}(D_1) = \mathbf{C}_{D_1}(U)$. To show that $G \leq \Gamma(V)$, it suffices to show that V is an irreducible D_1-module (see Lemma 2.2 and Step 1). So write $V = X_1 \oplus X_2$ for non-zero D_1-submodules X_i of V and let $0 \neq x \in X_1$. For $y \in X_2$, $C_{D_1}(x + y)$ contains a Sylow-q-subgroup of D_1 for some $q \in \pi$. Since $\mathbf{C}_{D_1}(x + y) \leq \mathbf{C}_{D_1}(x)$ for all y and since V_{D_1} is homogeneous, it follows from the minimality of π_1 that $\mathbf{C}_{D_1}(x)$ contains a Sylow-q-subgroup of D_1 for each $q \in \pi_1$. Since U acts fixed-point freely on V, $\mathbf{C}_U(x) = 1$. But D_1/U is a π_1-group and so $\mathbf{C}_{D_1}(x) \in \mathrm{Hall}_{\pi_1}(D_1)$. Choose $y \in X_2$ not centralized by $\mathbf{C}_{D_1}(x)$. Thus $\mathbf{C}_{D_1}(x + y) \notin \mathrm{Hall}_{\pi_1}(D_1)$. But since V_{D_1} is completely reducible, $V_{D_1} = Y_1 \oplus Y_2$ for D_1-invariant $Y_i \neq 0$ with Y_1 irreducible and $x + y \in Y_1$. The argument above for x shows that $\mathbf{C}_G(x + y) \in \mathrm{Hall}_{\pi_1}(D_1)$, a contradiction. Hence V is an irreducible D_1-module and $G \leq \Gamma(V)$, as desired. Step 4 follows.

Step 5. Let $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$. Then

(a) $|\mathbf{C}_V(P)| \leq |V|^{1/2}$;

(b) If $1 \neq P_1 \leq P \cap D$, then $|\mathbf{C}_V(P_1)| \leq |V|^{1/5}$ and $p \,|\, t \cdot \dim(W)$;

(c) $|G| \geq \Sigma_{p \in \pi} |\mathrm{Syl}_p(G)| \geq |V|^{1/2}$.

Proof. Let $1 \neq P_0 \leq P$ with $|P_0| = p$. Recall that $p \nmid |F|$. First suppose that $p \,|\, |D|$ and assume without loss of generality that $P_0 \leq D$. Since $U = \mathbf{C}_D(U)$ by Step 1 and $p \nmid |U|$, we may choose $1 \neq Y \leq Z$ with YP_0 a Frobenius group. Note $\mathbf{C}_V(Y) = 0$ because $Y \trianglelefteq G$. Then $\dim(V) = p \cdot$

$\dim(\mathbf{C}_V(P_0))$ by [Is, Theorem 15.16]. Since $\dim(V) = te\dim(W)$ and $p \nmid |F|$, in fact $p | t \cdot \dim(W)$. Parts (a) and (b) follow when $p||D|$. When $p \nmid |D|$, part (a) follows from [Wo1, Lemma 1.7].

Part (c) follows from Proposition 2.5, Step 2 (a) and part (a) of this step.

Step 6. (a) Set $C_i = \mathbf{C}_G(F_i/T)$. Assume that G/C_i has a normal Sylow-q-subgroup for all $q \in \pi$ and all i, $1 \le i \le l$. Then $|\pi_0(G:C)| \ge 4$.

(b) We may assume that $e_1 \ge 8$.

(c) If $e \ge 32$, then $e_1 = 9$, $e = e_1 = 5^2$, or $e = e_1 = 2^5$.

Proof. Now F_i/T is a faithful irreducible G/C_i = module of order e_i^2 for each i. Also

$$\bigcap_i C_i = C = \mathbf{C}_G(F/T).$$

If $e_i \in \{2,3,5,7\}$, then $\pi_0(G/C_i) = \emptyset$. By Step 4, some prime $q_0 \in \pi \subseteq \pi_0(G/F)$ does not divide $|D/U|$ and thus does not divide $|C/F|$. Thus $s := |\pi_0(G/C) \cap \pi|$ is at least one. We may assume $q_0||G/C_1|$ and thus $e_1 = 4$ or $e_1 \ge 8$.

(a) Since G/C_i has a normal Sylow-q-subgroup for all $q \in \pi$ and since $\bigcap_i C_i/F = C/F \le \mathbf{Z}(G/F)$, indeed G/F has a normal Sylow-q-subgroup for all $q \in \pi$. If $q \in \pi$ does not divide $|G/C|$, then each Sylow-q-subgroup Q of G lies in D and $|\mathbf{C}_V(Q)| \le |V|^{1/5}$ by Step 5 (b). If q does divide $|G/C|$, then $|\mathbf{C}_V(Q)| \le |V|^{1/2}$ by Step 5 (a) and $|\mathrm{Syl}_q(G)| \le |F:\mathbf{C}_F(Q)| \le e^2|U|$. Since $\sum_{q\in\pi}|\mathrm{Syl}_q(G)||\mathbf{C}_V(Q)| \ge |V|$ by Step 2 (a), we have that

$$se^2 \cdot |U| \cdot |V|^{1/2} + |D||V|^{1/5} \ge |V|$$

using Proposition 2.5 to bound $|\mathrm{Syl}(D)|$. Since $U = \mathbf{C}_D(U)$ is cyclic, indeed $|D| \le |U|^2$. Since W is an irreducible U-module and $e \ge 4$, it follows that $|U| < |W| < |W|^{3e/10} \le |V|^{3/10}$ and $|D||V|^{1/5} < |U||V|^{1/2}$. Then $(se^2+1)|U| > |V|^{1/2}$. But, for now, we may assume that $1 \le s \le 3$ and $3e^2 + 1 > |W|^{(te/2)-1}$. Since $|U|||W|-1$, then $|W| \ge 3$ and $e < 16$. If $\pi_0(G/C_i) \ne \emptyset$, it follows with help of Lemma 2.4 that $e_i \ge 4$ and $\pi_0(G/C_i)$ is a singleton. Since $e < 16$, then $\pi_0(G/C_j) = 0$ for $j \ge 2$ and $s = |\pi \cap \pi_0(G/C)| = 1$. Because $|\pi| \ge 2$, some $r \in \pi$ divides $|D/U|$ and $r|t\dim(W)$ by Step 4 (b). Thus $e^2 + 1 > (32)^{(e/2)-1}$, whence $e < 4$, a contradiction. This proves (a).

(b) If every $e_i \le 7$, it follows by Lemma 2.4 that $\pi_0(G/C_i) \subseteq \{5\}$ and G/C_i has a normal Sylow-5-subgroup for all i, contradicting (a). So we assume that $e_1 \ge 8$.

(c) Suppose now $e = e_1 \cdots e_l \le 32$. By part (b), it follows that $\pi_0(G/C_i) = \emptyset$ for all $i \ge 2$ and $\pi_0(G/C) = \pi_0(G/C_1)$. By part (a), it follows that $|\pi_0(G/C_i)| \geq 4$ or that G/C_1 does not have a normal Sylow-q-subgroup for some $q \in \pi$. Since $e \leq 32$, Lemma 2.4 yields that $e_1 = 9$, $e = e_1 = 5^2$, or $e = e_1 = 2^5$.

Step 7. Conclusion.

Proof. Since F/T is a faithful, completely reducible G/C-module or order e^2, it follows from [Wo2, Theorem 3.1] that $|G/C| < (e^2)^{9/4}/2$. Since $C/F \lesssim G/\mathbf{C}_G(Z)$ and Z is cyclic, $|C/F| \leq |Z| \leq |U|$. Also $|T:U| \leq 2$ with equality possible only when $2 \mid |U|$. Thus $|C/F||T| \leq |U|^2$ in all cases. Now

$$|G| \leq |G:C||C:F|||F:T||T| \leq e^{13/2}|U|^2/2.$$

By Step 2,

$$e^{13}|U|^4 \geq 4|V| = 4|W|^{te}.$$

Since $|U| \mid |W| - 1$ by Step 3 (d), indeed

$$e^{13} \geq 4|W|^{te-4} \geq 4 \cdot 3^{e-4} \tag{2.1}$$

and hence $e < 64$. Every prime divisor of e divides $|U|$ and $|W| - 1$. If $e > 32$, then e is divisible by a prime $p \geq 5$ or $6|e$, whence $|W| \geq 7$ and (2.1) gives a contradiction. So $e \leq 32$. If $e = 25$, then $|W| \geq 11$ and (2.1) gives a contradiction. By Step 6, either $e_1 = 9$ or $e = e_1 = 2^5$.

First suppose $e_1 = 3^2$. Since $e \leq 32$, Lemma 2.4 yields that $\pi_0(G/C) \subseteq \{5\}$. Since $|\pi| \geq 2$, some prime $q \geq 7$ in π divides $|D/U|$ and $t \dim(W)$ by Step 5 (b). If $t \geq 7$, then $e^{13} \geq 4 \cdot |W|^{7e-4}$, an easy contradiction. Thus $q | \dim(W)$. Since (2.1) implies that $|W| \leq 303$, indeed $|W| = 2^7$, a contradiction because $3 \mid |W| - 1$. So $e = e_1 = 2^5$.

By (2.1), it follows that $t = 1$, $|W| = 3$ and thus $|U| = 2$. Hence $U \leq \mathbf{Z}(G)$, $C = F$ and $D = T = U$. In particular, F is extra-special of order 2^{11}. By Lemma 2.4 (d),

$$G/F \lesssim \Gamma(2^5) \operatorname{wr} Z_2 \quad \text{or} \quad G/F \lesssim \Gamma(2^{10}).$$

Thus $\pi \subseteq \{5, 11, 31\}$. Routine arguments show that

$$|\mathrm{Syl}_{31}(G)| \leq 2^{10}, |\mathrm{Syl}_{11}(G)| \leq 2^{10} \quad \text{and} \quad |\mathrm{Syl}_5(G)| \leq 2^{10} \cdot 31^2 \leq 2^{20}.$$

By Step 5 (c), $3^{16} = |V|^{1/2} \leq 2^{20} + 2^{10} + 2^{10} < 2^{21}$, a contradiction. □

For completeness, we include the following, which was at least implicity inferred by Espuelas in his proof of Theorem 1.5 (b).

2.7 THEOREM. *Suppose V is a finite faithful irreducible G-module and $|G||V|$ is odd. Write $V = W^G$ where W is a primitive H-module, $H \leq G$. If $H/\mathbf{C}_H(W) \nleq \Gamma(W)$, then there exists $v \in V$ such that $\pi(G : C_G(v)) = \pi(G)$.*

Proof. If V is imprimitive, repeat the argument of Step 1 of Theorem 2.6 using Lemma 1.1 (b) instead of 1.1 (a). If V is primitive and $G \nleq \Gamma(V)$, Espuelas [Es, Lemma 2.1] proved that G has even a regular orbit. □

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UNIVERSITY OF HEIDELBERG
HEIDELBERG, GERMANY

OHIO UNIVERSITY
ATHENS, OHIO