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DEFORMATIONS AND DIFFEOMORPHISM TYPES OF HOPF MANIFOLDS

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1. Introduction

A generalized Hopf manifold or simply a Hopf manifold of complex dimension n is a compact complex manifold of which the universal covering is $\mathbf{C}^n - \{\mathbf{0}\}$, where n is a positive integer ($n \geq 2$).

The Hopf manifold, first introduced by H. Hopf, is well known as the first example of a non-Kähler manifold. In his essays [3] presented to R. Courant, H. Hopf referred to a complex manifold diffeomorphic to $S^1 \times S^{2n-1}$, which was originally called a Hopf manifold. The generalized definition above is due to K. Kodaira [6].

Perhaps one of the first fundamental problems concerning the Hopf manifold is to determine their diffeomorphism types. This was done for the case of $n = 2$ by M. Kato [4]. Later, in his paper [5], M. Kato studied submanifolds of Hopf manifolds and obtained a result on diffeomorphism types of Hopf manifolds (although the result is not fully stated as a theorem, it may be inferred from the results in the paper).

In this paper we study deformations of Hopf manifolds and give a short and direct proof of the theorem that a Hopf manifold of complex dimension n is diffeomorphic to a fiber bundle over S^1 with fiber S^{2n-1}/H, defined by a representation $\rho\colon \pi_1(S^1) \to N_{U(n)}(H)$ such that $\rho(1)$ is an element of finite order in $N_{U(n)}(H)$, where H is a finite unitary and fixed-pint-free group, and $N_{U(N)}$ is the nomalizer of H in $U(n)$. This theorem determines explicitly the diffeomorphism types of the Hopf manifolds.

We state here a conjecture that a compact complex manifold of which the universal covering is $\mathbf{C}^n$ is diffeomorphic to a manifold which has a torus or a non-toral nilmanifold as a finite covering. The first case is clearly a Kähler manifold and the second case is a non-Kähler manifold (cf. [2]).

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2. Fundamental properties of the covering transformation groups of Hopf manifolds

In this section, we will review some results of K. Kodaira [6] and M. Kato [4] on the basic properties of Hopf manifolds in the generalized form.

An analytic automorphism g over $\mathbf{C}^n$ which fixes the origin is called a contraction if the sequence $\{g^n\}$ converges uniformly to $\mathbf{0}$ on any compact neighborhood of the origin as n approaches infinity, or equivalently, if for any $r_1, r_2 \in \mathbf{R}_+$ there exists an $m \in \mathbf{N}$ such that

$$g^n(B(r_1)) \subset \mathrm{Int}(B(r_2))$$

holds for any $n \in \mathbf{N}$ $(n \geq m)$, where $\mathbf{R}_+$ is the set of positive real numbers, $\mathbf{N}$ is the set of positive integers, $B(r) = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n \mid |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \leq r^2\}$, and $\mathrm{Int}(B)$ is the interior of B. Note that we have defined a contraction in a slightly stronger form than the original one in [6].

Now let M be a Hopf manifold and G its covering transformation group. Then G is properly discontinuous and fixed-point-free. We regard M as the quotient manifold W/G where W denotes $\mathbf{C}^n - \{\mathbf{0}\}$. By Hartogs' Lemma, we can consider any element of G as an analytic automorphism over $\mathbf{C}^n$ which fixes the origin.

THEOREM 2.1. *Let G be the covering transformation group of a Hopf manifold. Then G contains an infinite cyclic subgroup, and any cyclic subgroup of G is generated by a contraction.*

Step 1. There exists a $g \in G$ such that $g(B(1)) \subset \mathrm{Int}(B(1))$. Thus $Z = \langle g \rangle$ is an infinite cyclic subgroup of G.

Proof. For simplicity, we write B in place of $B(1)$. Since G is properly discontinuous, $g(\partial(B)) \cap \partial(B) = \emptyset$ for all but finitely many $g \in G$, where $\partial(B)$ is the boundary of B. Since G is obviously infinite, there exists a $g \in G$ such that $g(\partial(B)) \cap \partial(B) = \emptyset$. As g fixes the origin, it follows that $g(B) \subset \mathrm{Int}(B)$ or $g^{-1}(B) \subset \mathrm{Int}(B)$.

Step 2. g obtained in step 1 is a contraction.

Proof. Suppose that g is not a contraction. Then there are $B_1 = B(r_1)$ and $B_2 = B(r_2)$ $(r_1, r_2 \in \mathbf{R}_+)$ such that $g^n(B_1) \not\subset \mathrm{Int}(B_2)$ for infinitely many $n \in \mathbf{N}$. Hence there exists a subsequence $\{k_n\}$ of $\mathbf{N}$ such that $g^{k_n}(B_1) \not\subset \mathrm{Int}(B_2)$ for all $n \in \mathbf{N}$. Since g fixes the origin and B_1 is connected, it follows that $g^{k_n}(B_1) \cap \partial(B_2) \neq \emptyset$ for all $n \in \mathbf{N}$. Therefore, we can take $z_n \in B_1$ $(z_n \neq \mathbf{0})$ such that $g^{k_n}(z_n) \in \partial(B_2)$ for each $n \in \mathbf{N}$. We will show that $\lim_{n\to\infty} z_n = \mathbf{0}$. Suppose that $\lim_{n\to\infty} z_n = a (a \neq \mathbf{0})$. Then $K = \{a\} \cap \{z_n\}$ is a compact subset of W and $g^{k_n}(K) \cap \partial(B_2) \neq \emptyset$ for all $n \in \mathbf{N}$. This contradicts the fact that $z = \langle g \rangle$ is properly discontinuous, and thus $\lim_{n\to\infty} z_n = \mathbf{0}$. Now, since $g^n(B) \subset \operatorname{Int}(B)$ for all $n \in \mathbf{N}$, $\{g^{k_n}\}$ $(n \in \mathbf{N})$ is uniformly bounded over B. And thus we can see by Cauchy's estimate that $\{g^{k_n}\}$ is equi-continuous at the origin. Therefore $\lim_{n\to\infty} g^{k_n}(z_n) = \mathbf{0}$, which contradicts the fact that $G^{k_n} \in \partial(B_2)$ for all $n \in \mathbf{N}$.

Step 3. Let Z be any infinite cyclic subgroup of G. Then Z is generated by a contraction.

Proof. Since $Z = \langle g \rangle$ is properly discontinuous, in the same manner as in step 1, there exists a $k \in \mathbf{N}$ such that $g^k(B) \subset \operatorname{Int}(B)$ or $g^{-k}(B) \subset \operatorname{Int}(B)$; thus g^k or g^{-k} is a contraction. Take g^{-1} as a generator of Z in the latter case. We will show that g is a contraction. Suppose that g is not a contraction. Then there exists B_1 and B_2 as in the proof of step 2 such that $g^n(B_1) \subset \operatorname{Int}(B_2)$ for infinitely many $n \in \mathbf{N}$. But then there exists $r \in N$ $(0 \leq r < k)$ such that

$$g^{kn+r}(B_1) = g^{kn}(g^r(B_1)) \not\subset \operatorname{Int}(B_2)$$

for infinitely many $n \in \mathbf{N}$. Since $g^r(B_1)$ is a compact neighborhood of the origin, this contradicts that g^k is a contraction.

COROLLARY 2.2. *Let Z be an infinite cyclic subgroup of G. Then $[G; Z]$ is finite.*

Proof. We may assume by Theorem 1 that g, the generator of Z, is a contraction, and thus for arbitrarily large $r \in \mathbf{R}_+$ there exists an $m \in \mathbf{N}$ such that $g^n(B(r)) \subset \operatorname{Int}(B)$ for all $n \in \mathbf{N}$ $(n \geq m)$. We can also see that

$$B - \{\mathbf{0}\} = \bigcup_{k=0}^{\infty} (g^k(B) - g^{k+1}(\operatorname{Int} B))$$

since $\bigcap_{k=0}^{\infty} g^k(\operatorname{Int} B) = \{\mathbf{0}\}$. Hence, the compact subset $B - g(\operatorname{Int} B)$ of W contains a fundamental domain for Z. Therefore, $\hat{M} = W/Z$ is compact, and thus the induced covering map from $\hat{M}$ to M is finite. It follows that $[G; Z]$ is finite.

THEOREM 2.3. *Let G be the covering transformation group of a Hopf manifold. Then G can be expressed as a semi-direct product of an infinite cyclic subgroup Z generated by a contraction and a finite normal subgroup H.*

Proof. Let u be a homomorphism from G to $\mathbf{R}_+$ defined by $u(x) = |\det d(\mathbf{x})(\mathbf{0})|$ where $d(x)(\mathbf{0})$ is the Jacobian matrix of x at the origin. Since G

contains a contraction g and clearly $u(g) < 1$, u is discrete. Hence, $u(G)$ is generated by an $a \in \mathbf{R}_+$ $(a \neq 0)$. Take a $g \in G$ such that $u(g) = a$, and let $Z = \langle g \rangle$ be an infinite cyclic subgroup generated by g. By theorem 1, we may assume that g is a contraction. Clearly u: $Z \to u(G)$ is an isomorphism. Let H be Ker u. Then H is a normal subgroup of G and $Z \in H = (I)$. Therefore, by the corollary to Theorem 1, H is finite. Since $u(G) = u(H)$, G is the semi-direct product of Z and H.

COROLLARY 2.4. *Let Z and H be the subgroups of G in Theorem 2. Then there exists an $m \in \mathbf{N}$ such that g^m belongs to the center of G. Thus $\hat{Z} = \langle g^m \rangle$ and $N = \hat{Z} \times H$ are normal subgroup of G.*

Proof. Let us consider the action of Z on H by conjugation. Since H is finite, it is clear that there exists an $m \in \mathbf{N}$ such that $g^{-m}hg^m = h$ for any $h \in H$. Therefore, it follows that g^m belongs to the center of G.

3. Deformations and diffeomorphism types of Hopf manifolds

Let x be an analytic automorphism over $\mathbf{C}^n$ which fixes the origin. Then x can be expressed in the power series

$$x = (x_1, x_2, \dots, x_n),$$

where

$$x_i = a^i_1 z_1 + a^i_2 z_2 + \cdots + a^i_n z_n + (\text{higher powers}) \ (i = 1, 2, \dots, n).$$

The non-singular $n \times n$ matrix (a^i_j) is called the linear part of x, and is denoted by $L(x)$. Note that $L(x) = d(x)(\mathbf{0})$ is the Jacobian matrix of x at the origin. Then the map L: $G \to GL(n, \mathbf{C})$ is a homomorphism, but not necessarily one-to-one. However, concerning the covering transformation groups of Hopf manifolds, we have the following result.

LEMMA 3.1. *Let G be the covering transformation group of a Hopf manifold. Then the homomorphism L: $G \to L(G)$ from G onto $L(G) \subset GL(n, \mathbf{C})$ is a group isomorphism.*

Proof. It is sufficient to prove that L is one-to-one. By Theorem 2, G is the semi-direct product of an infinite cyclic subgroup Z which is generated by a contraction g and a finite normal subgroup H. Now let $x = g^k h (h \in H)$ and $L(x) = L(g^k)L(h) = I$. Then since $\det(L(g)) < 1$ and $\det(L(h)) = 1$, k must be 0 and thus $L(h) = I$. But h is of finite order, it follows from Cartan's uniqueness theorem that $h = I$, and thus $x = I$. Therefore, L is one-to-one.

LEMMA 3.2. *$L(G)$, being a group of analytic automorphisms over W, is properly discontinuous and fixed-point-free.*

Proof. It is easily seen that $L(Z)$ is properly discontinuous and fixed-point-free. Since $[L(G); L(Z)]$ is finite, it follows that $L(G)$ is also properly discontinuous. We will show that $L(G)$ is fixed-point-free. If $L(x)$ $(x \in G)$ is of infinite order, then there is a $k \in \mathbf{N}$ such that $L(x)^k \neq I$ and $L(x)^k$ belongs to $L(Z)$. Since $L(Z)$ is fixed-point-free, $L(x)$ has no fixed point over W. If $L(x)$ $(x \in G)$ is of finite order, then so is x by Lemma 1. According to the generalized result of Cartan's uniqueness theorem [1], there exists an analytic coordinate transformation T on a neighborhood U of the origin such that $T^{-1}xT = L(x)$ on U. Suppose that $L(x)$ has a fixed point $p \in W$. Since $L(x)$ is a linear map, we may assume that $p \in U$. But then $T(p)$ is a fixed point of x, which is a contradiction. This completes the proof of the lemma.

THEOREM 3.3. *There exists a deformation which transforms $M = W/G$ to $W/L(G)$. And thus M is diffeomorphic to $W/L(G)$.*

Proof. For $x \in G$ and $t \in \mathbf{C}$ $(t \neq 0)$, let $x_t = T_t^{-1}xT_t$ and $G(t) = \{x_t | x \in G\}$ where T_t is an analytic automorphism over W of the following form:

$$T_t\colon (z_1, z_2, \ldots, z_n) \to (tz_1, tz_2, \ldots, tz_n).$$

$G(t)$ $(t \neq 0)$ is obviously group isomorphic to G, and properly discontinuous and fixed-point-free. And thus so is $G(0) = L(G)$ by the above lemmas. We will show that

$$\{M_t | M_t = W/G(t)\ (t \in \mathbf{C})\}$$

forms a complex analytic family. Then it follows from a theorem of deformation theory (cf. [7]), $M = W/G$ is diffeomorphic to $W/G(0) = W/L(G)$.

Now we define for $x \in G$ an analytic automorphism $\tilde{x}$ over $W \times \mathbf{C}$ as follows:

$$\tilde{x}\colon (z, t) \to (x_t(z), t)$$

where $z = (z_1, z_2, \ldots, z_n) \in W$ and $t \in \mathbf{C}$. Let $\tilde{G} = \{\tilde{x} | x \in G\}$. Then $\tilde{G}$ is a group of analytic automorphisms over $W \times \mathbf{C}$, and $\tilde{G} = \tilde{Z} \cdot \tilde{H}$ where $\tilde{Z} = \langle \tilde{g} \rangle$, g is a contraction which generates Z, and $\tilde{H} = \{\tilde{h} | h \in H\}$.

We first prove that $\tilde{G}$ is properly discontinuous and fixed-point-free. It is clear from the above argument that $\tilde{G}$ is fixed-point-free. By the definition of a contraction we see that for a given compact set K of W and a given point $\tau \in \mathbf{C}$, there exists an ε $(\varepsilon > 0)$ such that $g_t^m(K) \cap K = \emptyset$ holds for $t \in \mathbf{C}(|t - \tau| < \varepsilon)$ and for all but finitely many integers m. It follows that for a given compact set K of W and a given compact set I of $\mathbf{C}$, $\tilde{g}^m(K \times I) \cap (K \times I) = \emptyset$ for all but finitely many integers m. Hence $\tilde{Z}$ is properly discontinuous. Since $[\tilde{G} : \tilde{Z}]$ is finite, $\tilde{G}$ is also properly discontinuous.

Now let $\tilde{M} = W \times \mathbf{C}/\tilde{G}$ and π: $\tilde{M} \to \mathbf{C}$ be the canonical map induced from the projection Pr_2: $W \times \mathbf{C} \to \mathbf{C}$. Then $\tilde{M}$ is a complex manifold, π is holomorphic, and clearly the rank of the Jacobian of π is 1 at each point of $\tilde{M}$. Since $\pi^{-1}(t) = M_t$ for each $t \in \mathbf{C}$, $\{M_t | t \in \mathbf{C}\}$ forms a complex analytic family. This completes the proof of Theorem 3.3.

LEMMA 3.4. *Suppose that $A \in GL(n, \mathbf{C})$ is of the form*

$$A = A_1(a_1, n_1) + A_2(a_2, n_2) + \cdots + A_k(a_k, n_k)$$

where $A_i(a_i, n_i)$ is a $n_i \times n_i$ lower triangular matrix with eigenvalue a_i, $n_1 + n_2 + \cdots + n_k = n$, $a_i \neq 0$, and a_i are mutually distinct. Let B be any $n \times n$ matrix which commutes with A. Then B is of the same form as A:

$$B = B_1(n_1) + B_2(n_2) + \cdots + B_k(n_k)$$

where $B_i(n_i)$ is a $n_i \times n_i$ matrix.

Proof. Let V = $\mathbf{C}^n$ (an n-dimensional vector space over $\mathbf{C}$). Then

$$V = V_1 + V_2 + \cdots + V_k$$

where

$$V_i = \left\{ v \in V | (A - a_i I)^s v = 0 \text{ for some } s \in \mathbf{N} \right\}.$$

Since A and B commute, V_i is B-invariant, B being a linear endomorphism over V, for $i = 1, 2, \ldots, n$. Hence it follows that B has the above form.

THEOREM 3.5. *Suppose that G is the direct product of Z and H, then $M = W/G$ is diffeomorphic to $S^1 \times S^{2n-1}/U$, where U is a finite subgroup of $U(n, \mathbf{C})$ which is conjugate to $L(H)$ in $GL(n, \mathbf{C})$.*

Proof. We have proved that W/G is diffeomorphic to $W/L(G)$. For simplicity, we write G, Z, H in place of $L(G)$, $L(Z)$, $L(H)$. Now since G is a subgorup of $GL(n, \mathbf{C})$, we may assume by Lemma 3 that g is of the Jordan form and $h \in H$ is of the same form as g. Let

$$g_t = tg_n + g_s + tg + (1 - t)g_s$$

where g_n is the nilpotent part of g and g_s is the semi-simple part of g. Then since g and h $(h \in H)$ commute, g_t $(t \in \mathbf{C})$ and h also commute. Therefore,

g_t induces an analytic automorphism $\hat{g}_t$ over $\hat{W}$ where $\hat{W}$ denotes W/H. Let $M_t = \hat{W}/Z(t)$ where $Z(t) = \langle \hat{g}_t \rangle$. Then $\{M_t | t \in \mathbf{C}\}$ forms a complex analytic family. Accordingly, $M = W/G$ is diffeomorphic to W/G_0 where $G_0 = Z_0 \times H$, $Z_0 = (g_0)$, and g_0 is of the form

$$g_0: (z_1, z_2, \ldots, z_n) \to (a_1 z_1, a_1 z_2, \ldots, a_k z_n) \quad (0 < |a_i| < 1).$$

Now, consider a diffeomorphism F from $\mathbf{R} \times S^{2n-1}$ to W defined as follows:

$$F: (t, z_1, z_2, \ldots, z_n) \to \left(a_1^t z_1, a_1^t z_2, \ldots, a_k^t z_n\right).$$

Since H is a finite subgorup of $GL(n, \mathbf{C})$, taking a suitable linear coordinate transformation, we can assume that $H \subset U(n, \mathbf{C})$ while g is the same as before. The corresponding automorphisms to g and h over $\mathbf{R} \times S^{2n-1}$ are of the form

$$\bar{g}: (t, z_1, z_2, \ldots, z_n) \to (t+1, z_1, z_2, \ldots, z_n)$$

and

$$\bar{h}: (t, z_1, z_2, \ldots, z_n) \to \left(t, h(z_1, z_2, \ldots, z_3)\right),$$

respectively. Therefore, M is diffeomorphic to $S^1 \times S^{2n-1}/H$. In our first notation, M is diffeomorphic to $S^1 \times S^{2n-1}/U$ where U is a unitary group conjugate to $L(H)$ in $GL(n, \mathbf{C})$.

LEMMA 3.6. *Let $J(a, k)$ be a Jordan form of order k with eigenvalue a and $A = (a_{ij})$ be any $m \times n$ matrix. Then, $J(a, m)A = AJ(a, n)$ if and only if $a_{ij} = a_{i+1, j+1}, a_{in} = 0$ for i $(1 \le i \le m-1)$, and $a_{ij} = 0$ for j $(2 \le n)$.*

Proof. Let A_i denote the i-th row vector of A and A^j the j-th column vector of A. We define the inner product $(A_i, B_j) = A_i B_j^t$ and $(A^i, B^j) = A^{it} B^j$. Let $J(k) = J(0, k)$ for simplicity. It is clearly sufficient to show the assertion for $a = 0$. Now if $J(m)A = AJ(n)$, then

$$\begin{aligned} a_{ij} &= (E^i, AE^j) = \left(E^i, AJ(n)E^{j-1}\right) = \left(E^i, J(m)AE^{j-1}\right) \\ &= \left(J(m)^t E^i, AE^{j-1}\right) = \left(E^{j-1}, E^{j-1}\right) = a_{i-1, j-1}, \end{aligned}$$

and $a_{1j} = (J(m)^t E^1, AE^{j-1}) = 0$ for j $(2 \le j \le n)$. Similarly, $a_{in} = (E_i A, E_n) = (E_{i+1} J(m) A, E_n) = (E_{i+1} AJ(n), E_n) = (E_{i+1} A, E_n J(n)^t) = 0$ for i $(1 \le i \le m-1)$. The converse is obvious.

THEOREM 3.7. *Let M be a Hopf manifold and G its covering transformation group. Then M is diffeomorphic to a fiber bundle over S^1 with fiber S^{2n-1}/U, which has a certain explicit bundle structure (as described in the proof), where U is a finite subgroup of $U(n, \mathbf{C})$.*

Proof. We may assume as in the proof of Theorem 3.5 that G is a subgroup of $GL(n, \mathbf{C})$ which is the semi-direct product of an infinite cyclic subgroup Z which is generated by a contraction g and a finite normal subgroup H. According to Corollary 2.4, there exists a minimal positive integer m such that $\hat{g} = g^m$ belongs to the center of G. Since $\hat{g}$ and $h \in H$ commute, we may assume that $\hat{g}$ is of the Jordan form and h has the same form as $\hat{g}$. We will show that $M = W/G$ is diffeomorphic to $W/Z \cdot H$ where $Z = \langle g \rangle$ (g is a diagonal matrix). Since $\hat{g}$ and h have the same forms, it is sufficient to consider the case that $\hat{g}$ has only one eigenvalue a, that is, $\hat{g} = J(a, k_1) + J(a, k_2) + \cdots + J(a, k_s)$. We will show the assertion for $s = 2$. It is then easily proved for the general case. Now, for each $x \in G$ and $t \in \mathbf{C}$, let $x_t = T_t^{-1} x T_t$ where T_t is an analytic automorphism over W defined as follows:

$$T_t\colon (z_1, z_2, \ldots, z_n) \to \left(t^{k_1-1} z_1, t^{k_1-2} z_2, \ldots, z_{k_1}, t^{k_2-1} z_{k_1+1}, \ldots, z_{k_1+k_2}\right).$$

It follows from Lemma 4 that x_t is well defined. Thus

$$\{M_t | M_t = W/G(t)\}, G(t) = \{x_t | x \in G\}$$

forms a complex analytic family. Therefore M is diffeomorphic to $W/Z_0 \cdot H_0$ where $Z_0 = \langle g_0 \rangle$ and $\hat{g}_0 = g_0^m$ is a diagonal matrix. Then, taking a suitable linear coordinate transformation, g_0 is diagonalizable.

We have shown so far that M is diffeomorphic to W/G where $G = Z \cdot H, Z$ is generated by a diagonal matrix g, and H is a finite subgroup of $GL(n, \mathbf{C})$, all of which elements are of the same form as g^m. Therefore, g and $h \in H$ are of the following form:

$$g = ac$$

where $a = A_1(a_1, n_1) + A_2(a_2, n_2) + \cdots + A_k(a_k, n_k)$, $A(a_i, n_i)$ is a diagonal matrix with eigenvalue a_i ($0 < |a_i| < 1$, a_i are mutually distinct, and $n_1 + n_2 + \cdots + n_k = n$), and c is a diagonal matrix belonging to $N(H; GL(n, \mathbf{C}))$, all of whose entries are m-th roots of 1; and

$$h\text{: non-singular } n \times n \text{ matrix of the same form as } a.$$

Since H is a finite subgroup of $GL(n, \mathbf{C})$ and $c \in N(H; GL(n, \mathbf{C}))$, we can construct a semi-direct product $\langle c \rangle \cdot H$ which is also a finite subgroup of

$GL(n, \mathbf{C})$. Therefore, taking a suitable linear coordinate transformation, we can assume that $\langle c \rangle \cdot H \subset U(n, \mathbf{C})$ while a is the same as before.

Now consider the diffeomorphism F in the proof of Theorem 4:

$$F: (t, z_1, z_2, \ldots, z_n) \to (a_1^t z_1, a_2^t z_2, \ldots, a_n^t).$$

The corresponding automorphisms over $\mathbf{R} \times S^{2n-1}$ to g, g^m and $h \in H$ are of the form

$$\bar{g}: (t, z_1, z_2, \ldots, z_n) \to (t + 1, c(z_1, z_2, \ldots, z_n)),$$
$$\bar{g}^m: (t, z_1, z_2, \ldots, z_n) \to (t + m, z_1, z_2, \ldots, z_n),$$
$$\bar{g}: (t, z_1, z_2, \ldots, z_n) \to (t, h(z_1, z_2, \ldots, z_n)),$$

respectively. Therefore, M is diffeomorphic to the fiber bundle

$$S^1 \times_{Z/mZ} S^{2n-1}/H,$$

where the action of Z/mZ on S^1 is given by $s \cdot k = \exp(2\pi i/m) \cdot s$ and the action of Z/mZ on S^{2n-1}/H is given by $u \cdot k = \hat{c}(u)$, where $s \in S^1$, $u \in S^{2n-1}/H$, $k \in Z/mZ$, and $\hat{c}$ is an automorphism over S^{2n-1}/H of order m induced by c. This is our expected result.

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