ILLINOIS JOURNAL OF MATHEMATICS
Volume 37, Number 4, Winter 1993

FOLIATIONS INVARIANT UNDER THE MEAN CURVATURE FLOW

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Introduction

Let $\mathcal{F}$ be a foliation of a Riemannian manifold (M, g) equipped with the Levi-Civita connection ∇. The tangent bundle of M splits into the orthogonal sum $T\mathcal{F} \oplus T^{\perp}\mathcal{F}$ of the bundle tangent to F and its orthogonal complement, so any vector v decomposes into the sum $v^{\top} + v^{\perp}$ of vectors respectively tangent and perpendicular to $\mathcal{F}$. For any point $x \in M$, $H(x)$ denotes the mean curvature at x of the leaf L of $\mathcal{F}$ which passes through x. H is defined as the trace of the second fundamental form B of $\mathcal{F}$:

$$B(X,Y) = (\nabla_X Y)^{\perp} \tag{1}$$

for all vector fields X and Y tangent to $\mathcal{F}$ and

$$H = \sum_{i=1}^{p} B(X_i, X_i), \tag{2}$$

where $p = \dim \mathcal{F}$ and $X_1, \dots, X_p$ is a (local) orthonormal frame of vector fields tangent to $\mathcal{F}$. (For suitable background in Riemannian geometry we refer to [K], for the notions and results of the theory of foliations to [CN], [HH] and [T].)

In this paper, we are interested in those foliations $\mathcal{F}$ which are invariant under the local flows generated by the vector field H. Such foliations are said to be *mean curvature invariant*, or MCI for short. The infinitesimal condition sufficient and necessary for $\mathcal{F}$ to be MCI is that

$$\langle [H, X], N \rangle = 0 \tag{3}$$

for all vector fields X tangent to $\mathcal{F}$ and N orthogonal to $\mathcal{F}$. In other words, $\mathcal{F}$ is MCI iff H is parallel w.r.t. the (partial) Bott connection in $TM/T\mathcal{F} \cong T^{\perp}\mathcal{F}$, i.e. iff the Lie derivation $\mathcal{L}_H$ maps the module $\mathcal{X}(\mathcal{F})$ of vector fields

Received September 26, 1991.
1991 Mathematics Subject Classification. Primary 53C12; Secondary 57R30.

tangent to $\mathscr{F}$ or the ideal $\mathscr{I}(\mathscr{F})$ of forms vanishing along $\mathscr{F}$ into itself. For a transversely oriented codimension-q foliation $\mathscr{F}$ the condition (3) can be expressed in the form

$$\mathscr{L}_H \omega = f\omega, \tag{4}$$

where ω is a non-vanishing q-form defining $\mathscr{F}$. If $q = 1$ and $\omega(N) = 1$ for a unit vector field N orthogonal to $\mathscr{F}$, then the factor f in (4) is given by

$$f = \langle \nabla h, N \rangle, \tag{5}$$

where $h = \langle H, N \rangle$ is the (scalar) mean curvature of $\mathscr{F}$ and ∇u denotes the gradient of a smooth function $u\colon M \to \mathbf{R}$. For arbitrary q and ω with $\|\omega\| = 1$,

$$f = \operatorname{div} H + \|H\|^2.$$

The interest in MCI-foliations is motivated by the following.

Mean curvature of foliations of Riemannian manifolds was studied from the very beginning of the theory of foliations. G. Reeb [Re] observed that

$$\int_M h = 0 \tag{6}$$

for the mean curvature h of any codimension-one foliation $\mathscr{F}$ of any compact Riemannian manifold M and so either $h \equiv 0$ and all the leaves are minimal submanifolds of M or h is somewhere positive and elsewhere negative on M. This observation led to a number of papers concerned with the problem of "minimalizability" of foliations ([Su], [Ru], [Ha], etc.), several articles on minimal foliations and their properties ([H], [KT 1], etc.) and some results on prescribing mean curvature of foliations ([O 2], [W 1], etc.). Also, the "integral formula" (6) was generalized to the higher codimension case in different ways ([BLR], [R], [W 2], etc.).

On the other hand, since several years there is a lot of interest in deforming submanifolds of $\mathbf{R}^n$ by their mean curvature flow ([Br], [CGG], [ES], etc.; see [Hu 2] for a review of results). Being more precise, given a hypersurface L, the problem consists in building a one parameter family (f_t) of immersions of L such that for any $x \in L$ and t the variation vector $(s \mapsto f_s(x))^{\cdot}(t)$ is equal to the mean curvature vector of $f_t(L)$ at $f_t(x)$. For example, if L is convex and closed, then the family $(f_t;\ 0 < t < t_0)$ exists for some $t_0 > 0$ such that the hypersurfaces $L_t = f_t(L)$ shrink to a point p when $t \to t_0$ and form an MCI-foliation of the punctured convex body bounded by L [GH], [Hu 1].

In this article, we describe some elementary properties of MCI-foliations (Section 1), prove the existence of minimal leaves in some situations (Section 2), get some integral formulae analogous to those mentioned above (Section 3) and collect some examples of MCI-foliations (Section 4).

The above discussion and the examples of Section 4 show that MCI foliations appear in different geometric situations as well as in some problems of physics such as, for example, the structure of crystals [F] or perturbation theory related to phase transition phenomena ([Ga], [MS], etc.).

The paper was written during the author's stay at the Washington University in St. Louis where he enjoyed the hospitality of the faculty and the staff of the Department of Mathematics. Especially, he is indebted to Larry Conlon and Gary Jensen for arranging the visit and valuable discussions.

1. First observations

Let $\mathcal{F}$ be an MCI-foliation of M. If $H(x) = 0$ for a point x of a leaf L, then x is a fixed point of a local flow of H. Since this flow maps leaves to leaves, an open neighbourhood U of x in L consists of fixed points of this flow and $H \equiv 0$ on U. Since leaves are connected, H vanishes identically on L:

Proposition 1. *If L is a leaf of an MCI-foliation and the mean curvature vector of L vanishes at one point, then L is a minimal submanifold.*

For codimension one MCI-foliations it follows that the scalar mean curvature has constant sign along any leaf. Also, the above Proposition together with (6) lead immediately to the following.

Corollary 1. *Any codimension-one MCI-foliation of a compact manifold has minimal leaves.*

Assume now that the vector field H is complete (this is the case when M is compact) and denote its flow by (φ_t). If L is a non-minimal leaf of $\mathcal{F}$, then all the leaves $\varphi_t(L)$ are diffeomorphic to L. Moreover, if c: $[0, 1] \to L$ is a closed leaf curve, then the curves $\varphi_t \circ c$ are closed and lie on the leaves $\varphi_t(L)$. Therefore, the holonomy map h_c preserves all the points $\varphi_t(x)$, where $x = c(0)$. We get the following.

Proposition 2. *The mean curvature vector $H(x)$ of an MCI-foliation $\mathcal{F}$ at a point x of a non-minimal leaf L is preserved by the linear holonomy group of L. If codimension of $\mathcal{F}$ equals 1, then any non-minimal leaf of $\mathcal{F}$ has trivial holonomy.*

Let $\Omega^{\mathcal{F}}$ be the volume form of an oriented foliation $\mathcal{F}$. Elementary calculation shows that

$$\mathcal{L}_H \Omega^{\mathcal{F}} = -\|H\|^2 \Omega^{\mathcal{F}} \quad \text{on } T\mathcal{F} \tag{7}$$

(see [T], p. 67). It follows that if L is a compact leaf of an MCI-foliation $\mathcal{F}$ and (φ_t) is, as before, the mean curvature flow of $\mathcal{F}$, then

$$\operatorname{vol} \varphi_t(L) - \operatorname{vol}(L) = -\int_0^t \int_{\varphi_s(L)} \|H\|^2 \, d\Omega^{\mathcal{F}} \, ds. \tag{8}$$

Assume now that L is an arbitrary submanifold of a Riemannian manifold M. If $x \in L$ and $H(x) \neq 0$, then one can find a codimension one submanifold N of M foliated by p-dimensional submanifolds such that $x \in N$, $L \cap N$ is one of the leaves and the mean curvature vector of the foliation of N is everywhere transverse to N. In the same way, if L is compact and embedded, and $H(x) \neq 0$ for all the points $x \in L$, then there exists an embedded codimension one submanifold N foliated by compact p-dimensional leaves such that L is one of them and the mean curvature vector of the leaves is again transverse to N everywhere. The existence and uniqueness results for deformations by the mean curvature flow ([GH] and [CGG]) lead to the following.

PROPOSITION 3. (1) *If L is a submanifold of a Riemannian manifold M, $x \in L$ and $H(x) \neq 0$, then there exists an open neighbourhood U of x in M and an MCI-foliation $\mathcal{F}$ of U such that the connected component of $L \cap U$ containing x is one of the leaves of $\mathcal{F}$.*

(2) *If L is compact embedded and $H(x) \neq 0$ for any x of L, then there exists an open neighbourhood U of L and an MCI-foliation $\mathcal{F}$ of U such that L is a leaf of $\mathcal{F}$.*

(3) *If* $\operatorname{codim}(L) = 1$, *then the foliations of* (1) *and* (2) *are unique* (*up to the domain*).

It is clear that uniqueness established in the above Proposition does not hold in codimension > 1. For example, for any curve $c = (x, y, z)$: $\mathbf{R} \to \mathbf{R}^3$ such that the function z is strictly increasing and maps $\mathbf{R}$ onto $\mathbf{R}$ one can foliate the punctured planes $z =$ const. by concentric circles centered at the appropriate point of the curve c. All those foliations form an MCI-foliation of $\mathbf{R}^3 \setminus c(\mathbf{R})$. For different curves c one obtains different MCI-foliations which could coincide on some horizontal planes $z =$ const.

2. Minimal leaves

As we observed in Section 1, codimension-one MCI-foliations of compact manifolds must have minimal leaves. In this section, we are going to prove the existence of minimal leaves under different circumstances and to show how they are distributed over the foliated manifold.

Assume that an MCI-foliated manifold $(M, \mathscr{F})$ is compact. Its mean curvature flow (φ_t) acts in the manifold M as well as in the space of leaves $M/\mathscr{F}$. Therefore, for any leaf L one can consider the orbit $(\varphi_t(L))$ in $M/\mathscr{F}$ together with all associated objects and properties like limit sets, recurrency, wandering etc.

THEOREM 1. *If L is a compact leaf, then the ω-limit set of L contains minimal leaves.*

Proof. Assume that the ω-limit set of a compact leaf L does not contain minimal leaves and denote by F^+ the closure of the subset $\bigcup\{\varphi_t(L);\, t \geq 0\}$. Let $a > 0$ be the minimum over F^+ of the norm of H. Denote also by $v_{\min}$ the (positive) minimum of volumes of leaves of $\mathscr{F}$. From (8) it follows that

$$\operatorname{vol} \varphi_t(L) \leq \operatorname{vol}(L) - a^2 v_{\min} t \to -\infty,$$

when $t \to \infty$. Contradiction.

THEOREM 2. *If* $\operatorname{codim}(\mathscr{F}) = 1$ *and L is compact, then also the α-limit set of L contains minimal leaves.*

Proof. Assume to the contrary that $a = \min\{\|H(x)\|;\, x \in F^-\}$ is positive, where F^- is the closure of the negative half-orbit of L. Cover F^- with a finite number of charts bi-distinguished by $\mathscr{F}$ and H (compare [HH]).

The formula (8) shows that the orbit of L can meet each of the charts only once. In fact, otherwise the equality $\varphi_s(L) = \varphi_t(L)$ would hold for some reals $s < t$ implying the condition $\operatorname{vol}(\varphi_s(L)) = \operatorname{vol}(\varphi_t(L))$ which contradicts (8). It follows that the volumes of leaves in F^- are bounded from above (for example, by the product of the number of charts times the maximum of volumes of plaques of our cover).

On the other hand, if $v_{\min}$ has the same meaning as before, then (8) implies that

$$\operatorname{vol}(\varphi_t(L)) \geq \operatorname{vol}(L) - a^2 v_{\min} t \to \infty$$

when $t \to -\infty$. Contradiction.

The Novikov's results ([N], compare [HH]) on the structure of transversely oriented codimension-one foliations allow to formulate the following.

COROLLARY 2. *For any open Novikov component U of a transversely oriented codimension-one* MCI-*foliation of a compact Riemannian manifold M, the closure of U contains minimal leaves.*

Proof. The boundary of U consists of finite number of compact leaves $L_1, \dots L_k$. If one of them is minimal, we have done. If not, some of the limit sets of L_i's are contained in U and the statement follows from our Theorems 1 and 2.

Frankel [Fr] has shown that two compact minimal codimension-one submanifolds of a complete Riemannian manifold of positive Ricci curvature have to intersect. This fact combined with the above observations yields another result.

COROLLARY 3. *If a transversely oriented codimension-one MCI-foliation $\mathscr{F}$ of a compact Riemannian manifold M of positive Ricci curvature has at least two compact leaves, then $\mathscr{F}$ is given by a locally trivial fibration of M over S^1.*

Proof. If L_1 and L_2 are compact leaves of $\mathscr{F}$, then only one of them, say L_2, is minimal. The orbit U of L_1 under the mean curvature flow is open and contains L_2 in its closure. It follows that $M = U \cup L_2$, so all the leaves of M are compact of trivial holonomy and form fibres of a fibration $M \to S^1$.

Remark. In the proof of Theorem 2, the essential step consists in establishing an upper bound for the volumes of the leaves in F^-. The results and methods of [EMS] seem to be promising for the proof of an analogous theorem in the case $\operatorname{codim}(F) = 2$.

3. Integral formulae

Assume that a codimension q MCI-foliation $\mathscr{F}$ of M is transversely oriented and let ω be a q-form defining $\mathscr{F}$. The integrability of $T\mathscr{F}$ yields the existence of an 1-form η such that

$$d\omega = \eta \wedge \omega \quad \text{and} \quad \iota_H \eta = 0. \tag{9}$$

THEOREM 3. *The form $\iota_H \omega \wedge \eta \wedge d\eta$ is exact. In particular,*

$$\int_M \iota_H \omega \wedge \eta \wedge (d\eta)^k = 0 \tag{10}$$

when M is oriented and compact, and $\dim \mathscr{F} = 2k$.

Proof. Conditions (4) and (9) imply that

$$f\omega = d\iota_H\omega + \iota_H\, d\omega = d\iota_H\omega + \iota_H(\eta \wedge \omega)$$

and $\iota_H(\eta \wedge \omega) = -\eta \wedge \iota_H\omega$. Consequently,

$$d(f\omega) = -d\eta \wedge \iota_H\omega + \eta \wedge d\iota_H\omega$$

and

$$\eta \wedge d\eta \wedge \iota_H\omega = -\eta \wedge d(f\omega) = df \wedge d\omega = d(f d\omega).$$

Remark. Note that the form $\beta = \eta \wedge (d\eta)^q$ is closed and represents the Godbillon-Vey invariant gv of $\mathcal{F}$ ([GV], [T], etc.). If β is harmonic and $3q = \lim M$, then the above result could be interpreted by saying that the harmonic part of $\iota_H\omega$ represents the cohomology class w for which $w \cup gv = 0$.

Recall now the notion of a harmonic measure on a foliated Riemannian manifold as defined by Lucy Garnett [Ga]. For any foliation $\mathcal{F}$ of a Riemannian manifold M and any smooth function f on M, the foliated Laplacian Δ of f is defined at a point $x \in M$ as the value at x of the Laplacian of $f|L$ on the leaf $L(x \in L)$ equipped with the induced Riemannian structure. A finite measure μ on M is said to be *harmonic* if

$$\int_M \Delta(f)\, d\mu = 0$$

for any f. Note that a compact foliated Riemannian manifold always admits nontrivial harmonic measures. For examples and properties of harmonic measures we refer to [Ga].

THEOREM 4. *If $\mathcal{F}$ is a codimension-one transversely oriented MCI-foliation of a compact Riemannian manifold M and N is an unit vector field orthogonal to $\mathcal{F}$, then*

$$\int_M h\big(\mathrm{Ric}(N) + \|B\|^2 + \|\nabla_N N\|^2 - \langle \nabla h, N\rangle\big)\, d\mu = 0. \tag{11}$$

for any harmonic measure μ on M.

Proof. Let us compute the foliated Laplacian of the scalar mean curvature h of $\mathcal{F}$:

$$\Delta(h) = \mathrm{div}(\nabla h)^{\top}.$$

From (3) it follows that for any X tangent to $\mathcal{F}$ we have

$$\langle \nabla h, X \rangle = -\langle \nabla_H N, X \rangle, \tag{12}$$

so

$$(\nabla h)^{\top} = -h \nabla_N N.$$

Consequently,

$$\Delta(h) = -h \operatorname{div} \nabla_N N - \langle \nabla_N N, \nabla h \rangle. \tag{13}$$

Moreover, the formulae (2.16) and (2.17) of [BKO] (compare also [O 1]) yield

$$\operatorname{div} \nabla_N N = \operatorname{Ric}(N) + \|B\|^2 - \langle \nabla h, N \rangle. \tag{14}$$

Also, the formula (12) applied to $X = \nabla_N N$ yields

$$\langle \nabla h, \nabla_N N \rangle = -\|\nabla_N N\|^2. \tag{15}$$

Formulae (13)–(15) yield

$$\Delta(h) = -h\big(\operatorname{Ric}(N) + \|B\|^2 + \|\nabla_N N\|^2 - \langle \nabla h, N \rangle\big) \tag{16}$$

and this proves (11) according to the definition of harmonic measures.

Since the volume form of a compact leaf provides us with a harmonic measure we get the following.

COROLLARY 4. *If L is compact leaf on an MCI-foliation, then*

$$\int_L h\big(\operatorname{Ric}(N) + \|B\|^2 + \|\nabla_N N\|^2 - \langle \nabla h, N \rangle\big)\, d\operatorname{vol}_L = 0.$$

4. Examples

4.1. Two dimensional torus. C^2-foliations of $\mathbf{T}^2$ have been classified by Kneser ([Kn], see also [Go]) several years ago. If the leaves of a foliation $\mathcal{F}$ are not dense, then the torus splits into the family of annuli $A_1, \dots A_k$, $k = 1, 2, \dots, \infty$, such that the boundary circles of A_i's are leaves and the interiors are foliated in one of the following ways:

(1) All the leaves are closed.

(2) All the leaves are spiralling approaching the boundary components which are oriented in the same way.

(3) All the leaves are approaching the boundary components which are oriented in the opposite way (this is called *Reeb component*).

If the foliation like this is MCI (w.r.t. the standard flat metric), then from observations of Section 1 it follows that the boundary components of A_i's are geodesics. They do not intersect, so they should have the same length. Also all the leaves of components of type (1) should be geodesics otherwise they should have the strictly positive (or, strictly negative) curvature. The components of type (2) cannot appear: The curvature of any leaf contained in a component of this type should vanish somewhere, so the leaves should be geodesics but this is impossible. Finally, consider Reeb components of MCI-foliations. Lift the foliation to the universal covering $[0, \pi] \times \mathbf{R}$ and denote by X the unit vector field tangent to the lifted foliation:

$$X(x, y) = \sin \alpha(x, y) \frac{\partial}{\partial x} + \cos \alpha(x, y) \frac{\partial}{\partial y}. \tag{17}$$

The condition (3) can be expressed in terms of the function α as follows:

$$\sin^2 \alpha \cdot \alpha_{xx} + 2 \sin \alpha \cos \alpha \cdot \alpha_{xy} + \cos^2 \alpha \cdot \alpha_{yy} = 0. \tag{18}$$

The function α should be periodic in y $(\alpha(x, y + 1) = \alpha(x, y))$ and satisfy the initial conditions $\alpha(0, y) = 0$ and $\alpha(\pi, y) = \pi$ for all y. Obviously, the linear functions solve equation (18) and among them there is precisely one, $\alpha(x, y) = x$, which satisfies the above conditions. It follows that for any $n = 0, 1, 2, \ldots$ there are MCI-foliations of $\mathbf{T}^2$ with n Reeb components $A_1, \ldots, A_n$. The remaining part of the torus $\mathbf{T}^2 \setminus \bigcup A_i$ has to be foliated by geodesics of the same length. In general, the foliation obtained is of the class C^0. It is smooth of the class C^∞ iff either $n = 0$ or Reeb components are pairwise congruent and fill up the whole torus.

If the leaves of an MCI-foliation $\mathscr{F}$ are dense, then the curvature of leaves has to vanish identically, so $\mathscr{F}$ consists of dense geodesics.

Remarks. (i) It would be interesting to study equation (18) more deeply getting all its solutions satisfying the periodicity and initial conditions that should lead to a complete classification of MCI-foliations of $\mathbf{T}^2$.

(ii) One could easily modify the above consideration to get a similar description of MCI-foliations of the Klein bottle.

4.2. Three-sphere. Any codimension-one foliation $\mathscr{F}$ of $\mathbf{S}^3$ contains a Reeb component $\mathbf{D}^2 \times \mathbf{S}^1$ [N]. If $\mathscr{F}$ is MCI, its boundary $L = \mathbf{T}^2$ should be minimal. Assume that L is congruent to the Clifford torus $x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}$ and lift the Reeb component to the universal covering

$$A = \mathbf{D}^2(0, 1/\sqrt{2}) \times \mathbf{R},$$

equip A with the Riemannian structure lifted from $\mathbf{S}^3$ and consider the unit normal N of the lifted foliation. In cylindrical coordinates (r, θ, z), the condition (3) is equivalent to

$$r^2(1-r^2)\sin^2\alpha \cdot \alpha'' + r(1-2r^2)\cdot\alpha' + \sin\alpha\cos\alpha\cdot\frac{1-r^2+2r^4}{1-r^2} = 0 \tag{19}$$

for $r \in (0, 1/\sqrt{2})$ if N is given by

$$N = \frac{1}{c}\cos\alpha(r)\cdot\frac{\partial}{\partial r} + c\sin\alpha(r)\cdot\frac{\partial}{\partial z},$$

where $c = \sqrt{1-r^2}$. Studying the existence and uniqueness of solutions α of (19) satisfying the conditions $\alpha(0) = \pi/2$ and $\alpha(1/\sqrt{2}) = 0$ one could construct and classify three-dimensional MCI-Reeb components. Gluing together two Reeb components equipped with MCI-foliations one could construct an MCI-foliation of $\mathbf{S}^3$.

Moreover, our Corollary 3 shows that an MCI-foliation of S^3 has at most one compact leaf T^2. This leaf is unknotted according to the Lawson's result on closed minimal surfaces in S^3 [L]. Therefore, any MCI-foliation of S^3 should be diffeomorphic to a standard Reeb foliation obtained by gluing together two Reeb components ([HH], Part B, p. 42).

4.3. Homogeneous foliations. Let G be a Lie group equipped with a left invariant Riemannian metric, a closed connected subgroup K and a discrete subgroup Γ. Let $M = \Gamma \setminus G$ and denote by $\mathscr{F}$ the foliation of M obtained by projecting submanifolds aK, $a \in G$. If $\bar{g}$ and $\bar{k}$ denote the Lie algebras of G and K, respectively, then

$$H = \sum_{i \le p,\, \alpha > p} C^i_{\alpha, i} X_\alpha$$

and $\mathscr{F}$ is MCI iff

$$\sum_{i \le p,\, \alpha > p} C^i_{\alpha, i} C^\beta_{\alpha, j} = 0 \tag{20}$$

for all $i \le p$ and $\beta > p$. Here, $p = \dim K$ and $C^c_{a,b}$ are structure constants of $\bar{g}$ w.r.t. an orthonormal frame $(X_1, \dots, X_n)$ of $\bar{g}$ such that $X_i \in \bar{k}$ for $i = 1, \dots, p$. It is easy to see that the condition (20) is satisfied when K is a normal subgroup of G.

4.4. Riemannian foliations. A Riemannian foliation $\mathscr{F}$ is MCI w.r.t. a bundle-like metric g iff

$$\nabla_v^\perp H = -A^{\perp v} H \tag{21}$$

for any vector v tangent to $\mathscr{F}$, where $A^\perp$ is the Weingarten operator associated to the bundle $\mathrm{T}^\perp\mathscr{F}$ and $\nabla^\perp$ is the natural connection in this bundle induced by the Levi-Civita connection on M.

If the bundle $\mathrm{T}^\perp\mathscr{F}$ is integrable, then $A^\perp = 0$ and (21) reduces to

$$\nabla_v^\perp H = 0 \quad (v \in \mathrm{T}^\perp \mathscr{F}). \tag{22}$$

In other words, a Riemannian foliation with involutive normal bundle is MCI w.r.t. a bundle-like metric iff its leaves have parallel mean curvature vector.

In general, if $\mathscr{F}$ is Riemannian and MCI, then the norm $\|H\|$ is constant along the leaves. In fact, if X is a vector field tangent to $\mathscr{F}$, then

$$X\|H\|^2 = (\mathscr{L}_X g)(H, H) + 2\langle [X, H], H\rangle = 0.$$

The similar argument was used in [KT 2], where the authors proved (Lemma 1.17) that the mean curvature form of a Riemannian foliation is basic iff H is parallel w.r.t. the (partial) Bott connection in the bundle $TM/T\mathscr{F}$ identified under the canonical projection with the bundle $T^\perp\mathscr{F}$.

The results of [BKO] imply:

If codim $\mathscr{F} = 1$, *M is compact and the Ricci curvature of M is non-negative, then $\mathscr{F}$ is totally geodesic and the Ricci curvature of M in the normal direction vanishes.*

This result could be obtained also from our Theorem 4 and the integral formula of [W 2]. In fact, $\mathscr{F}$ has to be minimal: Otherwise, we could consider the set K where the mean curvature h attains its positive maximum (or, negative minimum). K is closed and saturated, and there exists a harmonic measure μ supported in K. Formula (11) implies that $h \equiv 0$ on K. The mentioned integral formula reduces now to

$$\int_M (\mathrm{Ric}(N) + \|B\|^2) = 0$$

and this yields our observation.

The first Structure Theorem for Riemannian foliations ([M], p. 155) says that the closures of the leaves of a Riemannian foliation $\mathscr{F}$ form a possibly singular foliation $\mathscr{F}_0$. If $\mathscr{F}$ is MCI, $\mathscr{F}_0$ is preserved by (φ_t) and $\|H\|$ is constant along the leaves of $\mathscr{F}_0$, however $\mathscr{F}_0$, need not be MCI. If codim $\mathcal{F} = 1$, then either $\mathcal{F}_0 = \mathcal{F}$ or the leaves of $\mathcal{F}$ are dense. In the last case, $\mathcal{F}$ is MCI iff minimal.

4.5. Principal foliations. A codimension-one hypersurface M of the Euclidean space $\mathbf{R}^{n+1}$ is called a *Dupin hypersurface* if the number and multiplicities of different principal curvatures are the same at all the points of M, and the principal curvature functions are constant along the leaves of the foliations induced by the corresponding distributions of principal vectors [CR]. (It is known that the distributions corresponding to principal curvature functions are integrable.)

Let $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k$ be the principal curvature functions on a Dupin hypersurface M. Let $\mathcal{F}$ be the principal foliation corresponding to λ and let B be its second fundamental tensor. If X and Y are tangent to $\mathcal{F}$ and Z is a principal vector corresponding to λ_i, $i \geq 0$, then [W 2]

$$(\lambda - \lambda_i)\langle B(X,Y), Z\rangle = \langle X,Y\rangle\langle\nabla\lambda, Z\rangle. \tag{23}$$

It follows that the mean curvature vector H of $\mathcal{F}$ is given by

$$H = \sum_{i=1}^{k} (\lambda - \lambda_i)^{-1} m_i (\nabla\lambda)_i, \tag{24}$$

where m_i is the multiplicity of λ_i and $(\nabla\lambda)_i$ is the component of $\nabla\lambda$ tangent to the principal distribution corresponding to λ_i. Note that if M is isoparametric (principal curvatures are constant), then $B = 0$ and $H = 0$; if $k = 1$, then the formula (24) reduces to

$$H = (\lambda - \lambda_1)^{-1} m_1 \nabla\lambda. \tag{25}$$

Recall also [W 1] the relation between the mean curvature vectors H and $\bar{H}$ of an arbitrary p-dimensional foliation of any manifold equipped with two conformally equivalent Riemannian metrics g and $\bar{g} = \varphi g$:

$$\bar{H} = \frac{1}{\varphi}\left(H + p(\nabla\varphi)^{\perp}\right). \tag{26}$$

Finally, note the following classification ([CR], Theorem 6.2) of connected complete Dupin hypersurfaces M of $\mathbf{R}^{n+1}$ with two principal curvatures: If M is non-compact and one principal curvature is identically zero then M is a standard product $S^k(r) \times \mathbf{R}^{n-k}$, otherwise M is obtained from a standard product $S^k(r) \times S^{n-k}(s)$, $r^2 + s^2 = 1$, in $S^{n+1}(1)$ via the stereographic projection $S^{n+1} \setminus \{x_0\} \to \mathbf{R}^{n+1}$ from a point x_0 of S^{n+1}. This classification together with an elementary calculation involving formulae (25) and (26) yield:

Principal foliations of a complete Dupin hypersurface with two principal curvatures are MCI.

4.6. Horocycle foliations. Let N be an n-dimensional complete Riemannian manifold of constant negative curvature -1 and let $M = T^1N$ be its unit tangent bundle. M carries the so called Sasaki Riemannian metric $\bar{g}$ induced by the Riemannian structure g of N and the Levi-Civita connection ∇ of (N, g). The tangent bundle TTN splits into the direct sum of the horizontal and vertical subbundles, so every vector ξ of TM can be written as the sum $\xi^h + \xi^v$ of its horizontal and vertical components. The bundle TM splits also ([K], Chapter 3) into the orthogonal sum of integrable subbundles

$$TM = E^s \oplus E^0 \oplus E^u$$

which could be described in the following way under the canonical identification of the horizontal and vertical fibres with the tangent spaces of M.

E^0 is the one-dimensional bundle spanned by the geodesic flow (g_t) of N, E^s is the stable subbundle consisting of all the vectors $\xi \in T_wM$, $w \in M$, for which $\xi^h = -\xi^v$ and $\xi^v \perp w$ and E^u is the unstable bundle which is defined in the similar way: $\xi \in E^u_w$ iff $\xi^h = \xi^v$ and $\xi^v \perp w$. The bundles E^s and E^u are invariant under the geodesic flow (g_t).

Let $\mathcal{F}$ be the strong stable foliation of M: $T\mathcal{F} = E^s$. From the known formulae [Ko] describing the Levi-Civita connection $\bar{\nabla}$ of the Sasaki metric it follows that

$$(27)\qquad \left(\bar{\nabla}_\xi \xi\right)^{\perp}_w = \mathscr{H}(\nabla_X X)_w - \mathscr{V}(\nabla_X X)_w + \mathscr{H}(R(X,w)X)_w,$$

where R is the curvature tensor on (N, g), $\mathscr{H}(u)_w$ and $\mathscr{V}(u)_w$ denote respectively the horizontal and vertical lifts of a vector u to the tangent space T_wTN, and $\xi = \mathscr{H}(X) - \mathscr{V}(X)$ is the vector field on TN determined by a vector field X on N. From (27) and the well known formula for the curvature tensor in the case of constant sectional curvature it follows that the mean curvature vector H of $\mathcal{F}$ is given by

$$H = (n-1)G,$$

where G is the vector field of the flow (g_t). Therefore:

The stable (and also the unstable) foliation of the unit tangent bundle of a complete Riemannian manifold of constant negative curvature is MCI.

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