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THE HILBERT TRANSFORM ALONG CURVES THAT ARE ANALYTIC AT INFINITY

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1. Introduction

It is known that if B denotes the unit ball of $\mathbf{R}^m, \gamma: B \to \mathbf{R}^n$ is an analytic function, $\gamma(0) = 0$, and k is a $C^\infty(\mathbf{R}^m - \{0\})$ function, homogeneous of degree $-m$, then the operator given by $Tf(x) = p \cdot v \cdot \int_B f(x - \gamma(t))k(t)\,dt$ is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$. See for example [2], [9]. We observe that in this case γ is "approximately homogeneous" at the origin in the sense given in [10].

The purpose now is to consider the analogous problem at infinity, for the case $m = 1$. More precisely we prove the following:

THEOREM 1.1. *Let $B^C = \{t \in \mathbf{R}: |t| > 1\}$ and let $\gamma: B^C \to \mathbf{R}^n$ be defined by*

$$\gamma(t) = \left(t^{a_1} + \alpha_1(t), \dots, t^{a_n} + \alpha_n(t)\right), a_i \in N, \qquad a_1 < \cdots < a_n,$$

where α_i is a real analytic function on B^C, $\alpha_i(t) = h_i(t) + P_i(t)$ with h_i analytic at infinity, and P_i a polynomial of degree at most $a_i - 1$. Then the operator

$$\mathscr{H}_\gamma f(x) = p \cdot v \cdot \int_{B^C} f(x - \gamma(t)) \frac{dt}{t}$$

is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$.

This result still holds if $\gamma(t) = (\gamma_1(t) + \alpha_1(t), \dots, \gamma_n(t) + \alpha_n(t))$ where $\gamma_i(t)$ are homogeneous functions of degree a_i, $a_i \in \mathbf{R}$, $1 \le a_1 < \cdots < a_n$, and asking weaker conditions about the behavior at infinity of $\alpha_i(t)$.

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2. Proof of the theorem

Let us consider $\mathbf{R}^n$ with the group of dilations given by $D_r(x) = (r^{a_1}x_1, \ldots, r^{a_n}x_n)$ for all $r > 0$, where $a_1 < \cdots < a_n$, $a_i \in N$, $i = 1, \ldots, n$. We set $D_r(x) = r \cdot x$.

Associated to $\{D_r\}_{r>0}$ we fix a homogeneous norm, i.e., a continuous function

$$|\ \ |: \mathbf{R}^n \to [0, \infty)$$

which is C^∞ on $\mathbf{R}^n - \{0\}$ and satisfies:
(a) $|x| = 0$ if and only if $x = 0$;
(b) $|-x| = |x|$;
(c) $|r \cdot x| = r|x|$ for all $x \in \mathbf{R}^n$, $r > 0$.
It can be proved that homogeneous norms always exist. Also it is known that

(2.1) $|x + y| \le c(|x| + |y|)$ for some constant $c > 0$, for all $x, y \in \mathbf{R}^n$.

For the proof of these facts see [4].
Let $a = a_1 + \cdots + a_n$ be the homogeneous degree of $\mathbf{R}^n$.

LEMMA 2.2. *Let $\{\psi_j\}$, $j \in Z$, be a family of functions in $L^1(\mathbf{R}^n)$ satisfying*:

(i) $$\int \psi_j = 0$$

and for some $c > 0$ and $0 < \delta < 1$;

(ii) $$\int |\psi_j(x + y) - \psi_j(x)|\, dx \le c|y|^\delta \ (L^1\text{-Hölder condition});$$

(iii) $$\int |x|^\delta |\psi_j(x)|\, dx \le c.$$

Let T_j be the operator of convolution by $2^{ja}\psi_j(2^j \cdot x)$, then for $n, m \in Z$, $n \le m$,

$$\left\| \left(\sum_n^m T_j \right)(f) \right\|_p \le c_p \|f\|_p, \qquad 1 < p < \infty, \text{ with } c_p \text{ independent of } n \text{ and } m.$$

Proof. By the Marcinkiewicz Interpolation Theorem and a usual duality argument, it is enough to check

$$(2.3)\qquad \left\| \left(\sum_n^m T_j \right) f \right\|_2 \le c_2 \|f\|_2, \qquad c_2 \text{ independent of } n \text{ and } m,$$

and

$$(2.4)\qquad \left| \left\{ x : \left| \left(\sum_n^m T_j \right) f(x) \right| > \lambda \right\} \right| \le \frac{c_1}{\lambda} \|f\|_1 \quad \text{(weak type 1-1)}$$

with c_1 independent of n and m.

To prove (2.3) we use Cotlar's Lemma, from [5]. Let $f_i(x) = 2^{ia}\psi_i(2^i \cdot x)$. The operator T_j^* is given by convolution with $g_j(y) = \overline{f_j(-y)}$. So, for $i < j$

$$\begin{aligned}
\|T_i T_j^*\|_{2,2} = \|f_i * g_j\|_1 &= \int \left| \int f_i(x-y) g_j(y)\, dy \right| dx \\
&= \int \left| \int (f_i(x-y) - f_i(x)) \overline{f_j(-y)}\, dy \right| dx \\
&\le \int 2^{ja} \int |\psi_i(x - 2^i \cdot y) - \psi_i(x)|\, dx\, |\psi_j(-2^j \cdot y)|\, dy \\
&\le c \int 2^{ja} 2^{i\delta} |y|^\delta |\psi_j(-2^j \cdot y)|\, dy \\
&= 2^{(i-j)\delta} \int |y|^\delta |\psi_j(y)|\, dy \le c 2^{(i-j)\delta}
\end{aligned}$$

The estimations for $\|T_i^* T_j\|_{2,2}$ when $i < j$ and the case $j < i$ are similar. So

$$\left\| \sum_{j=n}^m T_j \right\|_{2,2} \le c \sum_{i=-\infty}^{+\infty} 2^{-|i|\delta/2}$$

It is known that (2.4) follows if we check that there exists a constant A independent of n and m, such that, for $y \neq 0$,

$$\int_{|x| > 2c|y|} \left| \sum_{j=n}^m (f_j(x+y) - f_j(x)) \right| dx \le A$$

$\times$ where c is the constant in (2.1) [3].

Now

$$\int_{|x|>2c|y|}\left|\sum_{j=n}^{m}\left(f_j(x+y)-f_j(x)\right)\right|dx$$

$$\leq \sum_{j\in Z}\int_{|x|>2^{j+1}c|y|}|\psi_j(x+2^j\cdot y)-\psi_j(x)|\,dx$$

$$= \sum_{2^{j+1}c|y|<1} + \sum_{2^{j+1}c|y|\geq 1}$$

We use (ii) to get the first sum bounded by $\sum_{2^{j+1}c|y|<1}2^{j\delta}|y|^{\delta}$ and this geometric sum is bounded independently of y. Now

$$\sum_{2^{j+1}c|y|\geq 1}\int_{|x|>2^{j+1}c|y|}|\psi_j(x+2^j\cdot y)-\psi_j(x)|\,dx$$

$$\leq \sum_{2^{j+1}c|y|\geq 1}\left(\int_{|x|>2^j|y|}|\psi_j(x)|\,dx+\int_{|x|\geq 2^{j+1}c|y|}|\psi_j(x)|\,dx\right)$$

$$\leq \sum_{2^{j+1}c|y|\geq 1}\left(\int_{|x|>2^j|y|}|\psi_j(x)||x|^{\delta}|x|^{-\delta}\,dx+\int_{|x|\geq 2^{j+1}c|y|}|\psi_j(x)||x|^{\delta}|x|^{-\delta}\,dx\right)$$

$$\leq c\sum_{2^{j+1}c|y|\geq 1}\left(2^{-j\delta}|y|^{-\delta}+(2c)^{-\delta}2^{-j\delta}|y|^{-\delta}\right).$$

In the last inequality we use (iii). So we obtain another geometric sum bounded independently of y. ■

Remark 2.5. It can be proved that if $\{\psi_j\}_{j\in Z}$ is a family of functions as in Lemma 2.2, then

$$\Psi(f)=\sum_{j\in Z}2^{ja}\int\psi_j(2^j\cdot x)f(x)\,dx$$

defines a tempered distribution and thus we have just proved that the operator of convolution by Ψ is bounded on $L^p(\mathbf{R}^n)$, $1<p<\infty$.

Let $\gamma(t)$ be as in Theorem 1.1.

LEMMA 2.6. *Let*

$$\Gamma(t_1,\ldots,t_n)=\gamma(t_1)+\cdots+\gamma(t_n)$$

and

$$\mathscr{J}(t_1,\dots,t_n) = \det(D\Gamma)|_{(t_1,\dots,t_n)},$$

the determinant of the jacobian matrix of Γ *at* $(t_1,\dots,t_n)$.

(1) $\mathscr{J}(t_1,\dots,t_n) = P(t_1,\dots,t_n) + R(t_1,\dots,t_n)$, *where* P *is a homogeneous polynomial of degree* $a - n$ *and for some positive constant* A, R *is an analytic function in*

$$\{(z_1,\dots,z_n) \in C^n \colon |z_i| > A > 0, i = 1,\dots,n\}.$$

(2) *If* K *is a compact set in* C^n *contained in* $C - \{0\} \times \cdots \times C - \{0\}$ *then*

$$r^{(a-n)}R(r^{-1}z_1,\dots,r^{-1}z_n) \xrightarrow[r\to 0]{} 0 \quad \textit{uniformly on } K.$$

Proof. We do the proof by induction on n.

• $n = 1$ We have to check that $r^{a_1-1}\alpha_1'(r^{-1}z) \to 0$ uniformly on K as $r \to 0$. Since $\alpha_1(t) = h_1(t) + P_1(t)$, h_1 analytic at infinity, there exists $A > 0$ such that $\alpha_1(z) = \sum_{-\infty}^{a_i-1} b_k z^k$ in $|z| > A$.

So there exists $r_0 > 0$ such that $r_0^{-1}K \subset \{z \colon |z| > A\}$ and this implies that

$$r^{a_1}\alpha_1(r^{-1}z) \xrightarrow[r\to 0]{} 0 \quad \text{uniformly on } K.$$

By Cauchy's formula we have that if r is small enough and for $z \in K$,

$$\alpha_1'(r^{-1}z) = \frac{1}{2\pi i}\int_{|\zeta - r^{-1}z| = (r^{-1}|z|)/2} \frac{\alpha_1(\zeta)}{(\zeta - r^{-1}z)^2}\,d\zeta$$

and so

$$|\alpha_1'(r^{-1}z)| \le \frac{2r}{|z|} \sup_{|\zeta - r^{-1}z| = (r^{-1}|z|)/2} |\alpha_1(\zeta)|.$$

Then

$$r^{a_1-1}|\alpha_1'(r^{-1}z)| \le \frac{2r^{a_1}}{|z|} \sup_{|\zeta - r^{-1}z| = (r^{-1}|z|)/2} |\alpha_1(\zeta)|.$$

But $\alpha_1(\zeta) = \alpha_1(r^{-1}r\zeta)$ and $\frac{1}{2}|z| \le |r\zeta| \le \frac{3}{2}|z|$.

So $r\zeta$ belongs to a compact $\tilde{K}$ such that $0 \notin \tilde{K}$. Since

$$r^{a_1}|\alpha_1(r^{-1}w)| \xrightarrow[r\to 0]{} 0 \quad \text{uniformly on } \tilde{K},$$

we have

$$r^{a_1-1}|\alpha_1'(r^{-1}z)| \underset{r\to 0}{\longrightarrow} 0 \quad \text{uniformly on } K.$$

- We now assume that the statement of the lemma holds for $n-1$:

$$D\Gamma(t_1,\dots,t_n) = \begin{bmatrix} a_1 t_1^{a_1-1} + \alpha_1'(t_1) & \cdots & a_1 t_n^{a_1-1} + \alpha_1'(t_n) \\ \vdots & & \vdots \\ a_n t_1^{a_n-1} + \alpha_n'(t_1) & \cdots & a_n t_n^{a_n-1} + \alpha_n'(t_n) \end{bmatrix}$$

We develop the determinant by the first column and we obtain summands of the form

$$\begin{aligned}(a_j t_1^{a_j-1} + \alpha_j'(t_1))(P_{n-1}(t_2,\dots,t_n) + R_{n-1}(t_2,\dots,t_n)) \\ = a_j t_1^{a_j-1} P_{n-1}(t_2,\dots,t_n) + a_j t_1^{a_j-1} R_{n-1}(t_2,\dots,t_n) \\ + \alpha_j'(t_1) P_{n-1}(t_2,\dots,t_n) + \alpha_j'(t_1) R_{n-1}(t_2,\dots,t_n)\end{aligned}$$

where P_{n-1} is a homogeneous polynomial of degree

$$a_1 + \cdots + \check{a}_j + \cdots + a_n - (n-1),$$

and R_{n-1} satisfies

$$r^{a_1+\cdots+\check{a}_j+\cdots+a_n-(n-1)} R_{n-1}(r^{-1}z_2,\dots,r^{-1}z_n) \underset{r\to 0}{\longrightarrow} 0$$

on compact sets as those described in (2).

By inductive hypothesis and the estimate about α_j', the lemma follows. ∎

Proof of the Theorem 1.1. Following [7], for $f \in S$, we decompose

$$\mathscr{H}_\gamma f(x) = \left(\sum_{j=-\infty}^{0} \mu_j * f\right)(x)$$

where

$$\mu_j(f) = \int_{|t|>1} f(\gamma(t))\varphi_0(2^j|t|)\frac{dt}{t}$$

with $\varphi_0 \in C_0^\infty(\frac{1}{2}, 2)$ satisfying $\sum_{j\in Z}\varphi_0(2^j|t|) = 1$.

The theorem follows if we prove that

(2.7) $\mathscr{H}_\gamma^m f = \sum_{j=-m}^{0} \mu_j * f$ is bounded on $L^p(\mathbf{R}^n)$ independently of m.

For $x \in \mathbf{R}^n$, we define $\phi_0(x) = \varphi_0(|x|)$ and for $k \in Z$, let $\phi_k(x) = 2^{ka}\phi_0(2^k \cdot x)$. So, for each fixed j_0,

$$\delta_0 = \phi_{j_0} + \sum_{k=j_0}^{\infty} \phi_{k+1} - \phi_k.$$

Then

$$\mathscr{H}_\gamma^m f = \sum_{j=-m}^{0} \delta_0 * \mu_j * f = \sum_{j=-m}^{0} \left(\phi_j + \sum_{k=j}^{\infty} \phi_{k+1} - \phi_k\right) * \mu_j * f$$
$$= \sum_{j=-m}^{0} \phi_j * \mu_j * f + \sum_{k=0}^{\infty} \sum_{j=-m}^{0} \eta_{k+j} * \mu_j * f$$

where $\eta_k = \phi_{k+1} - \phi_k$. Thus

$$\mathscr{H}_\gamma^m f = \left(L_m + \sum_{k=0}^{\infty} M_k^m\right) * f$$

with

$$L_m = \sum_{j=-m}^{0} \phi_j * \mu_j \quad \text{and} \quad M_k^m = \sum_{j=-m}^{0} \eta_{k+j} * \mu_j$$

To prove (2.7) we first show that if $1 < p < \infty$,

(2.8) $\|L_m\|_{p,p} \le c_p, \quad \|M_k^m\|_{p,p} \le c_p 2^{k\varepsilon}, \qquad \varepsilon > 0,$

c_p independent of m.

and

(2.9) $\|M_k^m\|_{2,2} \le c 2^{-\sigma k}$ for some $\sigma > 0$, c independent of m.

($\|L_m\|_{p,p}$ denotes the convolution operator norm of L_m on $L^p(\mathbf{R}^n)$, and similarly for $\|M_k^m\|_{p,p}$.)

From (2.8) and (2.9) we obtain (2.7). Indeed, let p be a fixed exponent, $1 < p < 2$, and take p_0 such that $1 < p_0 < p < 2$. We use the Riesz convexity Theorem and so we interpolate between (2.9) and the estimate (2.8) for $\|M_k^m\|_{p_0, p_0}$. If we choose the exponent ε in (2.8) small enough, we obtain

$$\|M_k^m\|_{p,p} \leq c 2^{-\sigma k s} 2^{\varepsilon k(1-s)} \quad \text{where } \frac{1}{p} = \frac{s}{2} + \frac{1-s}{p_0}$$

and thus $\sum_{k=1}^{\infty} \|M_k^m\|_{p,p}$ is bounded independently of m.

For $2 < p < \infty$, (2.7) can be proved by duality.

To check (2.8) we observe that

$$L_m(x) = \sum_{j=-m}^{0} (\phi_j * \mu_j)(x) = \sum_{j=-m}^{0} 2^{ja}(\phi_0 * \nu_j)(2^j \cdot x)$$

and

$$M_k^m(x) = \sum_{j=-m}^{0} (\eta_{k+j} * \mu_j)(x) = \sum_{j=-m}^{0} 2^{ja}(\eta_k * \nu_j)(2^j \cdot x)$$

where $\nu_j(f) = \mu_j(f \circ D_{2^j})$.

It is easy to check that $\eta_k * \nu_j$ and $\phi_0 * \nu_j$ satisfy (i), (ii) and (iii) of Lemma 2.2. Moreover the constant $2^{k\varepsilon}$ in (2.8) comes from the L^1-Hölder condition of $\eta_k * \nu_j$.

To prove (2.9) we use Cotlar's Lemma and the iterative method in [1].

It is enough to check that if $j, l \in Z$,

$$\text{(2.10)} \quad \|\eta_{k+j} * \mu_j * (\eta_{k+l} * \mu_l)^*\|_{2,2} \leq c 2^{-\sigma k} 2^{-|j-l|\sigma} \quad \text{for some } \sigma > 0.$$

We verify this for $0 > j < l$.

To this end we recall that, if A and B are bounded linear operators on a Hilbert space, then

$$\|AB\| \leq \|A\|^{1/2} \|ABB^*\|^{1/2}$$

Iterating N times, we have

$$\|AB\| \leq \|A\|^{1-2^{-N}} \|A(BB^*)^{2^{N-1}}\|^{2^{-N}}.$$

Now

$$\|\eta_{k+j} * \mu_j * (\eta_{k+l} * \mu_l)^*\|_{2,2} \leq c \|\eta_{k+j} * \mu_j * \mu_l^*\|_{2,2}$$

and taking A and B as the operators of convolution by $\eta_{k+j} * \mu_j$ and μ_l^*

respectively, we obtain

$$\|\eta_{k+j} * \mu_j * \mu_l^*\|_{2,2} \le \|\eta_{k+j} * \mu_j\|_{2,2}^{1-2^{-N}} \|\eta_{k+j} * \mu_j * (\mu_l^* * \mu_l)^{2^{N-1}}\|_{2,2}^{2^{-N}}$$
$$\le c\|\eta_{k+j} * \mu_j * (\mu_l^* * \mu_l)^{2^{N-1}}\|_1^{2^{-N}}$$

since $\|\eta_{k+j} * \mu_j\|_1 \le c$ independently of k and j. So (2.10) follows if we check that for $0 > j > l$,

(2.11) $\|\eta_{k+j} * \mu_j * (\mu_l^* * \mu_l)^{2^{N-1}}\|_1 \le c2^{-\sigma k}2^{(l-j)\sigma}$ for some $\sigma > 0$.

Let

$$\Gamma(t_1, \dots, t_n) = -\gamma(t_1) + \gamma(t_2) + \cdots + (-1)^n \gamma(t_n)$$

and let

$$\mathscr{J}(t_1, \dots, t_n) = \det(D\Gamma)|_{(t_1, \dots, t_n)}.$$

It is clear that we can apply Lemma 2.6 to Γ. Thus if

$$\Gamma_l(t_1, \dots, t_n) = D_{2^l}\Gamma(2^{-lt_1}, \dots, 2^{-lt_n})$$

and

$$\mathscr{J}_l(t_1, \dots, t_n) = \det(D\Gamma_l)|_{(t_1, \dots, t_n)}$$

then

$$\mathscr{J}_l(t_1, \dots, t_n) = 2^{l(a-n)}\mathscr{J}(2^{-l}t_1, \dots, 2^{-l}t_n) = P(t_1, \dots, t_n)$$
$$+ 2^{l(a-n)}R(2^{-1}t_1, \dots, 2^{-l}t_n)$$

which converges to $P(t_1, \dots, t_n)$ when $l \to -\infty$ if $t_i \ne 0$ for $1 \le i \le n$.

Since $a_1 < \cdots < a_n$, $P \not\equiv 0$ and so $\mathscr{J}$ is not identically null. Furthermore

$$\mathscr{J}(t_1, \dots, t_n) \ne 0 \quad \text{a.e.}\ (t_1, \dots, t_n)$$

such that $|t_i| > 1$, since it is a real analytic function there.

Now we apply Proposition (2.1) in [7] to obtain

$$\mu_l^* * \mu_l * \cdots * \frac{\mu_l + \mu_l^*}{2} + (-1)^n\left(\frac{\mu_l - \mu_l^*}{2}\right)$$

is absolutely continuous since it is the transported measure of

$$w_l(t_1,\dots,t_n) = \prod_{i=1}^{n} \varphi_0(2^l t_i) 1/t_i$$

by $\Gamma(t_1,\dots,t_n)$. Moreover its density ρ_l satisfies an L^1-Hölder condition.

From now on we fix N such that $2^{N-1} \geq n$. Then it is enough to prove

$$\|\rho_l * \mu_j * \eta_{k+j}\|_1 \leq c2^{-\sigma k} 2^{(l-j)\sigma} \quad \text{for some } \sigma > 0. \tag{2.12}$$

Let

$$\tilde{w}(t_1,\dots,t_n) = 2^{-ln} w_l(2^{-l}t_1,\dots,2^{-l}t_n)$$

which doesn't depend on l. So if $\tilde{\rho}_l(y) = 2^{-la}\rho_l(2^{-l}\cdot y)$ we have that $\tilde{\rho}_l$ is the density of the transported measure by Γ_l of $\tilde{w}$.

If we prove that

$$\int |\tilde{\rho}_l(x+y) - \tilde{\rho}_l(x)|\, dx \leq c|y|^\sigma \tag{2.13}$$

for some $\sigma > 0$, c independent of l,

then

$$\int |\rho_l(x+y) - \rho_l(x)| \leq c2^{l\sigma}|y|^\sigma.$$

The same holds for $\rho_l * \mu_j$ since the total variation of μ_j is bounded independent of j. Also η_{k+j} has mean value zero and $\operatorname{supp} \eta_{k+j} \subset \{x\colon |x| \leq c2^{-(k+j)}\}$.

Thus

$$\begin{aligned}
\|\rho_l * \mu_j * \eta_{k+j}\|_1 &= \int |\rho_l * \mu_j * \eta_{k+j}(x)|\, dx \\
&= \int \left| \int (\rho_l * \mu_j)(x-y)\eta_{k+j}(y)\, dy \right| dx \\
&\leq \int\int |\rho_l * \mu_j(x-y) - (\rho_l * \mu_j)(x)|\, dx |\eta_{k+j}(y)|\, dy \\
&\leq c\int_{\operatorname{supp} \eta_{k+j}} 2^{l\sigma}|y|^\sigma\, dy \leq c2^{l\sigma}2^{-(k+j)\sigma}
\end{aligned}$$

which proves (2.12).

To prove (2.13) we first observe that

$$\int |\tilde{\rho}_l(x+y) - \tilde{\rho}_l(x)|\, dx$$

$$\leq c|y|^{\sigma}\left(\int_{\operatorname{supp}\tilde{w}} |\tilde{w}| + |\nabla \tilde{w}|\right)^{\sigma}\left(\int_{\operatorname{supp}\tilde{w}} \frac{|\tilde{w}|}{|\mathscr{J}_l|^{2\sigma/1-\sigma}}\right)^{1-\sigma}$$

for all $0 < \sigma < 1$ such that $\int_{\operatorname{supp}\tilde{w}} 1/|\mathscr{J}_l|^{2\sigma/1-\sigma} < \infty$ ([8]).

Thus we have to check

(2.14) There exists $\alpha > 0$ such that for $|l|$ large enough, $\int_{\operatorname{supp}\tilde{w}} |\mathscr{J}|^{-\alpha} \leq c$ independent of l.

Since $\mathscr{J}_l(t_1, \ldots, t_n) = P(t_1, \ldots, t_n) + 2^{l(a-n)}R(2^{-l}t_1, \ldots, 2^{-l}t_n)$, we will check that there exists $\alpha > 0$ such that

$$\int_{\operatorname{supp}\tilde{w}} |P(t) + r^{(a-n)}R(r^{-1}t)|^{-\alpha}\, dt \leq c \text{ for } r \text{ small enough.}$$

To see this we make use of Lemma (2.1) in [6].

Let

$$t_0 \in \operatorname{supp}\tilde{w} \subseteq [1/2, 2] \times \cdots \times [1/2, 2] \quad \text{and} \quad G_r(t) = r^{(a-n)}R(r^{-1}t).$$

For r small enough G_r is analytic in the neighborhood

$$t_0 + [-M, M]^n \text{ of } t_0 = (t_0^1, \ldots, t_0^n)$$

where $M = \min_i |t_0^i|/4$.

We will check that if $G_r(t) = \sum_I a_I^r (t - t_0)^I$ then

$$\sum_I |a_I^r| M^{|I|} \xrightarrow[r \to 0]{} 0$$

where

$$a_I^r = \frac{1}{i_1! \cdots i_n!} r^{(a-n)-|I|} \frac{\partial^{|I|}R}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}}(r^{-1}t_0).$$

Now by Cauchy's formula

$$\frac{\partial^{|I|}R}{\partial t_1^{i_1}\cdots\partial t_n^{i_n}}(r^{-1}t_0)$$
$$=\frac{i_1!\cdots i_n!}{(2\pi i)^n}\int_{\{\zeta/|\zeta_i-r^{-1}t_0^i|=(r^{-1}|t_0^i|)/2\}}\frac{R(\zeta)\,d\zeta_1\cdots d\zeta_n}{\left(\zeta_1-r^{-1}t_0^1\right)^{i_1+1}\cdots\left(\zeta_n-r^{-1}t_0^n\right)^{i_n+1}}$$

then

$$a_I^r\le 2^{|I|}|t_0^I|^{-1}r^{a-n}\sup_{\{\zeta/\zeta_i-r^{-1}t_0^i|=(r^{-1}|t_0^i|)/2\}}|R(\zeta)|.$$

We write $R(\zeta)=R(r^{-1}r\zeta)$. Since $r\zeta$ belongs to a compact set $\tilde{\mathscr{K}}$, satisfying (2) of Lemma 2.6, we have $a_I^r<\varepsilon 2^{|I|}|t_0^I|^{-1}$ for r small enough. So

$$\sum_I|a_I^r|M^{|I|}\le\varepsilon\sum_I 2^{-|I|}.$$

Now, Lemma, (2.1) in [6] states that for $\alpha<1/(a-n)$ there exist $c(t_0)$, $r(t_0)$ and a neighborhood $U(t_0)$ of t_0 such that

$$\int_{U(t_0)}|P(t)+G_r(t)|^{-\alpha}\le c(t_0)\text{ for }r\le r(t_0).$$

Since supp $\tilde{w}$ is compact, (2.14) follows.

3. Remarks

Remark 3.1. The theorem still holds if $\alpha_i(t)$ is a real analytic function for $|t|>1$, satisfying:

(i) For each t_0, $|t|>1$, the Taylor expansion of α_i converges in

$$\left\{\zeta\in C/|\zeta-t_0|\le\frac{|t_0|}{2}\right\}$$

(ii) For each t_0, $|t_0|>1$, $\lim_{r\to 0}r^{a_i}\alpha_i(r^{-1}\zeta)=0$ uniformly on

$$\left\{\zeta\in C/|\zeta-t_0|\le\frac{|t_0|}{2}\right\}$$

This result includes more curves than the Theorem; for example let $\alpha_i(t)=e^{-|t|}$ for $i=1,\dots,n$. We extend $\alpha_i(t)$ as e^{-z} for Re $z>0$ and e^z for Re $z<0$. So (i) and (ii) hold.

Proof of 3.1. As in the proof of the theorem, we must estimate

$$a_I^r = r^{(a-n)-|I|}\frac{1}{i_1!\cdots i_n!}\frac{\partial^{|I|}R}{\partial t_1^{i_1}\cdots\partial t_n^{i_n}}(r^{-1}t_0)$$

Reviewing Lemma 2.6 it is easy to see that the summands of $\mathscr{I}(t_1,\ldots,t_n)$ are either

$$P(t_{i_{k+1}},\ldots,t_{i_n})\alpha'_{j_1}(t_{i_1})\cdots\alpha'_{j_k}(t_{i_k}),$$

where P is homogeneous of degree

$$a-(a_{j_1}+\cdots+a_{j_k})-(n-k),$$

or

$$\alpha'_{j_1}(t_1)\cdots\alpha'_{j_n}(t_n).$$

Without lost of generality we assume

$$R(t_1,\ldots,t_n)=\alpha'_{j_1}(t_1)\cdots\alpha'_{j_k}(t_k)P(t_{k+1},\ldots,t_n).$$

We must estimate $\Sigma_I|a_I^r|M^{|I|}$ with M as in the theorem.

$$\begin{aligned}&\sum_I|a_I^r|M^{|I|}\\&\quad=\sum_{i_1}\frac{r^{a_{j_1}-1-i_1}}{i_1!}\left|\alpha_{j_1}^{(i_1+1)}(r^{-1}t_0^1)\right|M^{i_1}\cdots\sum_{i_k}\frac{r^{a_{j_k}-1-i_k}}{i_k!}\left|\alpha_{j_k}^{(i_k+1)}(r^{-1}t_0^k)\right|M^{i_k}\\&\qquad\cdot\sum_{I_2=(i_{k+1}\cdots i_n)}\left|\frac{D^{I_2}P(t_0^{k+1},\ldots,t_0^n)}{i_{k+1}!\cdots i_n!}\right|M^{|I_2|}.\end{aligned}$$

By Cauchy's formula

$$\begin{aligned}&\frac{r^{a_{j_1}-1-i_1}}{i_1!}\alpha_{j_1}^{(i_1+1)}(r^{-1}t_0^1)\\&\quad=r^{a_{j_1}-1-i_1}\frac{i_1+1}{2\pi i}\int_{|\zeta-r^{-1}t_0^1|=(r^{-1}|t_0^1|)/2}\frac{\alpha_{j_1}(\zeta)}{(\zeta-r^{-1}t_0^1)^{i_1+2}}\,d\zeta\end{aligned}$$

So

$$\left|\frac{r^{a_{j_1}-1-i_1}}{i_1!}\alpha_{j_1}^{(i_1+1)}\left(r^{-1}t_0^1\right)\right| \leq r^{a_{j_1}}(i_1+1)|t_0^1|^{-i_1-1}2^{i_1+1} \sup_{|\zeta-r^{-1}t_0^1|=(r^{-1}|t_0^1|)/2} |\alpha_{j_1}(\zeta)|$$

Since $|r\zeta - t_0^1| = |t_0^1|/2$ and $\alpha_j(\zeta) = \alpha_j(r^{-1}r\zeta)$, $r^{a_{j_1}} \sup|\alpha_{j_1}(\zeta)| \to 0$ as $r \to 0$ by (ii).

So by the choice of M, Σ_{i_1} converges and tends to zero with r. The same hold for the other sums. ∎

Remark 3.2. The theorem still holds if $\gamma(t) = (\gamma_1(t) + \alpha_1(t), \dots, \gamma_n(t) + \alpha_n(t))$ where γ_i is a homogeneous function of degree a_i, $a_i \in \mathbf{R}$, $1 \leq a_1 < \cdots < a_n$, and α_i satisfying the conditions of Remark 3.1.

References

1. M. Christ, *Hilbert transforms along curves, I. Nilpotent groups*, Ann. of Math. **122** (1985), 575–596.
2. M. Christ, A. Nagel, E. Stein and S. Wainger, *Singular and maximal Radon transform*, preprint.
3. R. Coifmann and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Lecture Notes in Math., no. 242, Springer-Verlag, New York, 1971, pp. 66–85.
4. G. Folland and E. Stein, *Hardy spaces on homogeneous groups*, Princeton Univ. Press, Princeton, N.J. 1982.
5. W. Knapp and E. Stein, *Intertwining operators for semisimple groups*, Ann. of Math. **93** (1971), pp. 489–578.
6. D. Müller, *Singular kernels supported by homogeneous submanifolds*, J. Reigne Angew. Math. **356** (1985), 90–118.
7. F. Ricci and E. Stein, *Harmonic analysis on nilpotent groups and singular integrals II. Singular kernels supported on submanifolds*, J. Func. Anal. **78** (1988), 56–84.
8. L. Saal, *A result about the Hilbert transform along curves*, Proc. Amer. Math. Soc. **110** (1990), 905–914.
9. L. Saal and M. Urciuolo, *Singular integrals supported on the image of an analytic function*, Math. Zeitschr. **206** (1991), 233–240.
10. E. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

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