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LOW DIMENSIONAL SECTIONS OF BASIC SEMIALGEBRAIC SETS[1]

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To the memory of M. Raimondo

Introduction

Let $X \subset \mathbb{R}^n$ be a real algebraic set, and let $\mathscr{P}(X)$ denote the ring of polynomial functions on X. Recall that a subset $S \subset X$ is called *semialgebraic* if there exist polynomials $f_{ij}, g_i \in \mathscr{P}(X)$ such that

$$S = \bigcup_{i=1}^{p} \{x \in X: f_{i1}(x) > 0, g_i(x) = 0\}.$$

As is well known, if S is open the g_i's in this expression can be omitted. Recall also that an open semialgebraic set is called *basic open* if furthermore $p = 1$. These basic open sets have attracted a lot of interest in recent times, till the proof of the beautiful theorem that states that a basic open set S has always a description

$$S = \{x \in X: f_1(x) > 0, \dots, f_s(x) > 0\}$$

with $s \leq \dim(x)$; see [Br2, 3, 4], [Sch], [Mh], [AnBrRz1]. However, the problem of understanding when a given semialgebraic set is basic open and, in that case, how many inequalities are needed to generate it, is far from solved. An immediate remark is that if S is basic open, then $S \cap \overline{(\bar{S} \setminus S)}^Z = \emptyset$, where $^-$ stands for the euclidean closure, and $^{-Z}$ for the Zariski closure. The only full characterization available is due to Bröcker and Scheiderer. To state it properly, let us say that a semialgebraic set S is *s-basic* if there are s polynomials $f_1, \dots, f_s \in \mathscr{P}(X)$ such that $S = \{f_1 > 0, \dots, f_s > 0\}$, and that S is *generically s-basic* if it is s-basic up to codimension 1, that is, there are s polynomials $f_1, \dots, f_s \in \mathscr{P}(X)$ and a nowhere dense algebraic subset $Z \subset X$ such that

$$S \setminus Z = \{f_1 > 0, \dots, f_s > 0\} \setminus Z.$$

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Now let S be an open semialgebraic set such that $S \cap \overline{(\bar{S} \setminus S)}^Z = \emptyset$; the Bröcker-Scheiderer criterion for the generation of basic sets can be stated as follows:

THEOREM 1. *The set S is s-basic if and only if for every irreducible subset $Y \subset X$ the intersection $S \cap Y$ is generically s-basic.*

Since the dimension bounds the number of inequalities needed to generate any basic set, being basic is equivalent to being d-basic, where $d = \dim(X)$. Hence the previous theorem has the following corollary:

COROLLARY 2. *The set S is basic if and only if for every irreducible subset $Y \subset X$ the intersection $S \cap Y$ is generically basic.*

Thus we come to the problem of whether there exists a distinguished family of subvarieties which suffices to characterize basicness. In fact, in [AnRzl] we proved:

THEOREM 3. *The set S is basic if and only if for every irreducible surface $Y \subset X$ the intersection $S \cap Y$ is basic.*

Since in dimension 1 every semialgebraic set is 1-basic, this was the best possible result concerning dimension, and the first suggestion that obstructions to the generation of basic sets should appear in the smallest predictable dimension. According to this idea, if a basic open set requires s inequalities, we should recognize it exactly in dimension $s + 1$, because in dimension $\leq s$ it certainly can be generated by s inequalities. The goal of this paper is the confirmation of this conjecture. We will prove:

THEOREM 4. *Suppose that S is basic. Then S is s-basic if and only if for every irreducible subset $Y \subset X$ of dimension $s + 1$ in intersection $S \cap Y$ is generically s-basic.*

The proofs of these results are always a combination of the theory of fans in spaces of orderings of function fields and the theory of the real spectrum. Fans are special sets of orderings which quite surprisingly play a dramatic role in the study of the previous questions and results. The definitions and basic properties of the theory of fans are collected in Section 1. What makes possible the improvements concerning dimension in Theorems 3 and 4 is a better analysis of the valuation theory behind the scenes. One key fact is that we can restrict our attention to discrete valuations of maximum rank, which lead to a special type of fan, defined by means of power series. These fans are called *algebroid*, and we prove that any fan can be arbitrarily approximated by an *algebroid* one (*approximation theorem for fans*).

The interest in valuations is not new in real algebraic geometry; see [An], [BrSch], [Rb], [Rz1] and the forthcoming [AnRz2]. Here we exploit systematically the notion of compatibility of a fan with a valuation, that is, the simultaneous compatibility of different orderings, as well as the general interplay between valuations and fans. Two essential tools in our proofs are resolution of singularities and Bertini's theorem. As a matter of fact the failure of the latter in the Nash or analytic category is the reason why our results do not extend to those categories (see the counterexample in [AnRzl]). Despite this failure, many interesting things can be said in the Nash and analytic case using the techniques of this paper. However, here we work only in the algebraic case and refer the reader to [AnBrRzl], [Rz2] and the forthcoming [AnRz3], [RzSh] for the other two.

The paper is organized as follows. Section 1 contains the definitions and some general facts concerning fans needed later. Section 2 describes the trivialization of fans by real valuations and the connection with power series. Section 3 is devoted to the approximation theorem for fans of function fields over the reals, which is the first step towards Theorem 4. In Section 4 we review the theory of the real spectrum that makes the connection between spaces of orderings and algebraic varieties. Finally, Section 5 contains the proof of Theorem 4.

1. Fans and basic sets

The abstract theory of spaces of orderings was developed by Marshall in the series of papers [Mr1-5]. A self-contained new presentation will appear in [AnBrRz2]. Here we only outline some basic facts.

Let K be a field, and consider its space of orderings $\Sigma = \operatorname{Spec}_r(K)$. Given $f \in K$ and $\sigma \in \Sigma$, we can see σ as a signature $\sigma\colon K \to \{-1, +1\}$ which maps the element f to $+1$ or -1 according to whether f is positive or negative in the ordering σ. To give a geometric meaning to the notation, we will write $f(\sigma) < 0$ instead of $\sigma(f) = +1$ and $f(\sigma) < 0$ instead of $\sigma(f) = -1$. A *constructible* subset of Σ is a set of the form

$$C = \bigcup_{i=1}^{p} \left\{\sigma \in \Sigma\colon f_{i1}(\sigma) > 0, \dots, f_{ir_i}(\sigma) > 0\right\},$$

where $f_{ij} \in K$. Such a set C is called *basic* if $p = 1$. The basic sets form a basis of *the Harrison topology* of Σ.

A (*finite*) *fan of* K is a finite set $F \subset \Sigma$ such that for any three orderings $\sigma_1, \sigma_2, \sigma_3 \in F$, their product $\sigma_4 = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$ is a well-defined ordering and belongs to F (we multiply orderings as signatures). Thus, subsets consisting of one or two orderings are always fans and are called *trivial fans*. A basic fact is that a fan F has the structure of an affine space over the field of two elements $\mathbb{F}_2 = \{-1, +1\}$, or, equivalently, for any $\sigma_0 \in F$, the set $\sigma_0 F$ is a vector space over $\mathbb{F}_2 = \{-1, +1\}$, with the product of signatures as inner operation and the natural scalar multiplication. In particular, it follows that $\#(F) = 2^k$, where k is the *affine dimension* of F, that is, $k+1$ is the minimal number of elements $\sigma_0, \dots, \sigma_k \in F$ such that any $\sigma \in F$ is the product of a (necessarily) *odd* number of σ_i's. An important property is that if F' is an affine subspace of F, then F' is again a fan.

In connection with basic sets, let us remark the immediate fact that for every basic set $C \subset \Sigma$, the intersection $F' = F \cap C$ is again a fan, and so $\#(F') = 2^l$, for some $l \leq k$.

Here is fundamental result concerning our problem.

THEOREM 1.1. *Let C be a constructible subset of Σ. The following assertions are equivalent*:

(a) *There are s elements $f_1, \dots, f_s \in K$ such that $C = \{f_1 > 0, \dots, f_s > 0\}$.*

(b) *For every fan $F \subset \Sigma$ with $\#(F) = 2^k$ and $F \cap C \neq \emptyset$ we have $\#(F \cap C) = 2^l$ with $0 \leq k - l \leq s$.*

Somehow suprisingly, 4-element fans are enough to check whether or not a set is basic.

THEOREM 1.2. *Let C be a constructible subset of Σ. The following assertions are equivalent*:

(a) *C is basic.*

(b) *For every fan $F \subset \Sigma$ with $\#(F) = 4$ we have $\#(F \cap C) \neq 3$.*

We will not use Theorem 1.2 here, since we are interested in the quantitative question. Let us remark that Theorem 1.1 is only a reformulation of the usual statement, and we still need a further modification.

COROLLARY 1.3. *Let C be a basic constructible subset of Σ. The following assertions are equivalent*:

(a) *There are s elements $f_1, \dots, f_s \in K$ such that $C = \{f_1 > 0, \dots, f_s > 0\}$.*

(b) *For every fan $F \subset \Sigma$ with $\#(F) = 2^k$ and $\#(F \cap C) = 1$ we have $k \leq s$.*

Proof. Since (b) is a particular case of (b) of Theorem 1.1 we only must prove (b) $\Rightarrow$ (a). For this, suppose $F \subset \Sigma$ is a fan. Since C is basic, the intersection $F' = F \cap C$ is a fan, say generated by $\sigma_1, \dots, \sigma_{l+1}$, and $\#(F') = 2^l$. Now, we can add to these σ_i's some others to get generators $\sigma_1, \dots, \sigma_{k+1}$ of F. Finally, consider the fan F'' generated by $\sigma_{l+1}, \dots, \sigma_{k+1}$; clearly,

$\#(F'') = 2^{k-1}$. Moreover, $F'' \cap C = F'' \cap F' = \{\sigma_{l+1}\}$ because these two intersections are affine subspaces of F of complementary dimensions containing the point σ_{l+1}, and both together generate F. Hence $\#(F'' \cap C) = 1$ and by (b), $k - l \leq s$. Now the result follows from Theorem 1.1. □

2. Fans and valuations

Let K be field and Σ its space of orderings as in Section 1. Let A be a subring of K and $\mathfrak{p}$ an ideal of A. An ordering $\sigma \in \Sigma$ makes $\mathfrak{p}$ *convex* if from $0 < f < g, f \in A, g \in \mathfrak{p}$ it follows $f \in \mathfrak{p}$. This implies that σ induces a unique ordering τ in the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ such that for every $f \in A \setminus \mathfrak{p}$, $f \bmod \mathfrak{p} >_\tau 0$ if $f >_\sigma 0$. In this situation, we say that σ *specializes to* τ or that τ is a *specialization of* σ, and we write $\sigma \to \tau$. The proper setting for this specialization relation is the theory of the real spectrum as we will see in Section 4. However this notion was first studied in the context of valuation theory which we discuss here. A valuation ring V of K is *compatible with* an ordering $\sigma \in \Sigma$ if σ makes convex the maximal ideal $\mathfrak{m}_V$ of V. Then σ specializes to an ordering τ in the residue field k_V of V: $\sigma \to \tau$. This kind of specializations are well understood by means of the Baer-Krull theorem [BCR]:

THEOREM 2.1. *Let Γ denote the value group V, and τ an ordering of k_V. Then there is a bijection between the set of orderings of K compatible with V and specializing to τ and the set of group homomorphisms $\phi: \Gamma \to \{+1, -1\}$.*

Note that this implies that V is compatible with some ordering if and only if its residue field is formally real. In that case we will say that V is *a real valuation ring*.

A particular situation in which the Baer-Krull theorem will be often applied is the following:

Example 2.2. Let B be a regular local ring with residue field L and quotient field K. Suppose $\dim(B) = m$ and consider a system of parameters $x_1, \ldots, x_m$. By induction on m, we define a valuation v_m in the quotient field K of B which has residue field L and value group $\mathbf{Z}^m$.

Indeed, if $m = 1$, then B is a discrete valuation ring and we have the corresponding discrete rank one valuation v_1. For $m > 1$, we consider the discrete valuation ring $W = B_{(x_m)}$, whose valuation is denoted by w. Then, the residue field k' of W is the quotient field of the local regular ring $B' = B/(x_m)$, and by induction we have in K' a valuation v_{m-1} with residue field L and value group $\mathbf{Z}^{m-1}$. Finally, we define v_m as the composite of w and v_{m-1}, and denote its valuation ring by V_m.

Now we fix an ordering τ in L and look for the set F of orderings σ of K compatible with V_m and specializing to τ. We claim that $\#(F) = 2^m$, and that every $\sigma \in F$ is completely determined by the signs in σ of the variables $x_1, \dots, x_m$. Indeed, by the Baer-Krull theorem (Theorem 2.1), we only have to exhibit 2^m orderings specializing to τ and having different signs at the parameters. Again this follows by induction (we use the notations introduce above). If γ is an ordering of K' compatible with V_{m-1}, we can lift it to two orderings γ_+, γ_- of K compatible with $B_{(x_m)}$ as follows: every $f \in B_{(x_m)}$ can be written as $f = ux_m^n$, where u is a unit of $B_{(x_m)}$, and we define,

$$\gamma_+(f) = \gamma(\bar{u}),$$
$$\gamma_-(f) = \gamma(\bar{u})(-1)^n,$$

(here $\bar{u}$ stands for the residue class of u in K'). Since V_m is the composite of V_{m-1} and $B_{(x_m)}$, γ_+ and γ_- are compatible with V_m and specialize to γ.

It can be checked directly that F is a fan, which can be identified with the affine space whose associated vector space is $\{-1, +1\}^m$. In fact, since any $\sigma \in F$ is completely determined by the values $\sigma(x_1), \dots, \sigma(x_m)$, the map

$$\varphi : \sigma_0 F \to \{-1, +1\}^m; \quad \sigma_0\sigma \mapsto (\sigma(x_1), \dots, \sigma(x_m)),$$

where σ_0 is given by $\sigma_0(x_1) = \cdots = \sigma_0(x_m) = +1$, is an isomorphism. With this identification, the elements $\sigma_0, \sigma_1, \dots, \sigma_m$ defined by the following table form a minimal system of generators of F ($*$ can be either $+1$ or -1).

In other words, keeping in mind that in $\mathbf{F}_2 = \{+1, -1\}$, $+1$ is the zero and -1 is the unit, geometrically we are taking σ_0 as the origin of F and the matrix of coordinates of $\sigma_0\sigma_1, \dots, \sigma_0\sigma_m$ is triangular, so that they are a basis of $\sigma_0 F$. All this can be seen as a particular case of a general situation which we describe now very briefly. □

We say that the valuation ring V is *compatible with* a fan $F \subset \Sigma$ if V is compatible with every ordering $\sigma \in F$. It is easily checked that the specializations of the orderings of F form a fan in k_V, possibly trivial. In fact, the main result concerning fans and valuations is the so-called *trivialization* theorem ([Brl], [AnBrRz2]):

Table 1

	x_1	$\cdots$	x_{m-2}	x_{m-1}	x_m
σ_0	$+1$	$\cdots$	$+1$	$+1$	$+1$
σ_1	$+1$	$\cdots$	$+1$	$+1$	-1
σ_2	$+1$	$\cdots$	$+1$	-1	$*$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
σ_m	-1	$\cdots$	$*$	$*$	$*$

THEOREM 2.3. *Let F be a fan of K. Then there exists a valuation ring V of K compatible with F such that the orderings of F have at most 2 distinct specializations in the residue field $k_v \circ V$.*

Conversely, given a fan $\bar{F}$ in k_v, the set of orderings of K which are compatible with V and specialize to an ordering of $\bar{F}$ is a fan called the *pull-back* of $\bar{F}$. This is extremely useful, since it gives an easy method to construct fans starting from trivial ones. For instance, the fan F constructed in Example 2.2 is the pull-back of the trivial fan of L consisting of the single ordering τ. We develop now a second example which will be very important later.

Example 2.4. Let L be a field and $x_1, \dots, x_m$ indeterminates. Consider the ring of formal power series $L[[x_1, \dots, x_m]]$ and its quotient field $L((x_1, \dots, x_m))$. We set $\mathfrak{m} = (x_1, \dots, x_m)$. Let V_m be the valuation ring of $L((x_1, \dots, x_m))$ constructed as in Example 2.2, that is, V_m is the composite of the discrete valuation ring $L[[x_1, \dots, x_m]]_{(x_1)}$ with the valuation ring V_{m-1} of the residue field $L((x_2, \dots, x_m))$.

(a) Fix an ordering τ in L and let F_τ be the set of orderings σ of K compatible with V_m that specialize to τ, that is, F_τ is the pull-back of τ by V_m. We mentioned in Example 2.2 that every $\sigma \in F_\tau$ is completely determined by τ and the signs $\sigma(x_1), \dots, \sigma(x_m)$. To make this precise, let $f \in L[[x_1, \dots, x_m]]$. We look at f as a series in x_1 with coefficients in $L[[x_2, \dots, x_m]]$, say

$$f = x_1^{\nu_{10}}\left(g_{10} + \sum_{l \geq 1} g_l x_1^l\right),$$

with $0 \neq g_{10} \in L[[x_2, \dots, x_m]]$. In particular $g_{10} + \Sigma g_\ell x_1^\ell$ is a unit in $L[[x_1, \dots, x_m]]_{(x_1)}$, namely, it coincides with $g_{10} \pmod{x_1}$. It follows that

$$\sigma(f) = \sigma(x_1)^{\nu_{10}} \gamma(g_{10})$$

where γ is the specialization of σ in $L((x_2, \dots, x_m))$. Now, to determine $\gamma(g_{10})$ we look at it as a series in x_2 and proceed as above. In this way, by induction we get

$$\sigma(f) = \sigma(x_1)^{\nu_{10}} \sigma(x_2)^{\nu_{20}} \cdots \sigma(x_m)^{\nu_{m0}} \tau(u_{\nu_0})$$

where $u_{\nu_0} \in L$ and $u_{\nu_0} x_1^{\nu_{10}} \cdots x_m^{\nu_{m0}}$ is the initial form of f when we consider in $\mathbf{N}^m$ the *lexicographic ordering*. In other words, the sign of f is completely determined by the sign of its initial form. In particular, if f and g have the same initial form (what happens if $h \equiv f \pmod{\mathfrak{m}^\nu}$ for ν high enough), then $\sigma(f) = \sigma(h)$.

(b) Now, we fix two distinct orderings γ_1, γ_2 in L. Let $\mathfrak{F}$ stand for the set of all orderings of $L((x_1, \dots, x_m))$ which are compatible with V_m and specialize to either of the γ_i's, that is, $\mathfrak{F}$ is the pull-back of $\{\gamma_1, \gamma_2\}$ by V_m. Then $\mathfrak{F}$ is a fan with $2 \cdot 2^m = 2^{m+1}$ elements, which is the union of the two fans F_{γ_1} and F_{γ_2} described in (a). In particular, if F_{γ_1} is generated by $\sigma_0, \dots, \sigma_m$, and $\sigma_{m+1} \in F_{\gamma_2}$, then $\mathfrak{F}$ is generated by $\sigma_0, \dots, \sigma_m, \sigma_{m+1}$.

After this preparation we introduce a key notion for our work.

Definition 2.5. Let $A \subset K$ be a subring of K and F a fan of K with $\#(F) = 2^k$. We say that F is *algebroid* if there is an embedding $K \hookrightarrow L((x_1, \dots, x_{k-1}))$ into a power series field such that F is the restriction to K of the fan $\mathfrak{F}$ of Example 2.4(b). We also say that F is parametrized over γ_1, γ_2 in L, and that L is the *coefficient field* of F. Finally, we say that F is finite on A if $A \subset L[[x_1, \dots, x_{k-1}]]$ under the above embedding.

A typical situation where we can construct algebroid fans is the following:

Example 2.6. Let B be a regular local ring with residue field L and quotient field K. Suppose $\dim(B) = k - 1$ and consider two orderings γ_1, γ_2 in L. Fix any system of parameters $x_1, \dots, x_{k-1}$. Then the adic completion $\hat{B}$ of B is isomorphic to $L[[x_1, \dots, x_{k-1}]]$ and this gives an embedding $K \hookrightarrow L((x_1, \dots, x_{k-1}))$. In the latter field we have the fan $\mathfrak{F}$ of Example 2.4b) and its restriction F to K is obviously an algebroid fan parametrized over γ_1, γ_2 in L. Clearly F is finite on B.

3. Approximation of fans

Again, let K be a formally real field and $\Sigma = \operatorname{Spec}_r(K)$. Fix an integer $k \geq 0$. Any fan of K with 2^k elements can be seen as a 2^k-tuple in the product

$$\sum_k = \overbrace{\Sigma \times \cdots \times \Sigma}^{2^k \text{ times}}.$$

Now the set Φ_k of all fans of K with 2^k elements can be seen as a subset of Σ_k. This identification is not bijective, unless we identify the tuples in Σ_k up to permutations, but we will not care about this technicality, because it is irrelevant for our purposes. Anyway, the set Σ_k carries the product topology of the Harrison topology of each factor space and, under our identification, the set Φ_k is endowed with the corresponding subset topology, which we still call the *Harrison topology*. Thus, the set Φ_k of all fans with 2^k elements is a topological space, and we can discuss approximation properties.

Assume now that K is a finitely generated extension of $\mathbf{R}$. Then K is a *function field* and its *dimension* is its transcendence degree over $\mathbf{R}$. The terminology comes from the fact that if $X \subset \mathbf{R}^p$ is any irreducible algebraic set whose field of rational functions $\mathscr{K}(X)$ is our field K, then the dimension of K coincides with the topological dimension of X; such an X is called a *model* of K. As is well known, any formally real finitely generated extension of $\mathbf{R}$ has a model. A useful remark which will be needed later is that we can always find a *compact* model. This is immediate by taking the projective closure of any given model; another way to see it is to take the one-point compactification, which is possible in the real case [BCR].

In this section we will show the following:

THEOREM 3.1. *Let K be a function field of dimension n and X a compact model of K. Let $k \geq 2$ and $F \in \Phi_k$ be a fan of K with 2^k elements. Then F can be arbitrarily approximated in the Harrison topology by an algebroid fan F' finite on $\mathscr{P}(X)$ and parametrized over a function field of dimension $n - k + 1$.*

Proof. Since X is compact, every polynomial is bounded on X, from which it follows that every real valuation ring of K contains the ring $\mathscr{P}(X)$ of polynomial functions of X. Let $F = (\sigma_i\colon 1 \leq i \leq 2^k)$ be the given fan, and $U = U_1 \times \cdots \times U_{2^k}$ an open neighborhood of F in Φ_k, with $U_i = \{f_{i1} > 0, \dots, f_{ir_i} > 0\}$, $f_{ij} \in \mathscr{P}(X)$. After shrinking the U_i's we may assume that they are pairwise disjoint, and we will say that the f_{ij}'s *separate* the orderings of F. By Theorem 2.3, there is a valuation ring V of K such that the σ_i's are compatible with V and induce two orderings τ_1, τ_2 in the residue field k_V of V (possibly $\tau_1 = \tau_2$); as remarked before, $V \supset \mathscr{P}(X)$.

Now we apply resolution of singularities *I* and *II* [Hk], so that after finitely many blowings-up we may assume that X is non-singular and all the f_{ij}'s are normal crossings. Let $\mathfrak{p} \subset \mathscr{P}(X)$ be the center of V in $\mathscr{P}(X)$: $\mathfrak{p} = \mathfrak{m}_V \cap \mathscr{P}(X)$, where $\mathfrak{m}_V$ is the maximal ideal of V. Then $A = \mathscr{P}(X)_{\mathfrak{p}}$ is a regular local ring of dimension, say, d, and has a regular system of parameters $x_1, \dots, x_d$ such that for all i, j

$$f_{ij} = u_{ij} x_1^{\alpha_{ij1}} \cdots x_d^{\alpha_{ijd}}$$

where the u_{ij} are units of A and the α_{ijl} are non-negative integers.

In this situation the residue field $\kappa(\mathfrak{p})$ of A is a subfield of the residue field k_V of V, and we denote also by τ_1, τ_2 the restriction to $\kappa(\mathfrak{p})$ of τ_1, τ_2. Notice that for each $p = 1, 2$ the signs of the elements f_{ij} in an ordering $\sigma \to \tau_p$ are completely determined by the signs of the parameters x_l in σ and the signs of the units (or more properly of their residue classes) in τ_p.

Next we split F into two disjoint sets F_1, F_2 as follows:

• If $\tau_1 = \tau_2$, we pick generators $\sigma_0, \dots, \sigma_k$ of F, and choose as F_1 the fan generated by $\sigma_0, \dots, \sigma_{k-1}$, and $F_2 = F \setminus F_1$. Note that $\#(F_1) = 2^{k-1} = \frac{1}{2}\#(F)$.

• If $\tau_1 \neq \tau_2$, take as F_1 the fan F_{τ_1} consisting of all orderings of F specializing to τ_1 and $F_2 = F_{\tau_2}$. By the Baer-Krull theorem (Theorem 2.1) there are as many orderings specializing to τ_1 as specializing to τ_2, so that $\#(F_1) = \#(F_2) = \frac{1}{2}\#(F) = 2^{k-1}$. As above we may assume that F_1 is generated by $\sigma_0, \dots, \sigma_{k-1}$, and that $\sigma_k \in F_2$, so that $\sigma_0 \dots, \sigma_{k-1}, \sigma_k$ generate the whole F.

CLAIM. *After some additional blowings-up, we find a regular local ring B dominating A, with the same residue field, and a regular system of parameters $y_1, \dots, y_d$ of B such that all f_{ij}'s are normal crossings in B with respect to them and for all $i = 0, \dots, k$ we have*

$$\sigma_i(y_j) = \begin{cases} +1 & \text{if } 1 \leq j \leq d - i \\ -1 & \text{if } j = d - i + 1 \end{cases}$$

(*compare Table* 1).

In fact, first, after changing x_j by $-x_j$ if necessary, we may assume that $\sigma_0(x_j) = +1$ for all j. Now, notice that since the functions f_{ij} separate the orderings of F_1 and all these orderings specialize to τ_1, two different $\sigma, \sigma' \in F_1$ cannot have the same sign at all the parameters $x_1, \dots, x_d$. In other words, the map

$$\varphi\colon \sigma_0 F_1 \to \{+1, -1\}^d$$

defined by

$$\sigma_0\sigma \mapsto (\sigma(x_1), \dots, \sigma(x_d)),$$

is a monomorphism of $\mathbf{F}_2$-vector spaces.

Thus, there is some j such that $\sigma_1(x_j) \neq \sigma_0(x_j)$. We reorder the parameters so that $\sigma_1(x_l) = +1$ for $1 \leq l < r$ and $\sigma_1(x_l) = -1$ for $r \leq l \leq d$. Consider the extension

$$A^{(1)} = A[x_r/x_d, \dots, x_{d-1}/x_d]_{(x_1, \dots, x_{r-1}, x_r/x_d, \dots, x_{d-1}/x_d, x_d)}.$$

We set $x_j^{(1)} = x_j$ for $1 \leq j \leq r - 1$, $x_j^{(1)} = x_j/x_d$ for $r \leq j \leq d - 1$ and $x_d^{(1)} = x_d$. Then $A^{(1)}$ is a regular ring dominating A, the residue fields of both rings coincide and $x_1^{(1)}, \dots, x_{d-1}^{(1)}, x_d^{(1)}$ is a regular system of parameters of

$A^{(1)}$. Furthermore the expression

$$f_{ij} = u_{ij} x_1^{\alpha_{ij1}} \cdots x_d^{\alpha_{ijd}}$$

can also be written as

$$f_{ij} = u_{ij}\left(x_1^{(1)}\right)^{\alpha_{ij1}} \cdots \left(x_{d-1}^{(1)}\right)^{\alpha_{ij(d-1)}}\left(x_d^{(1)}\right)^{\alpha_{ijr} + \cdots + \alpha_{ijd}}.$$

This means that the f_{ij} are still normal crossings in $A^{(1)}$, showing that all conditions verified by A are similarly verified by $A^{(1)}$. Moreover, we have $\sigma_1(x_l^{(1)}) = +1$ for $1 \leq l < d - 1$ and $\sigma_1(x_d^{(1)}) = -1$, so that we have completed the first step in the induction process. Assume now that we have already found a regular local ring $A^{(p)}$ dominating A with the same residue field that the latter and a system of parameters $x_1^{(p)}, \ldots, x_d^{(p)}$ such that the f_{ij}'s are normal crossings for them in $A^{(p)}$ and for all $0 \leq i \leq p$ it holds $\sigma_i(x_j^{(p)}) = +1$ if $1 \leq j \leq d - 1$ and $\sigma_i(x_{d-i+1}^{(p)}) = -1$. We construct $A^{(p+1)}$ as follows:

Consider σ_{p+1}. We claim that there is $j \leq d - p$ such that $\sigma_{p+1}(x_j) = -1$. For otherwise, a look at Table 1 shows at once that $\sigma_0\sigma_{p+1}$ would be in the subspace generated by $\sigma_0\sigma_1, \ldots, \sigma_0\sigma_p$, against our assumption that $\sigma_0, \ldots \sigma_k$ were affine independent. Then, after reordering $x_1^{(p)}, \ldots, x_{d-p}^{(p)}$ we may assume that $\sigma_p(x_j) = +1$ for $1 \leq j < r$ and $\sigma_1(x_j) = -1$ for $r \leq j \leq d - p$. Consider the extension

$$A^{(p+1)} = A^{(p)}\left[x_r^{(p)}/x_{d-p}^{(p)}, \ldots, x_{d-p-1}^{(p)}/x_{d-1}^{(p)}\right]_{\substack{(x_1^{(p)}, \ldots, x_{r-1}^{(p)}, x_r^{(p)}/x_{d-l}^{(p)}, \ldots, \\ x_{d-p-1}^{(p)}/x_{d-p}^{(p)}, x_{d-p}^{(p)}, \ldots, x_d^{(p)})}}$$

and set $x_j^{(p+1)} = x_j^{(p)}$ *for* $1 \leq j \leq r - 1$, $x_j^{(p+1)} = x_j^{(p)}/x_{d-1}^{(p)}$ for $r \leq j \leq d - p - 1$, and $x_j^{(p+1)} = x_j^{(p)}$ for $d - p \leq j \leq d$. An immediate computation shows that for $0 \leq i \leq p + 1$, $\sigma_i(x_j^{(p+1)}) = +1$ if $1 \leq j \leq d - i$ and $\sigma_i(x_{d-i+1}^{(p+1)}) = -1$, so that we have completed the step $p + 1$. Therefore, the claim is proved.

Once this is done, consider any $\sigma \in F$. There are two possibilities:

- $\sigma \in F_1$. Then $\sigma = \sigma_{i_1} \ldots \sigma_{i_s}$ with $0 \leq i_1 < \cdots < i_s \leq k - 1$, and s is necessarily odd. Let $1 \leq l \leq d - k + 1$; since $\sigma_{i_1}(y_l) = \cdots = \sigma_{i_s}(y_l) = +1$ we get $\sigma(y_l) = +1$.
- $\sigma \in F_2$. Then $\sigma = \sigma_{i_1} \cdots \sigma_{i_s} \cdot \sigma_k$ with $0 \leq i_1 < \cdots < i_s \leq k - 1$, and s is necessarily even. Let $1 \leq l \leq d - k + 1$; we get $\sigma(y_l) = \sigma_{i_1}(y_l) \cdots \sigma_{i_s}(y_l) \cdot \sigma_k(y_l) = \sigma_k(y_l)$.

In conclusion, for $\sigma \in F_1$ we have $\sigma(y_j) = +1$ for $1 \le j \le d-k+1$, while for $\sigma \in F_2$ we have $\sigma(y_j) = \sigma_k(y_j)$. This implies that we have two bijections $\varphi_p\colon F_p \to \{-1, +1\}^{k-1}$ given by $\sigma \mapsto (\sigma(y_{d-k+2}), \ldots, \sigma(y_d))$, $p = 1, 2$. In fact, since the functions f_{ij} separate the orderings of each F_p, the argument above shows that φ_p is injective, and since all sets involved have 2^{k-1} elements, the mappings are bijective.

Now we consider the following diagram

$$\begin{array}{ccc} B & \longrightarrow & C = B_{(y_{d-k+2}, \ldots, yd)} \subset K \\ \downarrow & & \downarrow \\ B/(y_{d-k+2}, \ldots, y_d) & \longrightarrow & k_C \\ \downarrow & & \\ k_A = k_B & & \end{array}$$

where k_B, k_C stand for the residue fields of B, C respectively. By construction, these two residue fields are finitely generated over $\mathbf{R}$. Now let k'_B be a quasicoefficient field of B, that is, a subfield $k'_B \subset B$ such that the extension $k'_B \subset k_B$ induced by the canonical homomorphism $B \to k_B$ is algebraic (even finite in our case). Then, since k_C is the quotient field of the ring $B/(y_{d-k+2}, \ldots, y_d)$, which is local regular of dimension $d-k+1$ and $k_B = k_A = \kappa(\mathfrak{p})$, we get

$$\begin{aligned} \operatorname{tr}\deg[k_C\colon \mathbf{R}] &= \operatorname{tr}\deg[k_C\colon k'_B] + \operatorname{tr}\deg[k'_B\colon \mathbf{R}] \\ &\ge (d-k+1) + \operatorname{tr}\deg[\kappa(\mathfrak{p})\colon \mathbf{R}] \\ &= (d-k+1) + \dim(\mathscr{P}(X)/\mathfrak{p}) \\ &= (d-k+1) + \dim(\mathscr{P}(X)) - \operatorname{ht}(\mathfrak{p}) \\ &= (d-k+1) + \dim(K) - \dim(B) \\ &= n-k+1. \end{aligned}$$

Now, we chase orderings through the diagram, starting in $k_B = \kappa(\mathfrak{p})$ with our τ_1, τ_2.

- Since $B/(y_{d-k+2}, \ldots, y_d)$ is local regular with parameters $y_1, \ldots, y_{d-k+1}$, we can lift τ_1 to an ordering γ_1 of k_C such that $\gamma_1(x_l) = +1$ for $1 \le l \le d-k+1$ (Example 2.2). Also we can lift τ_2 to an ordering γ_2 of k_C such that $\gamma_2(x_l) = \sigma_k(x_l)$ for $1 \le l \le d-k+1$.
- Since C is local regular with parameters $y_{d-k+1}, \ldots, y_d$, we can built up an algebroid fan F' of K parametrized over the two orderings γ_1, γ_2 in k_C (Example 2.6). Let F'_p be the set of orderings of F' specializing to γ_p, for $p = 1, 2$. Now, we also have two bijections $\varphi'_p\colon F'_p \to \{-1, +1\}^{k-1}\colon \sigma' \mapsto$

$(\sigma'(y_{d-k+2}),\dots,\sigma'(y_d))$, $p=1,2$, and consequently we obtain bijections $F_p \to F'_p$: $\sigma \mapsto \sigma'$ such that:

(*a*) $\sigma(y_l)=\sigma'(y_l)$ for $d-k+1<l\le d$.

(*b*) If $p=1$ and $\sigma\in F_1$, we have $\sigma(y_l)=+1=\gamma_1(y_l)=\sigma'(y_l)$ for $1\le l\le d-k$.

If $p=2$ and $\sigma\in F_2$, we have $\sigma(y_l)=\sigma_k(y_l)=\sigma'(y_l)$ for $1\le l\le d-k+1$.

(*c*) $\sigma,\sigma'\to\gamma_p$.

This gives another bijection ψ: $F\to F'$: $\sigma\mapsto\sigma'$, such that $\sigma(y_l)=\sigma'(y_l)$ for $1\le l\le d$ and $\sigma(u)=\sigma'(u)$ for any unit $u\in B$. Consequently, $\sigma(f_{ij})=\sigma'(f_{ij})$ for all i,j and since the f_{ij}'s define the neighborhood $U\subset\Phi_k$ of F fixed at the beginning, we conclude $F'\in U$, which completes the proof. □

We finish this section by pointing out that the restriction to compact models in the last theorem is essencial, as Example 4.5 will show.

4. Review on real spectra

In order to progress further we need the theory of the real spectrum. Here we just review the more basic facts, relying on [BCR] as general reference.

Let A be any commutative ring with unit. The real spectrum $\operatorname{Spec}_r(A)$ of A is the set of all pairs $\alpha=(\mathfrak{p}_\alpha,\le_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of A and $\le_\alpha$ is an ordering in the residue field $\kappa(\mathfrak{p}_\alpha)$; we denote by $\kappa(\alpha)$ the real closure of $\kappa(\mathfrak{p}_\alpha)$ with respect to $\le_\alpha$. Then, α can be seen as a homomorphism α: $A\to A/\mathfrak{p}_\alpha\subset\kappa(\mathfrak{p}_\alpha)\subset\kappa(\alpha)$: $f\mapsto f(\alpha)$. Now, let $\alpha,\beta\in\operatorname{Spec}_r(A)$. We say that α *specializes to* β, or that β is *a specialization of* α, and write $\alpha\to\beta$ if $f(\beta)>0$ implies $f(\alpha)>0$ for $f\in A$; more algebraically, $\alpha\to\beta$ if and only if $\mathfrak{p}_\alpha\subset\mathfrak{p}_\beta$ and the canonical map $A/\mathfrak{p}_\alpha\to A/\mathfrak{p}_\beta$ sends elements $\ge_\alpha 0$ to elements $\ge_\beta 0$. Of course, this is the same specialization introduced earlier in Section 2.

In the setting of the real spectrum we can impose sign conditions on the elements of A and use notations like $\{f_1>0,\dots,f_s>0\}\subset\operatorname{Spec}_r(A)$ for $\{\alpha\in\operatorname{Spec}_r(A): f_1(\alpha)>0,\dots,f_s(\alpha)>0\}$. Then we define in the obvious way the *constructible sets*, which are the sets of the form

$$C=\bigcup_{i=1}^{p}\{f_{i1}>0,\dots,f_{ir_i}>0,\ g_i=0\},$$

and, among them, we distinguish the *basic open* sets, which are the constructible sets of the form

$$C=\{f_{i1}>0,\dots,f_{ir_i}>0\}.$$

These basic open sets generate a topology, the *Harrison topology* of $\operatorname{Spec}_r(A)$, in terms of which the specialization relation introduced above behaves as a limit. For instance, if C is an open constructible set, $\beta \in C$ and $\alpha \to \beta$, then $\alpha \in C$.

We also define the *Zariski topology* of $\operatorname{Spec}_r(A)$ by analogy with the Zariski prime spectrum: a subbasis consists of all sets of the form $\{f \neq 0\}$; we distinguish the operations in this topology with an index Z.

If A is a field we find again the space of orderings described in Section 1. It is clear from the definitions that

$$\operatorname{Spec}_r(A) = \bigcup_{\mathfrak{p}} \operatorname{Spec}_r(\kappa(\mathfrak{p})),$$

where the $\mathfrak{p}$'s run among the prime ideals of A. This simple remark supports the idea of *patching* the informations obtained from the residue fields of A to learn about A itself. Actually, this method gives:

THEOREM 4.1. *Let A be a commutative ring with unit and C an open constructible subset of* $\operatorname{Spec}_r(A)$ *such that $S \cup \overline{(\overline{C} \setminus C)}^Z = \varnothing$. Let s be a positive integer. Suppose that for every prime ideal $\mathfrak{p}$ of A there are $g_1, \dots, g_s \in A$ such that*

$$C \cap \operatorname{Spec}_r(\kappa(\mathfrak{p})) = \{g_1 > 0, \dots, g_s > 0\} \cap \operatorname{Spec}_r(\kappa(\mathfrak{p})).$$

Then, there are $f_1, \dots, f_s \in A$ such that $C = \{f_1 > 0, \dots, f_s > 0\}$.

This theorem has a long history. It was first obtained by Bröcker [Br1], in case A was an algebra finitely generated over a real closed field R, but he could not control completely the number of equations involved. This was solved by Scheiderer in [Sch], who already remarked that the argument worked for any excellent ring A. At the same time Bröcker found a proof that only required A to be noetherian [Br3]. Finally, Marshall discovered how to modify all those proofs to obtain the result for arbitrary A [Mr6].

Now, let $A = \mathscr{P}(X)$ be the ring of polynomial functions of a real algebraic set $X \subset \mathbf{R}^n$. The tilde operator $S \mapsto \tilde{S}$ is the map that sends a semialgebraic set $S \subset X$ to the constructible set $\tilde{S} \subset \operatorname{Spec}_r(A)$ defined by any formula that also defines S. By Tarski's principle, this definition is consistent and we obtain a bijection that preserves inclusions and topological operations. This tilde operator is the main tool to translate semialgebraic problems and statements in terms of the real spectra.

Finally, suppose that X is irreducible and let $K = \mathscr{K}(X)$. Then $\operatorname{Spec}_r(K) \subset \operatorname{Spec}_r(A)$, and the tilde operation induces a mapping $S \mapsto \tilde{S} \cap \operatorname{Spec}_r(K)$, which is *generically* injective: if $S, T \subset X$ are semialgebraic sets such that $\tilde{S} \cap \operatorname{Spec}_r(K) = \tilde{T} \cap \operatorname{Spec}_r(K)$, then $S \setminus Z = T \setminus Z$ for some nowhere dense algebraic set $Z \subset X$. In this way we can mix the *geometric* and *algebraic* settings to study our problem. For all of this we refer to [BCR], [Br], [AnBrRz1, 2]. For instance, Theorem 1 of the introduction is just a translation of Theorem 4.1. We also use this strategy to deduce directly from Theorem 1.2 the following statement.

COROLLARY 4.2. *Let S be a semialgebraic subset of an irreducible real algebraic set $X \subset \mathbf{R}^n$. Then the following assertions are equivalent*:

(a) *S is generically basic.*

(b) *For every fan F of the field $\mathscr{K}(X)$ with $\#(F) = 4$ we have $\#(F \cap \tilde{S}) \neq 3$.*

This was our starting point in [AnRz1] to prove Theorem 3 of the introduction. Here we will work similarly to prove Theorem 4, using the following consequence of Corollary 1.3

COROLLARY 4.3. *Let S be a generically basic semialgebraic subset of an irreducible real algebraic set $X \subset \mathbf{R}^n$. Then the following assertions are equivalent*:

(a) *S is generically s-basic.*

(b) *For every fan F of the field $\mathscr{K}(X)$ with $\#(F) = 2^k$ and $\#(F \cap \tilde{S}) = 1$ we have $k \leq s$.*

The next step towards the proof of Theorem 4 of the introduction is the following result.

PROPOSITION 4.4. *Let S be a generically basic semialgebraic subset of a compact irreducible real algebraic set $X \subset \mathbf{R}^n$ of dimension d. Then the following assertions are equivalent*:

(a) *S is generically s-basic.*

(b) *For every algebroid fan F of the field $\mathscr{K}(X)$ finite over $\mathscr{P}(X)$ and parametrized over a function field of dimension $d - k + 1$ such that $\#(F) = 2^k$ and $\#(F \cap \tilde{S}) = 1$ we have $k \leq s$.*

Proof. We only have to prove (b) $\Rightarrow$ (a). So, suppose that S is not generically s-basic. By Corollary 4.3 there is a fan F of the field $\mathscr{K}(X)$ with $\#(F) = 2^k$ and $\#(F \cap \tilde{S}) = 1$, but $k > s$. Now let $f_1, \dots, f_r$ be the functions appearing in a description of S. For every $\sigma \in F$ we put $\varepsilon_{\sigma i} = \sigma(f_i)$, $1 \leq i \leq r$, and $U_\sigma = \{\varepsilon_{\sigma 1} f_1 > 0, \dots, \varepsilon_{\sigma r} f_r > 0\}$. Then $U = \prod_{\sigma \in F} U_\sigma$ is a neighborhood of F and by Theorem 3.1 there is an algebroid fan $F' \in U$ finite over $\mathscr{P}(X)$ and parametrized over a function field of dimension $d - k + 1$. It is obvious from our definition of U that $\#(F' \cap \tilde{S}) = \#(F \cap \tilde{S}) = 1$. Since $k > s$, we are done. $\square$

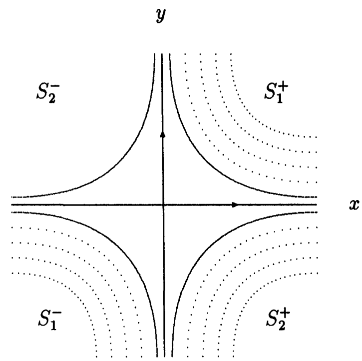

FIG. 1

Example 4.5. We construct a semialgebraic set $S \subset \mathbf{R}^2$ which is not generically basic, but the obstruction can only be read through fans compatible with valuations of $\mathscr{K}(\mathbf{R}^2) = \mathbf{R}(x, y)$ that are not finite on $\mathscr{P}(\mathbf{R}^2) = \mathbf{R}[x, y]$. Consequently those fans cannot be approximated by others finite on $\mathbf{R}[x, y]$.

To define S consider the sets (Figure 1)

$$S_1 = \{xy \geq 1\}, S_1^+ = S_1 \cap \{x \geq 0\}, S_1^- = S_1 \cap \{x \leq 0\},$$

$$S_2 = \{xy \leq -1\}, S_2^+ = S_2 \cap \{x \geq 0\}, S_2^- = S_2 \cap \{x \leq 0\},$$

and put $S = S_1 \cup S_2^+$. We will denote by $\tilde{S}_1$ the constructible subset of the space of orderings of $\mathscr{K}(\mathbf{R}^2)$ defined by the same equations as S_1, and similarly $\tilde{S}_1^+, \tilde{S}_1^-$, etc.

Now, to prove our previous claim, let $F = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a fan with $\#(F \cap \tilde{S}) = 3$. Since S_1 is generically 1-basic, $\#(F \cap \tilde{S}_1) = 0$ or 2; analogously, $\#(F \cap \tilde{S}_2) = 0$ or 2. Since S_2^+ is generically basic, $\#(F \cap \tilde{S}_2^+) = 0$, 1 or 2. Hence $\#(F \cap \tilde{S}_1) = 2$ and $\#(F \cap \tilde{S}_2^+) = \#(F \cap \tilde{S}_2^-) = 1$, say $\sigma_1, \sigma_3 \in \tilde{S}_1$, $\sigma_2 \in \tilde{S}_2^+$ and $\sigma_4 \in \tilde{S}_2^-$. Suppose now that there is a valuation V of $\mathbf{R}(x, y)$ compatible with F such that $\mathbf{R}[x, y] \subset V$. Then the maximal ideal $\mathfrak{m}_V$ of V lies over a real prime ideal $\mathfrak{p}$ of $\mathbf{R}[x, y]$, and the σ_i's make $\mathfrak{p}$ convex and specialize to at most two orderings τ_1, τ_2 in the residue field $\kappa(\mathfrak{p})$. Now we will argue using the real spectrum of the ring $\mathbf{R}[x, y]$. We distinguish two possible cases:

- $\mathrm{ht}(\mathfrak{p}) = 2$. Then $\mathfrak{p}$ is a maximal ideal, that is, the ideal of a point $z \in \mathbf{R}^2$. If $z \notin S_j$, since S_j is closed, no σ_i would be in S_j. Hence $z \in S_1 \cap S_2 = \varnothing$, which is absurd. Thus this case is impossible.

• $\mathrm{ht}(\mathfrak{p}) = 1$. Then $\mathfrak{p}$ is the ideal of an irreducible curve Z. Suppose that, say, $\sigma_4 \to \tau_2$. Then τ_2 is not an inner point of $\tilde{S}$, for otherwise, since the interior of $\tilde{S}$ is an open constructible set, the generization σ_4 would belong to $\tilde{S}$ too. Also, we have $\sigma_i \to \tau_2$ for some other σ_i, say σ_1. Since $\sigma_1 \in \tilde{S}_1$ and $\tilde{S}_1$ is closed, it follows that $\tau_2 \in \tilde{S}_1$. Although, we see that τ_2 belongs to the boundary $\partial\tilde{S}_1$ of $\tilde{S}_1$, which by construction is the hyperbola $xy - 1 = 0$. This means that $xy - 1 \in \mathfrak{p}$, or equivalently that $Z \subset \{xy - 1 = 0\}$. Now we have $\sigma_2 \to \tau_1$, and arguing as above we get $\tau_1 \in \partial\tilde{S}_2$, or equivalently $Z \subset \{xy + 1 = 0\}$. Since the two hyperbolas are disjoint we get a contradiction. The rest of the cases are treated similarly.

In conclusion, we always come to a contradiction, which shows that there is no valuation V compatible with F and finite over $\mathbf{R}[x, y]$. Finally we have to prove that F does exist if we do not require the finiteness condition. To do this we work in the projective plane with coordinates $(x_0 : x_1 : x_2)$, where $x = x_1/x_0$, $y = x_2/x_0$. Actually, we work at the point $(0:0:1)$, or better in the affine chart $x_2 \neq 0$. We put $u = x_0/x_2$ $v = x_1/x_2$ and our sets are given (birationally) by the sign conditions that follow:

$$S_1 = \{v \geq u^2\},\ S_1^+ = S_1 \cap \{u \geq 0\},\ S_1^- = S_1 \cap \{u \leq 0\},$$
$$S_2 = \{v \leq -u^2\},\ S_2^+ = S_2 \cap \{u \leq 0\},\ S_2^- = S_2 \cap \{u \geq 0\},$$

and of course $S = S_1 \cup S_2^+$ (Figure 2). Now, we obtain the fan F starting with a valuation compatible with it. Namely, the discrete rank 1 valuation $\mathbf{R}[u, v]_{(u)}$, whose residue field $\mathbf{R}(v)$ has two orderings: τ_1, with v positive and infinitesimal with respect to $\mathbf{R}$, and τ_2, with v negative and infinitesimal with respect to $\mathbf{R}$. Then F will consist of the four liftings σ_1, σ_3 and σ_2, σ_4 of τ_1 and τ_2 defined by $\sigma_1(t) = \sigma_4(t) = +1$, $\sigma_3(t) = \sigma_2(t) = -1$. Clearly $\sigma_1, \sigma_2, \sigma_3 \in \tilde{S}$ and $\sigma_4 \notin \tilde{S}$. Furthermore, there are two valuations compatible

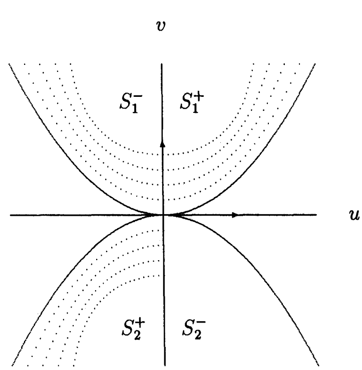

FIG. 2

with F. One is $\mathbf{R}[u, v]_{(u)}$, and the other is the composite of this with $\mathbf{R}[v]_{(v)}$. The centers of these valuations are, respectively, the line $u = 0$ and the point $u = v = 0$. In projective coordinates they are the line $x_0 = 0$ and the point $(0:0:1)$, in both cases they are at infinity with respect to the affine (x, y)-plane.

5. Proof of the main result

Theorem 4 of the introduction will be an immediate consequence of the following result.

THEOREM 5.1. *Let S be a generically basic semialgebraic subset of an irreducible real algebraic set $X \subset \mathbf{R}^n$. Let $Z \subset X$ be any proper algebraic subset containing the singular locus of X and the boundary of S, $\partial S = \overline{S} \setminus S^0$. Then the following assertions are equivalent*:

(a) *S is generically s-basic.*

(b) *For any irreducible algebraic set $Y \subset X$ of dimension $s + 1$, not contained in Z, the intersection $S \cap Y$ is generically s-basic.*

Proof. Assume first that S is generically s-basic, and let $Y \subset X$ be an irreducible subset not contained in Z. Denote by $\mathfrak{p} \subset \mathscr{P}(X)$ the ideal of Y. Since Y is not contained in the singular locus of X the localization $\mathscr{P}(X)_{\mathfrak{p}}$ is a regular local ring of dimension, say, d, whose residue field is $L = \mathrm{qf}(\mathscr{P}(X)/\mathfrak{p}) = \mathscr{K}(Y)$. Now suppose that $S \cap Y$ is not generically s-basic. Then, by Corollary 1.3, there is a fan $F = (\sigma_i : 1 \leq i \leq 2^k)$ of L such that $\#(F \cap \tilde{S}) = 1$ and $k > s$. This F lifts to a fan $F' = (\sigma_i' \colon 1 \leq i \leq 2^k)$ of $\mathscr{K}(X)$ with $\sigma_i' \to \sigma_i$. Indeed, as we explained in Example 2.2, using a regular system of parameters $x_1, \dots, x_d$ of $\mathscr{P}(X)_{\mathfrak{p}}$ we can lift any ordering σ of L to 2^d different orderings of K, each corresponding to a choice of signs for the given parameters. Hence we fix all parameters positive and lift every σ_i to σ_i'. It is very easy to check that the σ_i''s form a fan and of course $\#(F') = 2^k$. We claim that $\#(F' \cap \tilde{S}) = 1$. Indeed, suppose $\sigma_i \in \tilde{S}$. Since Z does not contain Y, we have $\sigma_i \notin \tilde{Z}$, and, since $\partial S \subset Z$, $\sigma_i \notin \widetilde{\partial S}$. Thus we have $\sigma_i \in \tilde{S}^0$, which is constructible and open. Since $\sigma_i' \to \sigma_i$, we get $\sigma_i' \in \tilde{S}^0 \subset \tilde{S}$. On the other hand, suppose $\sigma_i \notin \tilde{S}$. Again we have $\sigma_i \notin \tilde{Z}$, so that $\sigma_i \notin \tilde{Z} \cup \tilde{S}$, and we get $\sigma_i \notin \tilde{\overline{S}}$. Since this set is constructible and closed, it follows that $\sigma_i' \notin S$ either. Whence, $\#(F \cap \tilde{S}) = 1$ and S is not generically s-basic, as claimed. Note that for this implication we do not need any special type of fan.

In order to prove the converse implication, we can substitute X by its one-point compactification or, in other words, assume that X is compact. Now suppose S is not generically s-basic. Then, by Corollary 1.3, there is a fan F of $\mathscr{K}(X)$ such that $\#(F) = 2^k$, $\#(F \cap \tilde{S}) = 1$ and $k > s$. Let us see how this leads to a contradiction with (b). We will separate the argument in several steps.

I. *Realization of the obstruction to basicness by an algebroid fan.* By Proposition 4.4 we may suppose that F is an algebroid fan of the field $K = \mathscr{K}(X)$, finite on $\mathscr{P}(X)$ and parametrized over a function field L of dimension $d - k + 1$. This means that F is defined by an embedding

$$\phi: K \hookrightarrow L((x_1, \dots, x_{k-1}))$$

and two orderings γ_1, γ_2 in L, and that the ring $\mathscr{P}(X)$ of polynomial functions of X is contained in the ring $L[[x_1, \dots, x_{k-1}]]$ via ϕ. Now, since L is a function field, there is an irreducible algebraic set $W \subset \mathbf{R}^m$ whose field of rational functions $\mathscr{K}(W)$ is L, that is, L is the quotient field of the ring $\mathscr{P}(W)$ of polynomial functions on W.

II. *Choice of a hyperplane section in the coefficient field of the algebroid fan.* Let H stand for a generic hyperplane section of W and $\mathfrak{p}$ for the ideal of H in $\mathscr{P}(W)$. By Bertini's theorem [Jn], [BCR], H is a nonsingular irreducible subset of W, and $\mathfrak{p}$ a real prime ideal. Note that the field $\mathscr{K}(H)$ of rational functions of H is the residue field of $\mathfrak{p}$, that is, the quotient field of $\mathscr{P}(W)/\mathfrak{p}$. With all of this we have the following diagram

$$\mathscr{P}(X) \subset V \xrightarrow{\phi} \mathscr{K}(W)[[x_1, \dots, x_{k-1}]]$$
$$\mathscr{P}(W)_{\mathfrak{p}}[[x_1, \dots, x_{k-1}]] \xrightarrow{\varphi} \mathscr{K}(H)[[x_1, \dots, x_{k-1}]]$$

where the homomorphism φ is the obvious extension of the canonical mapping $\mathscr{P}(W)_{\mathfrak{p}} \to \mathscr{K}(H)$.

Since the ring $\mathscr{P}(X)$ is an algebra finitely generated over $\mathbf{R}$, we can pick finitely many generators $f_1, \dots, f_q$ in $\mathscr{P}(X)$; we add to these the equations, say $f_{q+1}, \dots, f_s$, involved in a description of the semialgebraic set S and an equation of Z. All these functions f_i are in $\mathscr{K}(W)[[x_1, \dots, x_{k-1}]]$, and so they have power expansions $f_i = f_i(x) = \Sigma_\nu (g_{i\nu}/h_{i\nu}) x^\nu$, where $\nu \in \mathbf{N}^{k-1}$ and $g_{i\nu}, h_{i\nu} \in \mathscr{P}(W)$. As our hyperplane section H is generic, we can suppose no $g_{i\nu}, h_{i\nu}$ vanishes on H (although there are infinitely many $g_{i\nu}, h_{i\nu}$'s, their number is countable, and working over the reals we can use Baire's theorem). In particular, $h_{i\nu} \notin \mathfrak{p}$ implies that the $f_i(x)$'s are well defined elements of $\mathscr{P}(W)_{\mathfrak{p}}[[x_1, \dots, x_{k-1}]]$. Finally, since the f_i's generate $\mathscr{P}(X)$ we get $\phi(\mathscr{P}(X)) \subset \mathscr{P}(W)_{\mathfrak{p}}[[x_1, \dots, x_{k-1}]]$ and consequently we have a *formal* homomorphism

$$\psi = \varphi\phi: \mathscr{P}(X) \to \mathscr{K}(H)[[x_1, \dots, x_{k-1}]].$$

Moreover $g_{i\nu} \notin \mathfrak{p}$ implies that the coefficients of the $f_i(x)$'s are units in $\mathscr{P}(W)_{\mathfrak{p}}$ and so

$$\psi(f_i) = \varphi(f_i(x)) = \Sigma_l(g_{i\nu}/h_{i\nu} \bmod \mathfrak{p})x^{\nu}$$

is a non-zero element of $\mathscr{K}(H)[[x_1, \ldots, x_{k-1}]]$.

III. *Perturbation of the hyperplane section to obtain a new algebroid fan.* We set $F = F_1 \cup F_2$, where F_p contains the orderings of F that specialize to γ_p, $p = 1, 2$. Then every ordering $\sigma \in F_p$ is determined by a sign condition ε: $\{x_1, \ldots, x_{k-1}\} \to \{+1, -1\}$. Also we know from Example 2.4(a) that the sign of f_i in any such ordering is completely determined by its initial form (with respect to the lexicographic ordering in the exponents) say $(g_{i\nu_{0i}}/h_{i\nu_{0i}})x^{\nu_{0i}}$. Let $\tilde{G}_p \subset \operatorname{Spec}_r(\mathscr{K}(W))$ be the open neighborhood of γ_p defined by

$$\{\eta_1 g_{1\nu_{01}}/h_{1\nu_{01}} > 0, \ldots, \eta_s g_{s\nu_{0s}}/h_{s\nu_{0s}} > 0\},$$

where η_i is the sign of $g_{i\nu_{0i}}/h_{i\nu_{0i}}$ in γ_p. Then, the lifting σ' of an ordering $\gamma'_p \in G_p$ corresponding to the sign condition ε has at the f_i's the same signs that σ. This implies that for any two orderings $\gamma'_1 \in \tilde{G}_1$ and $\gamma'_2 \in \tilde{G}_2$ the fan F' parametrized over them verifies also $\#(F' \cap \tilde{S}) = 1$ and $k > s$ (cf. Examples 2.2 and 2.6).

Now, we denote by G_1, G_2 the two open semialgebraic subsets of W corresponding to the neighborhoods just constructed. These semialgebraic sets are Zariski dense in W, which guarantees that we can choose the generic hyperplane section H to meet both of them. This implies that there are $\gamma'_1 \in G_1$ and $\gamma'_2 \in G_2$ which make the ideal $\mathfrak{p}$ of H convex. In other words, γ'_1 and γ'_2 induce two orderings τ_1 and τ_2 in the residue field of $\mathfrak{p}$, which is $\mathscr{K}(H)$. Next, over τ_1 and τ_2 we parametrize a fan F'' of $\mathscr{K}(H)[[x_1, \ldots, x_{k-1}]]$. We have bijections $F \to F' \to F''$: $\sigma \mapsto \sigma' \mapsto \sigma''$ such that σ, σ' and σ'' are all defined by the same sign condition ε: $\{x_1, \ldots, x_{k-1}\} \to \{+1, -1\}$.

After this preparation, we have the following diagram

$$\begin{tikzcd}
 & \mathscr{P}(X) \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & & \\
\mathscr{P}(W)_{\mathfrak{p}}[[x_1, \ldots, x_{k-1}]] \arrow[rr] \arrow[d] & & \mathscr{K}(H)[[x_1, \ldots, x_{k-1}]] \arrow[d] & \sigma' \arrow[r] \arrow[d] & \sigma'' \arrow[d] \\
\mathscr{P}(W)_{\mathfrak{p}} \arrow[rr] & & \mathscr{K}(H) & \gamma'_{\mathfrak{p}} \arrow[r] & \tau_{\mathfrak{p}}
\end{tikzcd}$$

where σ' and σ'' are defined by the same sign condition ε_{σ}, as explained above.

Now consider the kernel $\mathfrak{q}$ of the homomorphism ψ. Its zero set is an algebraic set $Y \subset X$ with $\mathscr{P}(Y) = \mathscr{P}(X)/\mathfrak{p}$, and $\dim(Y) = \dim(X) - ht(\mathfrak{q})$.

Furthermore, the fan F'' consisting of the σ'''s restricts to a fan F^* in $\mathscr{K}(Y)$ such that $\#(F^* \cap \tilde{S}) = 1$, because by construction the signs of σ'' at the $\psi(f_j)$'s coincide with those of σ'. Consequently, the semialgebraic set $S \cap Y$ is not generically s-basic. Furthermore, since among the f_i's there is an equation of Z, and no $\psi(f_i)$ is zero, Y is not contained in Z. Hence it only remains to show that we can impose the further condition $\dim(Y) < \dim(X)$ and from that the proof will end by induction.

IV. *Approximation of the algebroid fan by other whose coefficient field has smaller dimension.* In fact, we will approximate the formal homomorphism ψ. Notice that since $f_1, \dots, f_s$ generate $\mathscr{P}(X)$, ψ is completely determined by the images $\psi(f_1), \dots, \psi(f_s)$. Consider now r power series $a_1(x), \dots, a_r(x) \in \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]$ such that $a_i(x) \equiv \psi(f_i) \pmod{\mathfrak{m}^n}$ for a suitable $n \in \mathbf{N}$ and suppose that we may define a new homomorphism

$$\psi' : \mathscr{P}(X) \to \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]$$

by $\psi'(f_i) = a_i(x)$. Let $\mathfrak{q}'$ denote the kernel of ψ', and F^* the fan induced by F'' in $\mathscr{P}(X)/\mathfrak{q}'$. Since the signs are determined by the initial forms, and, for n large enough, the initial forms of the $\psi(f_i)$'s coincide with those of the $\psi'(f_i)$'s, it follows that $\#(F^* \cap \tilde{S}) = 1$. In other words, the approximation ψ' of ψ defines a subvariety Y' in which $S \cap Y'$ is not basic: Y' is the zero set of the kernel $\mathfrak{q}'$ of ψ', and $\mathscr{P}(Y') = \mathscr{P}(X)/\mathfrak{q}'$. We will see next that if the approximation ψ' is *algebraic*, that is, the a_i's are algebraic power series, then $\dim(Y') < \dim(X)$.

Indeed, if the a_i's are algebraic we have

$$\psi'(\mathscr{P}(X)) \subset \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]_{\mathrm{alg}},$$

and ψ' induces an embedding

$$\mathscr{P}(Y') \hookrightarrow \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]_{\mathrm{alg}}$$

which extends to the quotient fields $\mathscr{K}(Y') \hookrightarrow \kappa(\mathfrak{p})((x_1, \dots, x_{k-1}))_{\mathrm{alg}}$. Counting transcendence degrees over $\mathbf{R}$ we find

$$\begin{aligned} \dim(Y') &= \operatorname{tr deg}[\mathscr{K}(Y') : \mathbf{R}] \le \operatorname{tr deg}[\kappa(\mathfrak{p})((x_1, \dots, x_{k-1}))_{\mathrm{alg}} : \mathbf{R}] \\ &= (k-1) + \operatorname{tr deg}[k(\mathfrak{p}) : \mathbf{R}] = (k-1) + \dim(H) \\ &< (k-1) + \dim(W) = \dim(X) \end{aligned}$$

as wanted.

V. *Construction of an algebraic approximation by the formal homomorphism.* Let us see, finally, how to construct the algebraic approximation ψ'. Consider the prime ideal $\mathfrak{n} = \psi^{-1}(x_1, \dots, x_{k-1})$ and the corresponding localization $A = \mathscr{P}(X)_{\mathfrak{n}}$. The homomorphism ψ extends to the henselization A^h. Now, $\mathscr{P}(X)$ is a quotient of a polynomial ring, say, $\mathscr{P}(X) = \mathbf{R}[T_{11}, \dots, T_m]/\mathscr{T}$. We denote by $\mathscr{N}$ the ideal of $\mathbf{R}[T_1, \dots, T_m]$ corresponding to $\mathfrak{n}$, and fix a regular system of parameters $Z_1, \dots, Z_r \in \mathscr{N}$ of the regular local ring $\mathbf{R}[T_1, \dots, T_m]_{\mathscr{N}}$; note that in this situation, $A^h = (\mathbf{R}[T_1, \dots, T_m]_{\mathscr{N}})^h/\mathscr{T}^h$. Now, we have

$$(\mathbf{R}[T_1, \dots, T_m]_{\mathscr{N}})^h = \kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]_{\mathrm{alg}},$$

so that A^h is a quotient of an algebraic power series ring. In fact this can be seen as follows: By Noether's Normalization Lemma, we may assume that $\mathbf{R}[T_1, \dots, T_m]/\mathscr{N}$ is finite over $\mathbf{R}[T_1, \dots, T_{m-r}]$, so that the field $\kappa(\mathfrak{n})$ is an algebraic extension of $k = \mathbf{R}(T_1, \dots, T_{n-r})$. Thus, we have the two ring extensions

$$k[Z_1, \dots, Z_r] \subset \mathbf{R}[T_1, \dots, T_m]_{\mathscr{N}}$$

and

$$k[Z_1, \dots, Z_r] \subset \kappa(\mathfrak{n})[Z_1, \dots, Z_r]_{(Z_1, \dots, Z_r)},$$

which are both algebraic, since the three domains have the same transcendence degree m over $\mathbf{R}$. Then, since the henselizations of the two local regular rings

$$\mathbf{R}[T_1, \dots, T_m]_{\mathscr{N}} \quad \text{and} \quad \kappa(\mathfrak{n})[Z_1, \dots, Z_r]_{(Z_1, \dots, Z_r)}$$

are the algebraic closures in their common completion $\kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]$, we conclude that those henselizations coincide with the algebraic closure of $k[Z_1, \dots, Z_r]$ in $\kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]$. Hence,

$$(\mathbf{R}[T_1, \dots, T_m]_{\mathscr{N}})^h = \kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]_{\mathrm{alg}}$$

as claimed.

Let $\bar{\psi}$: $\kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]_{\mathrm{alg}} \to \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]$ be the composition of ψ and the canonical epimorphism $\kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]_{\mathrm{alg}} \to A^h$, and let $g_1, \dots, g_t$ be a system of generators of the ideal $\mathscr{T}^h$. We follow the method of [Tg, Chap. III, Section 5, page 64]. Set $z_i(x) = \bar{\psi}(Z_i)$. Then we have

$$g_i(z(x)) = 0 \quad \text{for all} \quad i = 1, \dots, m.$$

By M. Artin's approximation theorem (cf. [BCR, Theorem 8.3.1, page 154]) there are

$$y_1(x), \dots, y_m(x) \in \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]$$

arbitrarily close to $z_1(x), \dots, z_m(x)$ in the $\mathfrak{m}$-adic topology of $\kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]$ such that

$$g_i(y(x)) = 0 \quad \text{for all} \quad i = 1, \dots, m.$$

This means that the homomorphism $\bar{\psi}'$: $\kappa(\mathfrak{n})[[Z_1, \dots, Z_r]]_{\text{alg}} \to \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]$ defined by $Z_i \mapsto y_i(x)$ factors through A^h. In this way we can approximate arbitrarily ψ by

$$\psi' \colon A^h \to \kappa(\mathfrak{p})[[x_1, \dots, x_{k-1}]]_{\text{alg}},$$

as required. □

We finish the paper with the following:

Proof of Theorem 4. It is clear that if S is s-basic any intersection $S \cap Y$ with an irreducible subset $Y \subset X$ is also s-basic, and so generically s-basic. Conversely, suppose S is not s-basic. By the Bröcker-Scheiderer criterion (Theorem 1) there is an irreducible subset $X' \subset X$ such that $S \cap X'$ is not generically s-basic. Then, by Theorem 5.1 there is an irreducible subset $Y \subset X'$ of dimension $s + 1$ such that $S \cap Y$ is not generically s-basic, and we are done.

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