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CONJUGATE EXPANSIONS FOR HERMITE FUNCTIONS[1]

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1. Introduction

In the last chapter of [St], Stein discusses the notion of a Hilbert transform associated with a general Sturm-Liouville operator. In particular, let

$$L = \frac{d^2}{dx^2} + a(x)\frac{d}{dx}$$

with $a'(x) \leq 0$ and let

$$q(x) = \exp\left(\int_0^x a(t)\,dt\right).$$

Let $\{\varphi_n\}$, $n \geq 0$, be a complete orthonormal set of eigenfunctions of L with eigenvalue $-\lambda_n^2$ for the Hilbert space $L^2(\mathbf{R}, q(x)\,dx)$. Then Stein points out that

$$\left\{\frac{1}{\lambda_n}\frac{d\varphi_n}{dx}\right\}$$

is also an orthonormal system for $L^2(\mathbf{R}, q(x)\,dx)$ and the suggested Hilbert transform is given by the mapping

$$\varphi_n \to \frac{1}{\lambda_n}\frac{d\varphi_n}{dx},\ n \geq 0.$$

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[1] *Added on November 28, 1993.* The authors have recently become aware of the paper by I. Joó, *On Hermite-Fourier series,* Periodica Math. Hung. **24**(1992), 87-118, where another notion of conjugacy for Hermite function expansions is defined and investigated. Specifically the formal Hilbert transform that emerges is given by the mapping $h_n \to h_{n-1}$. The statements of main results from our paper and that of Joó though identical deal with different objects. Also the proofs in each case which are modelled on Muckenhoupt's technique are similar.

[2] The paper was written while the second author was visiting the Department of Mathematics, University of Georgia, during the 1990-1991 academic year.

One of the simplest and well-known examples occurs when $a(x) = -2x$. In this case one obtains the normalized hermite polynomials.

$$\varphi_n(x) = \tilde{H}_n(x) = (2^n n! \pi^{1/2})^{-1/2} H_n(x), \tag{1.1}$$

where

$$H_n(x) = \exp(x^2)(-1)^n \frac{d^n}{dx^n}(\exp(-x^2)).$$

The Hermite polynomials satisfy

$$LH_n = -2nH_n,$$

where

$$L = \frac{d^2}{dx^2} - 2x\frac{d}{dx},$$

and the system $\{\varphi_n\}$ in (1.1) is a complete orthonormal system on $L^2(\mathbf{R}, \exp(-x^2)\,dx)$. Thus $\lambda_n = \sqrt{2n}$, and by the identity $H_n' = 2nH_{n-1}$, one finds that the suggested Hilbert transform is given by the mapping $\tilde{H}_n \to \tilde{H}_{n-1}$, $n \geq 0$.

A natural question which arises is that of L^p mapping properties of such a Hilbert transform. One method of studying such properties is through the theory of Poisson and conjugate Poisson integrals in which the Hilbert transform is obtained as a boundary value of the conjugate Poisson integral. For the Hermite polynomials, this program was carried out by Muckenhoupt [Mu 1], [Mu 2] where the exponential $\exp(-x^2)$ is treated as a weight function.

Instead of treating $\exp(-x^2)$ as a weight function, it is possible to consider expansions with respect to the system of Hermite functions $\{h_n\}$ defined by

$$h_n(x) = (2^n n! \pi^{1/2})^{-1/2} \exp(-x^2/2) H_n.$$

The system $\{h_n\}$ is a complete orthonormal system on $L^2(\mathbf{R}, dx)$. The Hermite functions are eigenfunctions of the Hermite operator $L = d^2/dx^2 - x^2$. More specifically one has

$$Lh_n = -(2n+1)h_n.$$

Based solely on the facts that $\{d_n\}$ is orthonormal in $L^2(\mathbf{R}, dx)$,

$$Lh_n = -(2n+1)h_n = -\lambda_n^2 h_n,$$

and the h_n's are of rapid decrease at $\pm\infty$, it is not difficult to prove that the system $\{\psi_n\}$ where ψ_n are defined by

$$\psi_n(x) = \left(\lambda_n^2 - 1\right)^{-1/2}(h'_n + xh_n),\ n \geq 1,$$

is also an orthonormal system in $L^2(\mathbf{R}, dx)$. From the decay at $\pm\infty$ and orthonormality of $\{h_n\}$ one has

$$0 = \int_{-\infty}^{\infty} (h_n h_m x)'\,dx = \int_{-\infty}^{\infty} h'_n h_m x\,dx + \int_{-\infty}^{\infty} h_n h'_m x\,dx + \delta_{mn}.$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n \psi_m\,dx &\\ &= \left((\lambda_n^2 - 1)(\lambda_m^2 - 1)\right)^{-1/2} \int_{-\infty}^{\infty} (h'_n + xh_n)(h'_m + xh_m)\,dx \\ &= \left((\lambda_n^2 - 1)(\lambda_m^2 - 1)\right)^{-1/2}\left(\int_{-\infty}^{\infty} h'_n h'_m\,dx + \int_{-\infty}^{\infty} h_n h_m x^2\,dx - \delta_{mn}\right).\end{aligned}$$

Integrating by parts in the first integral and using $Lh_n = -\lambda_n^2 h_n$ shows that $\{\psi_m\}$ is orthonormal.

This suggests that the Hilbert transform for the Hermite functions be defined by the mapping $h_n \to (\lambda_n^2 - 1)^{-1/2}(h'_n + xh_n) = h_{n-1}$. The last equality comes from the explicit form of h_n. However, a closer examination of the situation reveals that things are not so simple. In the classical Sturm-Liouville case

$$\frac{d^2}{dx^2} + a(x)\frac{d}{dx},$$

the eigenvalues $-\lambda_n^2$ arise in both the definition of the Poisson integral

$$f(x, y) = \sum_0^{\infty} \exp\left(-\lambda_n^{1/2} x\right)\langle f, \varphi_n\rangle \varphi_n(y)$$

as well as the suggested Hilbert transform

$$\varphi_n \to \frac{1}{\lambda_n}\varphi'_n.$$

In the case of Hermite functions, λ_n still occurs in the Poisson integral while $(\lambda_n^2 - 1)^{-1/2}$ occurs in the suggested Hilbert transform. To deal with this fact, we define the Hilbert transform for Hermite functions by mapping

$$h_n \to \left(\frac{2n}{2n+1}\right)^{1/2} h_{n-1}. \tag{1.2}$$

While this mapping takes an orthonormal system into only an orthogonal system, the definition (1.2) will be crucial in the analogue of the Cauchy-Riemann equations in §3.

We now describe the main results of this paper. The primary objective is to develop a theory of Poisson and conjugate Poisson integrals associated with the Hermite functions $\{h_n\}$ and to obtain L^p mapping properties of the Hilbert transform defined by (1.2). In §2 we discuss the heat-diffusion integral for the Hermite operator and prove the appropriate estimates. We also apply a subordination principle to obtain the Poisson kernel. In §3 we derive the corresponding Cauchy-Riemann equation and obtain the conjugate Poisson kernel. In §4, 5, and 6, we closely follow Muckenhoupt's program [Mu 1], [Mu 2] to establish estimates for the conjugate Poisson integral and Hilbert transform. In §7, we state the main theorem of this paper.

Let $f^*(x)$ and $H^*(x)$ denote the Hardy-Littlewood maximal function and the ordinary maximal Hilbert transform on $\mathbf{R}$ respectively:

$$f^*(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)|\, dt,$$

$$H^*(x) = \sup_{\varepsilon > 0} \left| \int_{|x-t| > \varepsilon} \frac{f(t)}{x-t}\, dt \right|.$$

Muckenhoupt's program is based on the following two lemmas:

Lemma A. *Assume* $Tf(y) = \int_{-\infty}^{\infty} f(z) L(y, z)\, dz$, $L(y,z) \geq 0$, *and* $\|L(y, \cdot)\|_1 \leq C$ *where* C *is independent of* y. *Suppose* $L(y, \cdot)$ *is monotone increasing for* $z < y$ *and monotone decreasing for* $z > y$. *Then* $|Tf(y)| \leq Cf^*(y)$.

Lemma B. *Assume* $Tf(y) = \int_{-\infty}^{\infty} f(z) K(y, z)\, dz$ *where* $K(y, y+h) = -K(y, y-h)$ *and the function* $h \to hK(y, y+h)$ *has total variation bounded by* C *where* C *is independent of* y. *Then* $|Tf(y)| \leq CH^*(y)$.

As in [Mu 2], we will show that the conjugate Poisson kernel $Q(x, y, z)$ (here x denotes the distance above the boundary $\mathbf{R}$) can be written as $J(x, y, z) + K(x, y, z)$ where $|J(x, y, z)| \leq L(y, z)$ and $L(y, z)$ and $K(x, y, z)$ satisfy the hypotheses of Lemmas A and B with estimates uniform in both x and y.

Finally, it is interesting to note that Thangavelu [Th 1], [Th 2] recently obtained the strong type (p, p) result for the Hilbert transform defined by (1.2) by transference techniques and conjectured that the weak-type $(1, 1)$ result was probably valid. Our results imply the weak-type $(1, 1)$ result and are obtained by entirely different and perhaps more straightforward methods. Throughout this paper C will denote a constant which may vary from line to line and $\|f\|_p$ will denote the L^p norm of f with respect to ordinary Lebesgue measure on $\mathbf{R}$.

2. Heat Diffusion and Poisson Integrals

We begin this section by recalling the following result of Markett [Ma]:

LEMMA 2.1 (Markett). *Let $\{h_n\}$ denote the sequence of normalized Hermite functions. Then*

$$\|h_n\|_p \sim \begin{cases} n^{\frac{1}{2p}-\frac{1}{4}}, & 1 \le p < 4, \\ n^{-\frac{1}{8}}(\log n)^{1/4}, & p = 4, \\ n^{-\frac{1}{6p}-\frac{1}{12}}, & 4 < p \le \infty. \end{cases}$$

In particular, we $\|h_n\|_p \le Cn^{\varepsilon(p)}$. Let $f \in L^p(\mathbf{R})$, $1 \le p \le \infty$, and let $\sum_{n=0}^{\infty} a_n h_n(x)$ denote the expansion of f with respect to the Hermite functions. Then if $1/p + 1/q = 1$,

$$(2.1) \qquad |a_n| = \left|\int_{-\infty}^{\infty} f(t)h_n(t)\,dt\right| \le \|f\|_p\|h_n\|_q \le Cn^{\varepsilon(q)}.$$

We define the heat-diffusion integral of f by

$$(2.2) \qquad g(x, y) = \sum_{n=0}^{\infty} e^{-(2n+1)x} a_n h_n(y), \qquad x > 0.$$

In view of (2.1) if follows that for any fixed $x > 0$, the series in (2.2) converges uniformly in y. We obtain an integral form of $g(x, y)$ by writing

$$\begin{aligned} g(x, y) &= \sum_{n=0}^{\infty} e^{-(2n+1)x}\left(\int_{-\infty}^{\infty} f(z)h_n(z)\,dz\right)h_n(y) \\ &= \int_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty} e^{-(2n+1)x}h_n(y)h_n(z)\right)f(z)\,dz \\ &\equiv \int_{-\infty}^{\infty} P(x, y, z)f(z)\,dz. \end{aligned}$$

Interchanging the order of summation and integration is justified by Lebesgue's dominated convergence theorem since

$$\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} e^{-(2n+1)x}|h_n(y)h_n(z)f(z)|\,dz \le \sum_{n=0}^{\infty} e^{-2(n+1)x}\|h_n\|_\infty\|h_n\|_q\|f\|_p$$

and both $\|h_n\|_\infty$ and $\|h_n\|_q$ grow polynomially in n by Lemma 2.1.

A preliminary result concerning the heat-diffusion integral is contained in the following

LEMMA 2.2. *For $f \in L^p(\mathbf{R})$, $1 \le p \le \infty$, the heat-diffusion integral $g(x, y)$ is a C^∞ function on $\mathbf{R}^+ \times \mathbf{R}$ satisfying the differential equation*

$$\left(L_y - \frac{\partial}{\partial x}\right)g = 0, \tag{2.3}$$

where

$$L_y = \frac{\partial^2}{\partial y^2} - y^2.$$

Proof. The estimate from Lemma 2.1 with $p = \infty$ justifies differentiation of the series (2.2) with respect to x. Thus we have

$$\frac{\partial^k}{\partial x^k} g(x, y) = \sum_{n=0}^{\infty} (-1)^k(2n+1)^k e^{-(2n+1)x} a_n h_n(y). \tag{2.4}$$

Since $h'_n(y) = (2n)^{1/2}h_{n-1}(y) - yh_n(y)$, we have $|h'_n(y)| \le Cn^{\varepsilon(A)}$ uniformly for $y \in [-A, A]$, $A > 0$. Thus for x fixed, the series (2.4) can be differentiated termwise with respect to y. A similar argument holds for higher derivatives and thus $g(x, y)$ is C^∞ on $\mathbf{R}^+ \times \mathbf{R}$. Differentiating term by term shows that $g(x, y)$ satisfies (2.3) and the lemma is proved.

Instead of working with the kernel $P(x, y, z)$, it is more convenient to introduce the kernel

$$U(r, y, z) = \sum_{n=0}^{\infty} r^n h_n(y)h_n(z), \qquad 0 \le r < 1. \tag{2.5}$$

Then $P(x, y, z) = e^{-x}U(e^{-2x}, y, z)$ and estimates obtained with the kernel

U can easily be converted into estimates for $g(x, y)$. By Mehler's formula [Sz, p. 380] we have

$$\text{(2.6)} \qquad U(r, y, z) = \frac{1}{\pi^{1/2}(1-r^2)^{1/2}} \times \exp\left(-\frac{1}{2}\left(\frac{1+r^2}{1-r^2}\right)(y^2+z^2) + \frac{2ryz}{1-r^2}\right)$$

and

$$\text{(2.7)} \qquad \int_{-\infty}^{\infty} U(r, y, z)\, dz = \left(\frac{2}{1+r^2}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{1-r^2}{1+r^2}\right)y^2\right).$$

To obtain (2.7) from (2.6), one completes the square and makes the appropriate change of variables. Let $L(r, y, z)$ be defined by

$$U\left(r, y, \frac{2r}{1+r^2}y\right)$$

for z in the interval with endpoints $(2r/(1+r^2))y$ and y and by $U(r, y, z)$ outside this interval. The main estimates for the heat-diffusion integral are readily obtained from the following

LEMMA 2.3. *For $0 \le r < 1$, $U(r, y, z) \le L(r, y, z)$, the function $L(r, y, z)$ as a function of z is increasing on $(-\infty, y]$ and decreasing on $[y, \infty)$, and satisfies*

$$\text{(2.8)} \qquad \int_{-\infty}^{\infty} L(r, y, z)\, dz \le C,$$

where C is independent of r and y.

Proof. We consider only the case $y > 0$, and note that the case $y < 0$ can be treated in a similar manner. The function $f(z) = \exp(-az^2 + bz)$ is increasing on $(-\infty, b/2a]$ and decreasing on $[b/2a, \infty)$. From this fact the monotonicity of $L(r, y, z)$ readily follows. To establish (2.8) it suffices to prove

$$\text{(2.9)} \qquad \left(y - \frac{2ry}{1+r^2}\right)U\left(r, y, \frac{2ry}{1-r^2}\right) \le C.$$

A simple calculation shows that the left hand side of (2.9) equals

$$\frac{1-r}{(1+r)(1+r^2)^{1/2}}\cdot y\left(\frac{1-r^2}{1+r^2}\right)^{1/2}\exp\left(-\frac{1}{2}y^2\left(\frac{1-r^2}{1+r^2}\right)\right)$$

and since $u\exp(-u^2/2)$ is bounded on $[0,\infty)$, the lemma is proved.

The main results of this section are contained in the following

THEOREM 2.4. *Let $f\in L^p(\mathbf{R})$, $1\le p<\infty$, and let $g(x,y)$ denote the heat-diffusion integral defined by* (2.2). *Then*

(a) $|g(x,y)|\le Ce^{-x}f^*(y)$,
(b) $\|g(x,\cdot)\|_p\le(\cosh 2x)^{-1/2}\|f\|_p$, $1\le p\le\infty$,
(c) $\|g(x,\cdot)-f(\cdot)\|_p\to 0$ *as* $x\to 0$, $1\le p<\infty$,
(d) $g(x,y)\to f(y)$ *a.e. as* $x\to 0$, $1\le p<\infty$.

Proof. (a) We have

$$\begin{aligned}|g(x,y)|&=\left|\int_{-\infty}^{\infty}P(x,y,z)f(z)\,dz\right|\\&=e^{-x}\left|\int_{-\infty}^{\infty}U(e^{-2x},y,z)f(z)\,dz\right|\\&\le e^{-x}\int_{-\infty}^{\infty}L(e^{-2x},y,z)|f(z)|\,dz\\&\le Ce^{-x}f^*(y)\end{aligned}$$

by Lemma A and Lemma 2.3.

(b) For $1\le p<\infty$, using $P(x,y,z)=e^{-x}U(e^{-2x},y,z)$ and (2.7), we obtain

$$\int_{-\infty}^{\infty}P(x,y,z)\,dz=(\cosh 2x)^{1/2}\exp\left(-\frac{1}{2}\tanh 2x\right). \tag{2.10}$$

By Hölder's inequality and (2.10) it follows that

$$|g(x,y)|^p\le(\cosh 2x)^{-p/2q}\int_{-\infty}^{\infty}|f(z)|^pP(x,y,z)\,dz.$$

Integrating with respect to y and using Fubini's theorem yields

$$\begin{aligned}\|g\|_p^p&\le(\cosh 2x)^{-p/2q}(\cosh 2x)^{-1/2}\|f\|_p^p\\&=(\cosh 2x)^{-p/2}\|f\|_p^p.\end{aligned}$$

Thus (b) is proved when $p < \infty$ and is obvious when $p = \infty$. To prove (c) and (d) we use standard arguments and the fact that the space of polynomials multiplied by $e^{-x^2/2}$ is dense in $L^p(\mathbf{R})$ (see [Mu 2]). This completes the proof of the theorem.

We conclude this section by defining the Poisson integrals. Let $f \in L^p(\mathbf{R})$, $1 \leq p \leq \infty$, and let $\sum_{n=0}^{\infty} a_n h_n$ denote the expansion of f in Hermite functions. The Poisson integral $f(x, y)$, $x > 0$, is defined by

$$f(x, y) = \sum_{n=0}^{\infty} e^{-(2n+1)^{1/2} x} a_n h_n(y). \tag{2.11}$$

The series in (2.11) converges uniformly in y for x fixed. An integral representation for $f(x, y)$ may be obtained by using the subordination formula (cf. [Go, p. 78])

$$e^{-\beta} = \frac{\beta}{\sqrt{4\pi}} \int_0^{\infty} s^{-3/2} e^{-s} e^{-\beta^2/4s}\, ds. \tag{2.12}$$

We have

(2.13)
$$\begin{aligned}
f(x, y) &= \sum_{n=0}^{\infty} e^{-(2n+1)^{1/2} x} \left(\int_{-\infty}^{\infty} f(z) h_n(z)\, dz \right) h_n(y) \\
&= \frac{x}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(z) \int_0^{\infty} \left(\sum_{n=0}^{\infty} e^{-(2n+1)s} h_n(y) h_n(z) \right) s^{-3/2} e^{-x^2/4s}\, ds\, dz \\
&= \frac{x}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(z) \int_0^{\infty} P(s, y, z) s^{-3/2} e^{-x^2/4s}\, ds\, dz \\
&= \int_{-\infty}^{\infty} f(z) R(x, y, z)\, dz
\end{aligned}$$

where

$$R(x, y, z) = \frac{x}{\sqrt{4\pi}} \int_0^{\infty} P(s, y, z) s^{-3/2} e^{-x^2/4s}\, ds \tag{2.14}$$

is the Poisson kernel associated with the Hermite functions $\{h_n\}$. As before, interchanging integration and summation is justified by the dominated convergence theorem. The same arguments used for the heat-diffusion integral imply that $f(x, y)$ is C^∞ on $\mathbf{R}^+ \times \mathbf{R}$ and satisfies

$$\left(L_y + \frac{\partial^2}{\partial x^2}\right) = 0$$

Where L_y is the Hermite operator. Finally, we note that the results in Theorem 2.4 remain valid for the Poisson integral providing the factors e^{-x} and $(\cosh 2x)^{-1/2}$ are dropped in (a) and (b). In fact from (2.13) it follows that

$$f(x, y) = \frac{x}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-x^2/4s} g(s, y)\, ds. \tag{2.15}$$

The point-wise estimate for $g(s, y)$, the identity

$$\frac{x}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-x^2/4s}\, ds = 1,$$

and Minkowski's integral inequality imply (a) and (b). The results in (c) and (d) for the Poisson integral follow by standard arguments (cf. [Mu 1]).

Remark 2.5. It is interesting to note here that $P(x, y, z)$, the heat kernel associated to the Hermite operator, may be majorized point-wise by the Gauss-Weierstrass kernel on $\mathbf{R}$, i.e. the heat kernel associated to the one-dimensional Laplacean. This is in accordance to a more general principle that takes place for Schrödinger operators and the Feynman-Kac formula is used to prove it. We are grateful to Andrzej Hulanicki to whom we owe this information. Specifically we have

$$P(x, y, z) \le W_x(y - z) \tag{2.16}$$

where $W_x(y) = (4\pi x)^{-1/2} \exp(-y^2/4x)$. This clearly gives another proof of Theorem 2.4, conceptually equivalent to our previous argument, except for the fact that our more careful analysis also gives the exponential factors in (a) and (b). To check (2.16) we note that

$$P(x, y, z) = (2\pi)^{-1/2} (\sinh(2x))^{-1/2} \exp(-\varphi_x(y, z))$$

where

$$\varphi_x(y, z) = \frac{1}{2}(y - z)^2 \coth 2x + yz \tanh x$$

and thus, since $\sinh t > t$ and $\coth t > 1/t$, for $yz \ge 0$ (2.16) is obvious.

Assuming now $yz < 0$ it suffices to check that

$$\psi_x(y,) \geq (y-z)^2/4x$$

and, by a homogeneity argument, this is reduced to the inequality

$$A \coth 2x - \tanh x \geq A/2x$$

which, for $A \geq 2$, is clearly valid for all $x > 0$.

3. Conjugate Poisson Integrals

Let $f \sim \sum_{n=0}^{\infty} a_n h_n$. Recall that $\tilde{f}$, the Hilbert transform of f, is formally defined by

$$\tilde{f} \sim \sum_{n=0}^{\infty} a_n \left(\frac{2n}{2n+1} \right)^{1/2} h_n.$$

Let $f(x, y)$ be the Poisson integral given by (2.11). We define the conjugate Poisson integral by

$$\tilde{f}(x,y) = \sum_{n=0}^{\infty} a_n \left(\frac{2n}{2n+1} \right)^{1/2} e^{-(2n+1)^{1/2}x} h_{n-1}(y). \tag{3.1}$$

The same arguments used for the heat-diffusion integral show that $\tilde{f}(x, y)$ is C^∞ on $\mathbf{R}^+ \times \mathbf{R}$ and satisfies

$$\left(L_y + \frac{\partial^2}{\partial x^2} \right) \tilde{f} = 2\tilde{f}.$$

where L is the Hermite operator. The fact that $h_n'(y) + y h_n(y) = (2n)^{1/2} h_{n-1}(y)$ immediately shows that $f(x, y)$ and $\tilde{f}(x, y)$ are related by the 'Cauchy-Riemann' equation

$$\frac{\partial f}{\partial y} + yf = -\frac{\partial \tilde{f}}{\partial x}. \tag{3.2}$$

We now use (3.2) to find an integral formula for $\tilde{f}(x, y)$. Using the subordination formula (2.12), taking $\beta = (2n+1)^{1/2}x$, making the change of variables $s \to (2n+1)s$, and then substituting $r = e^{-2s}$ leads to the formula

$$\text{(3.3)} \qquad e^{-(2n+1)^{1/2}x} = \int_0^1 \frac{x \exp\left(\dfrac{x^2}{2\log r}\right)}{(2\pi)^{1/2} r(-\log r)^{3/2}} r^{n+1/2}\,dr$$
$$\equiv \int_0^1 T(x,r) r^{n+1/2}\,dr.$$

Then if $R(x,y,z)$ denotes the Poisson kernel (2.14), we have

$$\begin{aligned} R(x,y,z) &= \sum_{n=0}^{\infty} e^{-(2n+1)^{1/2}x} h_n(y) h_n(z) \\ &= \sum_{n=0}^{\infty} h_n(y) h_n(z) \int_0^1 T(x,r) r^{n+1/2}\,dr \\ &= \int_0^1 T(x,r) \left(\sum_{n=0}^{\infty} r^n h_n(y) h_n(z) \right) r^{1/2}\,dr \\ &= \int_0^1 T(x,r) U(r,y,z) r^{1/2}\,dr. \end{aligned}$$

Combining this and (2.6) we obtain

$$\text{(3.4)} \quad \frac{\partial R}{\partial y} + yR = \exp\left(-\frac{1}{2}(y^2+z^2)\right) \int_0^1 \frac{2^{1/2}(z-ry)x \exp\left(\dfrac{x^2}{2\log r}\right)}{\pi(-\log r)^{3/2}(1-r^2)^{3/2}}$$
$$\times \exp\left(\frac{-r^2y^2 + 2ryz - r^2z^2}{1-r^2}\right) r^{1/2}\,dr.$$

Now

$$\text{(3.5)} \qquad \frac{\partial f}{\partial y} + yf = \int_{-\infty}^{\infty} \left(\frac{\partial R}{\partial y} + yR\right)(x,y,z) f(z)\,dz.$$

From (2.1) it is easy to check that $\tilde{f}(x,y) \to 0$ as $x \to \infty$ and so

$$\tilde{f}(x,y) = -\int_x^{\infty} \frac{\partial \tilde{f}}{\partial x}(t,y)\,dt.$$

Using (3.4), (3.5), and the Cauchy-Riemann equation (3.2) we find after integrating with respect to x and changing the order of integration

$$\tilde{f}(x, y) = \int_{-\infty}^{\infty} Q(x, y, z) f(z)\, dz$$

where

$$Q(x, y, z) = \exp\left(-\frac{1}{2}(y^2 + z^2)\right) Q_1(x, y, z) \tag{3.6}$$

and

$$Q_1(x, y, z) = \int_0^1 \frac{z - ry}{(1 - r^2)^2} \exp\left(\frac{-r^2y^2 + 2ry - r^2z^2}{1 - r^2}\right) w_1(x, r)\, dr \tag{3.7}$$

with

$$w_1(x, y) = \frac{2^{1/2}}{\pi} \left(\frac{1 - r^2}{-\log r}\right)^{1/2} \exp\left(\frac{x^2}{2 \log r}\right) r^{1/2}.$$

Interchanging the order of integration is justified since

$$\int_x^{\infty} t \exp\left(\frac{t^2}{2 \log r}\right) dt$$

merely brings out the factor $-\log r$. We call $Q(x, y, z)$ the conjugate Poisson kernel associated with the Hermite functions $\{h_n\}$. We note that $Q(x, y, z)$ differs from the corresponding kernel associated with the Hermite polynomials (see [Mu 2]) by the exponential factor $\exp(-(y^2 + z^2)/2)$ and the additional factor of $r^{1/2}$ inside the integral. Our goal is to show that the same techniques used by Muckenhoupt [Mu 1], [Mu 2] remain effective with these modifications. In many cases, the estimates from [Mu 2] can be applied directly, but in a few cases the argument is more delicate and the explicit form of our kernel must be taken into account.

As in [Mu 2] the general program is to write $Q(x, y, z) = J(x, y, z) + K(x, y, z)$ with $|J(x, y, z)| \le L(y, z)$ where L satisfies the conditions of Lemma A and with $K(x, y, z)$ being an odd function of z about y and satisfying the conditions of Lemma B uniformly in x and y. Specifically, we define $J(x, y, z)$ by

(3.8)

$$J(x, y, z) = \begin{cases} \frac{1}{2}(Q(x, y, z) + Q(x, y, 2y - z)), & |y - z| < \min(1, 1/|y|), \\ Q(x, y, z), & |y - z| \ge \min(1, 1/|y|), \end{cases}$$

and $K(x, y, z)$ by

(3.9)
$$K(x,y,z) = \begin{cases} \frac{1}{2}(Q(x,y,z) - Q(x,y,2y-z)), & |y-z| < \min(1, 1/|y|), \\ 0, & |y-z| \geq \min(1,1/|y|). \end{cases}$$

The function $L(x, y)$ is defined for $y \geq 2$ by

(3.10)
$$L(y,z) = \begin{cases} \exp(-z^2/2)/y\exp(y^2/2), & z \leq 0, \\ \exp(z^2/2)/y\exp(y^2/2), & 0 < z \leq y/2, \\ \exp(z^2/2)\left(y^{-1} + \left(y(y-z)^3\right)^{-1/2}\right)/\exp(y^2/2), & \\ & y/2 < z \leq y - 1/y, \\ y\exp(z^2/2)(1 - \ln[y(y-z)]/\exp(y^2/2), & \\ & y - 1/y < z < y, \\ y(1 - \ln[y(z-y)]), & y < z \leq y + 1/y, \\ y\exp(-z^2/2)\exp(y^2/2), & y + 1/y < z, \end{cases}$$

and for $0 \leq y \leq 2$ by

$$(3.11)\quad L(y,z) = \begin{cases} y(1 - \ln[y|y-z|]) + 1, & 0 < |y-z| \leq 1, \\ 1, & 1 < |y-z| \leq 2, \\ \exp(-z^2/2), & 2 < |y-z|. \end{cases}$$

A careful inspection of (3.10) and (3.11) shows that $L(y, z)$ as a function of z is increasing on $(-\infty, y)$ and decreasing on (y, ∞). A straightforward calculation also shows that

$$(3.12)\qquad \int_{-\infty}^{\infty} L(x,y)\,dz \leq C$$

where C is independent of $y \geq 0$. Since $Q(x, y, z) = -Q(x, -y, -z)$, it suffices to consider only the case $y \geq 0$.

The following lemma is crucial and the next three sections are devoted to its proof.

LEMMA 3.1. *Let $Q(x, y, z) = J(x, y, z) + K(x, y, z)$ where J and K are given by* (3.8) *and* (3.9). *Then $|J(x, y, z)| \leq CL(|y|, z)$ where L is defined by* (3.10) *and* (3.11) *and C is a fixed constant. Moreover the total variation of the* *function $h \to hK(x, y, y + h)$, (defined to be 0 for $h = 0$), is bounded by a constant independent of $x > 0$ and $y \in \mathbf{R}$.*

4. The estimate for $J(x, y, z)$ when $|y - z| < \min(1, 1/|y|)$

In this section, we need not distinguish between the cases $0 \leq y \leq 2$ and $y > 2$. When $|y - z| < \min(1, /|y|)$, from (3.8) we have

$$2J(x, y, z) = \exp\left(\frac{1}{2}(z^2 - y^2)\right)\Big[Q_1(x, y, z)\exp(-z^2) + Q_1(x, y, 2y - z) \times \exp\left(-(2y - z)^2\right)\exp(2y(y - z))\Big].$$

Applying (3.7) and noticing that $|w_1(x, r)| \leq 1$, we estimate $2|J(x, y, z)|$ by

$$\exp\left(\frac{1}{2}(z^2 - y^2)\right)\int_0^1 \left| \frac{z - ry}{(1 - r^2)^2}\exp\left(-\frac{(z - ry)^2}{1 - r^2}\right) + \frac{y(1 - r) + (y - z)}{(1 - r^2)^2}\exp\left(-\frac{y(1 - r) + (y - z)^2}{(1 - r^2)^2}\right)e^{2y(y-2)}\right| dr.$$

Adding and subtracting

$$\frac{y(1 - r) + (y - z)}{(1 - r^2)^2}\exp\left(-\frac{(z - ry)^2}{1 - r^2}\right)$$

inside the integral above and using the triangle inequality allows us to majorize $2|J(x, y, z)|$ by the sum of

$$Cy\exp\left(\frac{1}{2}(z^2 - y^2)\right)\int_0^1 \exp\left(-\frac{(z - ry)^2}{1 - r^2}\right)\frac{dr}{1 - r} \tag{4.1}$$

and

$$C\exp\left(\frac{1}{2}(z^2 - y^2)\right)\int_0^1 \frac{y(1 - r) + |y - z|}{(1 - r)^2}\exp\left(-\frac{(z - ry)^2}{1 - r^2}\right) \times \left|\exp\left(2y(z - y)\frac{1 - r}{1 + r}\right) - 1\right| dr. \tag{4.2}$$

Using Lemma 5, (4.1) from [Mu 2] then shows that (4.1) is bounded by

$$Cy\exp(z^2/2)(1-\ln[y|y-z|])/\exp(y^2/2). \tag{4.3}$$

Since $y|y-z|<1$, the mean value theorem implies

$$\left|\exp\left(2y(z-y)\frac{1-r}{1+r}\right)-1\right|\le Cy|z-y|.$$

Another application of Lemma 5 [Mu 2] now shows that (4.2) is bounded by (4.3). This is the required estimate in the cases $0\le y\le 2$ and $y>2$ with $y-1/y<z<y$. The estimate for the remaining interval $y<z<y+1/y$ also follows if we note that $\exp(1/2(z^2-y^2))\le C$. This completes the proof of the estimate $|J(x,y,z)|\le CL(|y|,z)$ for $|y-z|<\min(1,1/|y|)$.

5. Estimates for J on intervals away from y

With the exception of the case $1/2y<z\le y-1/y$ Muckenhoupt obtains his estimates by passing the absolute value inside the integral and making the estimate $|w(x,r)|\le 1$. In our case

$$Q(x,y,z)=\exp\left(-\frac{1}{2}(y^2+z^2)\right)Q_1(x,y,z)$$

where $Q_1(x,y,z)$ is Muckenhoupt's kernel with $w(x,r)$ replaced by $w_1(x,r) = w(x,r)r^{1/2}$. Since $|w(x,r)|\le 1$ implies $|w_1(x,r)|\le 1$ for $0<r<1$, the same estimates which are made for Muckenhoupt's kernel remain valid for $Q_1(x,y,z)$. When these estimates are multiplied by $\exp(-1/2(y^2+z^2))$, precisely the estimates of (3.10) and (3.11) are obtained.

For the interval $1/2y<z\le y-1/y$, we must integrate by parts and do a slightly more careful analysis. Following Muckenhoupt we let

$$S(r,y,z)=\frac{z-ry}{(1-r^2)^{3/2}}\exp\left(-\frac{r^2y^2+2ryz-r^2z^2}{1-r^2}\right).$$

We then write

$$\begin{aligned}Q(x,y,z)&=\exp\left(-\frac{1}{2}(y^2+z^2)\right)Q_1(x,y,z)\\&=\exp\left(-\frac{1}{2}(y^2+z^2)\right)\\&\quad\times\left\{Q_1(x,y,z)-\int_E\frac{w_1(x,r)}{(1-r^2)^{1/2}}\right.\\&\quad\left.\times S(r,y,z)\,dr+\int_E\frac{w_1(x,r)}{(1-r^2)^{1/2}}S(r,y,z)\,dr\right\}\end{aligned}$$

where

$$E = \left[1 - \frac{3(y-z)}{2y}, 1 - \frac{(y-z)}{2y}\right].$$

The estimates for

$$(5.1) \quad \exp\left(-\frac{1}{2}(y^2+z^2)\right)\left|Q_1(x,y,z) - \int_E \frac{w_1(x,r)}{(1-r^2)^{1/2}} S(r,y,z)\,dr\right|$$

and

$$(5.2) \qquad \exp\left(-\frac{1}{2}(y^2+z^2)\right)\left|\int_E \frac{w_1(x,r)}{(1-r^2)^{1/2}} S(r,y,z)\,dr\right|$$

are established separately. For (5.1) the difference inside the absolute value signs is estimated precisely as in [Mu 2] (see 5.4, [Mu 2]). It is written as the sum of two integrals over disjoint intervals, the absolute value is moved inside each integral, and then the estimate $|w_1(x,r)| \le 1$ is used. When the estimate (5.4) of [Mu 2] is multiplied by $\exp(-1/2(y^2+z^2))$, we obtain precisely the estimate in (3.12) corresponding to $1/2 < z \le y - 1/y$. To obtain the estimate for (5.2) we integrate by parts. For the boundary terms we obtain the same estimate as on the right had side of (5.5) in [Mu 2] since, once again, the fact that $|w_1(x,r)| \le 1$ is used. When this estimate is multiplied by the exponential term, the desired estimate is obtained. The term that remains to be estimated is

$$(5.3) \qquad \left|\int_E \left[\frac{d}{dr}\left(\frac{w_1(x,r)}{(1-r^2)^{1/2}}\right)\right]\int_r^1 S(t,y,z)\,dt\,dr\right|.$$

Recall that $w_1(x,r) = w(x,r)r^{1/2}$ where $w(x,r)$ is the function in [Mu 2]. The integral in (5.3) is majorized by

$$(5.4) \qquad \left|\int_E \left[\frac{d}{dr}\left(\frac{w(x,r)}{(1-r^2)^{1/2}}\right)r^{1/2}\int_r^1 S(t,y,z)\,dt\,dr\right|\right.$$
$$+\left|\int_E \frac{w(x,r)}{(1-r^2)^{1/2}}\frac{1}{2r^{1/2}}\int_r^1 S(t,y,z)\,dt\,dr\right|.$$

For the first integral in (5.4) the absolute value signs are put inside the integral, $r^{1/2}$ is replaced by 1, and the same estimates as in [Mu 2] are applied. For the second integral in (5.4), we note that $r \in E$ implies $r \geq 1/4$ and hence the term $1/2r^{1/2}$ is uniformly bounded. Now for $r \in E$, we have

$$\left|\frac{w(x,r)}{(1-r^2)^{1/2}}\right| \leq \frac{1}{(1-r)^{3/2}} \leq C\left(\frac{y}{y-z}\right)^{3/2}. \tag{5.5}$$

For the last inequality in (5.5) see (5.13) of [Mu 2]. For the second integral in (5.4), after putting the absolute value signs inside the integral, using the uniform bound for $1/2r^{1/2}$, the estimate (5.5), and the estimate in (5.12) [Mu 2] for $|\int_r^1 S(t,y,z)\,dt|$ when $r \in E$ to obtain

$$\begin{aligned}
&\left|\int_E \frac{w(x,y)}{(1-r^2)^{1/2}} \frac{1}{2r^{1/2}} \int_r^1 S(t,y,z)\,dt\,dr\right| \\
&\qquad \leq C\int_E \frac{y\exp(z^2)}{(y-z)^2} \exp\left(-\frac{y(z-ry)^2}{3(y-z)}\right) dr.
\end{aligned}$$

A substitution shows that this last integral is majorized by

$$\frac{C\exp(z^2)}{\left(y(y-z)^3\right)^{1/2}}.$$

Finally, multiplying by the exponential factor $\exp(-1/2(y^2+z^2))$ produces the required estimate in (3.12). This completes the estimate for the interval $y/2 < z < y - 1/y$.

6. Estimates for the total variation of $K(x,y,z)$

In order to apply Lemma B to the kernel $K(x,y,z)$ given by (3.9) we need to check that the total variation of the function $h \to hK(x,y,y+h)$ is bounded uniformly in $x, y \geq 0$. Following the argument from [Mu 2, §6], it is sufficient to prove that

$$\int_{|z-y|<m} \left|\frac{\partial}{\partial z}[(y-z)Q(x,y,z)\right| dz, \; m = \min(1, 1/y), \tag{6.1}$$

is bounded independently of $x, y \geq 0$. First we observe that Lemma 6 in [Mu 2, p. 250] remains valid for

$$p_1(x,r) = \frac{\partial w_1}{\partial r}(x,r)$$

since $w_1(x, r) = w(x, r)r^{1/2}$. That means

$$(6.2) \qquad \int_0^1 |p_1(x, r)|\, dr \le C.$$

From the definition of $Q(x, y, z)$ we have

$$(6.3) \quad (y - z)Q(x, y, z) = \int_0^1 p_1(x, t) \int_t^1 \frac{(y - z)(z - ry)}{(1 - r^2)^2} \times \exp\left(-\frac{1}{2}\left(\frac{1 + r^2}{1 - r^2}\right)(y^2 + z^2) + \frac{2ryz}{1 - r^2}\right) dr\, dt.$$

Differentiating (6.3) with respect to z shows that the integrand in (6.1) may be written in the form

$$(6.4) \quad \exp\left(\frac{1}{2}(z^2 - y^2)\right)\left|\int_0^1 p_1(x, t) \int_t^1 B(r, y, z) \exp\left(-\frac{(ry - z)^2}{1 - r^2}\right) dr\, dt\right|$$

where

$$(6.5) \quad B(r, y, z) = \frac{y(1 + r) - 2z}{(1 - r^2)^2} - \frac{2(z - ry)^2(y - z)}{(1 - r^2)^3} + \frac{z(y - z)(z - ry)}{(1 - r^2)^2}.$$

The condition $|z - y| < m$ implies $|z^2 - y^2| \le C$ and therefore the exponential factor before the integral in (6.4) can be ignored. Moreover the first two summands in (6.5) give rise to identical integrals as treated by Muckenhoupt (c.f. (6.2), [Mu 2]) with $p(x, t)$ replaced by $p_1(x, t)$. In view of (6.2) above, it follows that the integrals with these summands are bounded. We now consider the last summand in (6.5) and are reduced to estimating

$$(6.6) \quad |z|\left|\int_0^1 p_1(x, t)\left[\int_t^1 \frac{(y - z)(z - ry)}{(1 - r^2)^2} \exp\left(-\frac{(ry - z)^2}{1 - r^2}\right) dr\right] dt\right|.$$

Since $z - ry = (z - y) + y(1 - r)$, the inner integral in (6.6) is majorized by

$$(6.7) \quad (y - z)^2 \int_0^1 \frac{1}{(1 - r^2)^2} \exp\left(-\frac{(ry - z)^2}{1 - r^2}\right) dr + y|y - z| \int_0^1 \frac{1}{1 - r} \exp\left(-\frac{(ry - z)^2}{1 - r^2}\right) dr.$$

Applying Lemma 5 from [Mu 2] it follows that (6.7) is bounded by

$$C[1 + y|y - z|(1 - \log(y|y - z|))]. \tag{6.8}$$

For $y > 2$ we note that $y|y - z| \leq 1$ and that (6.6) is majorized by

$$C(1 + y)[1 - \log(y|y - z|)]. \tag{6.9}$$

It is easy to check that for $y > 2$

$$C(1 + y)\int_{|z-y|<m} (1 - \log(y|y - z|))\,dz$$

is bounded uniformly in y. For $y \leq 2$ we have $|y - z| \leq 1$ and $|z| \leq 3$. Thus in this case (6.6) is majorized by

$$C[1 - y\log(y|y - z|)].$$

and again it is easy to check that for $y \leq 2$

$$y\int_{|z-y|<m} (1 - \log(y|y - z|))\,dz$$

is bounded uniformly in y. This completes the estimates for the integral with the last summand of $B(r, y, z)$. This also establishes the uniform boundedness of (6.1) and completes the proof of Lemma 3.1.

7. Main results

The main results of this paper concerned with the conjugate Poisson integral $\tilde{f}(x, y)$ and its boundary value $\tilde{f}(y)$ for $1 < p < \infty$ are summarized in the following

THEOREM 7.1. *Let $f \in L^p(\mathbf{R})$, $1 < p < \infty$, and let $f \sim \Sigma a_n h_n$ where $\{h_n\}$ is the sequence of Hermite functions. Then*

a) $\|\tilde{f}(x, y)\|_p \leq A_p\|f\|_p$,

b) *$\tilde{f}(x, y)$ has an L_p limit as $x \to 0^+$ denoted by $\tilde{f}(y)$. Moreover*

$$\|\tilde{f}(x, y) - \tilde{f}(y)\|_p \to 0 \text{ as } x \to 0^+,$$

c) $\|\sup_{x>0}|\tilde{f}(x, y)|\,\|_p \leq A_p\|f\|_p$,

d) *$\tilde{f}(x, y) \to \tilde{f}(y)$ a.e. (y) as $x \to 0^+$,*

e) *$\tilde{f}(y)$ has the expansion $\Sigma_{n=0}^{\infty} a_n(2n/(2n + 1))^{1/2} h_n$.*

In the case $p = 1$, we have

THEOREM 7.2. *Let $f \in L^1(\mathbf{R})$ and let $\Sigma a_n h_n$ denote the expansion of f with respect to the Hermite functions. Then*

a) $|\{y : \sup_{x>0}|\tilde{f}(x, y)| > \lambda\}| \leq C\lambda^{-1}\|f\|_1$,

b) $\lim_{x\to 0^+} \tilde{f}(x, y)$ *exists a.e.* (y) *and is denoted by* $\tilde{f}(y)$,

c) $\{y: |\tilde{f}(y) > \lambda\} \leq C\lambda^{-1}\|f\|_1$.

For the proof we note that Lemma 3.1 allows us to use Lemmas A and B. Since the Hardy-Littlewood maximal function and maximal Hilbert transform are of strong type (p, p) and of weak type $(1, 1)$, the results for $\tilde{f}(x, y)$ following immediately. The existence of the boundary value $\tilde{f}(y)$ follows from the estimate for $\tilde{f}(x, y)$ and the fact that the space of polynomials multiplied by $\exp(-x^2/2)$ are dense in L^p, $1 \leq p < \infty$.

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