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DEFINING FRACTALS IN A PROBABILITY SPACE

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1. Introduction

In [1] Billingsley defined Hausdorff dimension for subsets of a probability space, and developed relations with entropy and information theory. In [2] he explored the effect of varying the probability measure in the definition. In revisiting his work we will sharpen his density theorems so that they give results for measures as well as dimension (Lemmas 5.1 and 5.2).

In Euclidean space $\mathbf{R}^d$ there is no generally accepted definition of a fractal, even though fractal sets are widely used as models for many physical phenomena. The idea behind these models is that of self-similarity or affineness which is based on the linear structure of $\mathbf{R}^d$. These and other geometrical notions have no obvious meaning in an abstract probability space. One of us [11] proposed a measure-theoretic definition for subsets $E \subset \mathbf{R}^d$: E should be called a fractal if

$$\dim(E) = \operatorname{Dim}(E), \tag{1.1}$$

where $\dim(E)$ is the familiar Hausdorff dimension and $\operatorname{Dim}(E)$ denotes packing dimension as defined in [10], using efficient packing by disjoint balls with center in E.

The first objective of this paper is to define packing measure and dimension in a probability space $(\Omega, \mathscr{F}, \mu)$. This is done in two stages: In Section 3 we produce a premeasure and obtain the analogue $\Delta(E)$ of the upper Minkowski index in Ω, while Section 4 completes the definition of packing measure and dimension. We can then use (1.1) as the definition of a fractal set in $(\Omega, \mathscr{F}, \mu)$ with respect to μ. In Section 5 we complete the development of density arguments relevant to both Hausdorff and packing measures and use these in Section 6 to provide (Corollary 6.4) a useful criterion for $A \subset \Omega$ to be a fractal of dimension c. In Section 7 we use ideas of Cutler [4] to analyse a measure ν on $(\Omega, \mathscr{F})$ with respect to μ. This leads to the analogue for probability spaces of the results obtained in [12] for $\mathbf{R}^d$.

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The Billingsley formulation (which we use) depends critically on a fixed stochastic process $\{X_1, X_2, \ldots\}$ taking values in a finite or countable state space S. No information about a subset $E \subset \Omega$ has any relevance unless it depends on the values $\{X_n(\omega)\}$. This means that we would lose nothing by assuming that Ω is an infinite product of a sequence of copies of S, which would make a typical point $\omega = (x_1, x_2, \ldots, x_n, \ldots)$, where each $x_n \in S$. This idea is developed and used extensively by Cajar [3]. We keep it in mind and note that it is the reason why so many of our arguments are simpler than in the Euclidean case. However, we find it more intuitive to build on the original formulation and notation of Billingsley.

In the final Section 8, we specialise Ω to the unit interval $[0,1]$ with Lebesgue measure for μ. Whenever S is a finite set of s elements $0, 1, 2, \ldots, s-1$ and the process $\{X_n\}$ consists of independent random variables taking each of these values with probability s^{-1}, the obvious mapping using expansions to base s provides a connection between the theories of this paper and the usual definitions in $\mathbf{R}^1$. We exploit this connection to show that certain exceptional sets of paths from the Polya random walk are fractals, and we can determine their dimension.

We start by collecting useful preliminary ideas and results in Section 2. As usual, we denote finite positive constants by $c, c_1, \ldots$. The values of these constants may change in different contexts.

2. Preliminaries

We start with a fixed stochastic process $\{X_n, n \in \mathbf{N}\}$ on a probability space $(\Omega, \mathscr{F}, \mu)$ taking values in a finite or countable state space S. A cylinder set C (of rank n) is of the form $C = \{\omega\colon X_i(\omega) = k_i, i = 1, 2, \ldots, n\}$, with $k_i \in S$. For each $\omega_0 \in \Omega$ there is a unique cylinder set of rank n, denoted by $u_n(\omega_0)$, which contains ω_0. Thus

$$u_n(\omega_0) = \{\omega\colon X_i(\omega) = X_i(\omega_0), i = 1, 2, \ldots, n\}. \tag{2.1}$$

We assume the process is $\mathscr{F}$-measurable, that is, that $\mathscr{C} \subset \mathscr{F}$, where $\mathscr{C}$ is the class of all cylinder sets. We use sets in $\mathscr{C}$ for both covering and packing. Many details of classical proofs are greatly simplified because $\mathscr{C}$ is nested; that is, given $C_1, C_2 \in \mathscr{C}$, either $C_1 \cap C_2 = \emptyset$ or $C_1 \subset C_2$ or $C_2 \subset C_1$.

Any function $\phi\colon [0, \delta] \to [0,1]$ which is continuous, monotone increasing, with $\phi(0) = 0$, is called a measure function. The Billingsley [1] definition of Hausdorff ϕ-measure follows. For $\delta > 0$, $A \subset \Omega$, define

$$L_\mu(A, \phi, \delta) = \inf\Big\{\sum_i \phi(\mu(C_i))\colon A \subset \cup C_i, C_i \in \mathscr{C}, \mu(C_i) < \delta\Big\}, \tag{2.2}$$

$$L_\mu(A, \phi) = \lim_{\delta \downarrow 0} L_\mu(A, \phi, \delta). \tag{2.3}$$

For fixed $\phi, \mu, L_\mu(\cdot, \phi)$ is an outer measure in Ω. When $\phi(s) = s^\alpha$ we write

$$L_\mu^\alpha(A) = L_\mu(A, s^\alpha), \tag{2.4}$$

and think of L_μ^α as an α-dimensional measure in Ω. The Hausdorff dimension of a subset $A \subset \Omega$ is then given by

$$\dim_\mu(A) = \inf\{\alpha: L_\mu^\alpha(A) = 0\}. \tag{2.5}$$

Whenever $\dim_\mu(A) > 0$, it is easy to see that

$$\dim_\mu(A) = \sup\{\alpha: L_\mu^\alpha(A) > 0\}.$$

Billingsley [1] proved that the index $\dim_\mu(\cdot)$ satisfies:
(i) $0 \leq \dim_\mu(A) \leq 1, \mu(A) > 0 \Rightarrow \dim_\mu(A) = 1$;
(ii) $A \subset B \Rightarrow \dim_\mu(A) \leq \dim_\mu(B)$;
(iii) $\dim_\mu(\bigcup_{i=1}^\infty A_i) = \sup_i\{\dim_\mu(A_i)\}$.
Thus $\dim_\mu(\cdot)$ is a σ-stable index (see [13)]) which compares the size of subsets of Ω of probability zero.

It is worth observing that we can use $\mathscr{C}$ to introduce a pseudo metric in Ω. Given $\omega_1, \omega_2 \in \Omega$, let $u_0(\omega) = \Omega$ for all ω, and

$$\begin{aligned} n = n(\omega_1, \omega_2) &= \sup\{k \in N: u_k(\omega_1) = u_k(\omega_2)\}, \\ \rho(\omega_1, \omega_2) &= 2^{-n}. \end{aligned}$$

(We allow $n = +\infty$ in the definition, so that $\rho(\omega_1, \omega_2) = 0$ if ω_1, ω_2 are not distinguished by the sets of $\mathscr{C}$.) The closure of $A \subset \Omega$ is then

$$\bar{A} = \{\omega: \rho(\omega, A) = 0\},$$

where

$$\rho(\omega, A) = \inf\{\rho(\omega, \omega'): \omega' \in A\};$$

then it is easy to check that $A \in \mathscr{H} = \sigma(\mathscr{C})$, the sigma field generated by the cylinder sets. $\mathscr{H}$ plays the role of Borel sets in the topology generated by the metric ρ. The sets of $\mathscr{C}$ are both open and closed in this topology, and each $C = u_n(\omega)$ can be thought of as a closed ball of radius 2^{-n} centered at ω. The condition that $\mathscr{H}$ has no μ-atoms is equivalent to

$$\mu(u_k(\omega)) \to 0 \text{ as } k \to \infty \tag{2.6}$$

for all $\omega \in \Omega$. Since any point ω_0 for which (2.6) is false has positive L_μ^α measure for every α, the set of such *atoms*, if it is not empty, has to be treated separately. We say that μ is $\mathscr{H}$-continuous if (2.6) holds for every ω, and we will assume this condition holds throughout the remainder of the paper.

We are now ready to introduce new set functions and indices.

3. Packing premeasure and the index Δ_μ

We use a definition analogous to that of $\phi - P^*$ in Euclidean space, as given in [10]. Our cylinder sets of rank n correspond to dyadic cubes of side 2^{-n}. We have already observed that, in the metric $\rho, C = \overline{C}$. For $\delta > 0$, define

$$P_\mu(A, \phi, \delta) = \sup\left\{ \sum_i \phi(\mu(C_i)) : C_i \text{ disjoint}, \right.$$
$$\left. \mu(C_i) < \delta \text{ and } C_i = u_n(\omega) \text{ with } \omega \in A \right\}.$$

We can think if $P_\mu(A, \phi, \delta)$ as the *most efficient* packing of A by cylinder sets of measure less than δ which touch A. Since we clearly have monontonicity in δ, we can define

$$P_\mu(A, \phi) = \lim_{\delta \to 0} P_\mu(A, \phi, \delta), \tag{3.2}$$

and we call the set function $P_\mu(\cdot, \phi)$ packing ϕ-premeasure with respect to μ. In general this set function fails to be countably subadditive, so it is not an outer measure. We collect the properties of $P_\mu(A, \phi)$ in a lemma.

LEMMA 3.1. *The set function $P_\mu(\cdot, \phi)$ defined for all subsets of Ω by* (3.2) *satisfies*:

(i) $A_1 \subset A_2 \Rightarrow P_\mu(A_1, \delta) \le P_\mu(A_2, \phi)$;
(ii) $P_\mu(A_1 \cup A_2, \phi) \le P_\mu(A_1, \phi) + P_\mu(A_2, \phi)$, *with equality if S is finite and* $\rho(A_1, A_2) > 0$;
(iii) *If μ is $\mathscr{H}$-continuous*, $P_\mu(\{\omega_0\}, \phi) = 0$;
(iv) $L_\mu(A, \phi) \le P_\mu(A, \phi)$;
(v) *For any* $A \subset \Omega, C \in \mathscr{C}$, $P_\mu(A, \phi) = P_\mu(A \cap C, \phi) + P_\mu(A \setminus C, \phi)$;
(vi) $P_\mu(\overline{A}, \phi) = P_\mu(A, \phi)$.

Proof. All these properties follow easily from the definitions. We give details only for (iv) and (v).

(iv) If $L_\mu(A, \phi) = 0$ there is nothing to prove, so assume $L_\mu(A, \phi) > 0$, so that $L_\mu(A, \phi, \delta) = \eta > 0$. Since the definition of $L_\mu(a, \phi, \delta)$ is a lower bound for all coverings by cylinder sets of measure less than δ we may clearly omit any covering set which does not intersect A, and for each pair of cylinder sets which intersect A we may remove the smaller one (since $\mathscr{C}$ is nested). This produces a more efficient covering of A by disjoint cylinder sets each of which intersects A. Thus the covering is also a packing, and

$$\begin{aligned} L_\mu(A,\phi,\delta) &= \inf\left\{\sum_i \phi(\mu(C_i)) : A \subset \cup C_i, \mu(C_i) < \delta\right\} \\ &= \inf\left\{\sum_i \phi(\mu(C_i)) : A \subset \cup C_i, C_i \text{ disjoint}, \right. \\ &\qquad\qquad \left. A \cap C_i \neq \emptyset, \mu(C_i) < \delta\right\} \\ &\le \sup\left\{\sum_i \phi(\mu(C_i)) : C_i \text{ disjoint}, A \cap C_i \neq \emptyset, \mu(C_i) < \delta\right\} \\ &= P_\mu(A,\phi,\delta). \end{aligned}$$

The result follows by letting $\delta \to 0$.

(v) Now suppose $\{C_i\}$ and $\{D_i\}$ are any packings of $A \cap C$ and $(A \setminus C)$ by cylinder sets of measure at most δ. Because $\mathscr{C}$ is nested at most one $C_i \supset C$ and the remainder are subsets of C. Similarly, at most one $D_i \supset C$ and the rest are disjoint from C. Hence, if we leave out at most two sets, each of μ-measure less than δ, the combination of $\{C_i\}$ and $\{D_i\}$ is a packing of A. Thus

$$P_\mu(A,\phi,\delta) \ge P_\mu(A \cap C,\phi,\delta) + P_\mu(A \setminus C,\phi,\delta) - 2\phi(\delta).$$

The result now follows by letting $\delta \downarrow 0$ and applying (ii).

Remark 3.2. The conditions obtained in Lemma 3.1 include those required for $P_\mu(\cdot,\phi)$ to be pre-measure. However, P_μ is not an outer measure because (ii) does not extend to a countable union of sets. To see this consider the rationals $\{r_1, r_2, \dots\}$ as a subset E of $[0,1]$ with μ equal to Lebesgue measure. If $\{X_n\}$ are independent random variables taking each of two values with probability $\frac{1}{2}$, then it is easy to check that, for $0 < \alpha < 1$, $P_\mu^\alpha(\{r_n\}) = 0$, but $P_\mu^\alpha(E) = +\infty$.

In this remark we have already used the notation when $\alpha > 0, \phi(s) = s^\alpha$,

$$P_\mu^\alpha(A) = P_\mu(A, s^\alpha).$$

The usual arguments now show that, when Ω is $\mathscr{H}$-continuous and $0 < \alpha < \beta$,

$$P_\mu^\alpha(A) < +\infty \Rightarrow P_\mu^\beta(A) = 0; \tag{3.3}$$

$$P_\mu^\beta(A) > 0 \Rightarrow P_\mu^\alpha(A) = +\infty. \tag{3.4}$$

It follows that, for any $A \subset \Omega$, there is a unique index $\alpha_0 = \Delta_\mu(A)$, given by

$$\alpha_0 = \inf\{\alpha : P_\mu^\alpha(A) = 0\}. \tag{3.5}$$

By (3.30, if $\alpha_0 > 0$, we also have

$$\alpha_0 = \sup\{\alpha\ P_\mu^\alpha(A) = +\infty\}.$$

Remark 3.3. We use the notation Δ_μ for this index because it is analogous to the upper Minkowski index in Euclidean space for which Tricot [13] uses Δ.

LEMMA 3.4. *The index $\Delta_\mu(\cdot)$ has the following properties*:
(i) $A \subset B \Rightarrow \Delta_\mu(A) \le \Delta_\mu(B)$;
(ii) $0 \le \dim_\mu(A) \le \Delta_\mu(A) \le 1 = \Delta_\mu(\Omega)$;
(iii) $\Delta_\mu(A \cup B) = \max\{\Delta_\mu(A), \Delta_\mu(B)\}$.

Proof. (i) and (iii) follow from (3.4) and (i), (ii) of Lemma 3.1. Clearly, $P_\mu^1(\Omega) = 1$, so (i) and (3.3) now give $\Delta_\mu(A) \le 1$, and (iv) of Lemma 3.1 ensures $\dim_\mu(A) \le \Delta_\mu(A)$.

Note. The example in Remark 3.2 shows that the index $\Delta_\mu(\cdot)$ is stable but not σ-stable.

4. Packing measure and packing dimension

We can apply the method of Munroe (see [7], Theorem 11.3 or [9], Theorem 4) to the pre-measure $P_\mu(\cdot, \phi)$ to obtain packing measure $\hat{P}_\mu(\cdot, \phi)$:

$$\hat{P}_\mu(A, \phi) = \inf\left\{ \sum_{i=1}^{\infty} P_\mu(A_i, \phi) : A \subset \cup A_i \right\}. \tag{4.1}$$

THEOREM 4.1. *The set function $\hat{P}_\mu(\cdot, \phi)$ given by* (4.1) *has the following properties*:
(i) $\hat{P}_\mu(A, \phi) \le P_\mu(A, \phi)$.
(ii) $\hat{P}_\mu(\cup_{i=1}^\infty A_i, \phi) \le \sum_{i=1}^\infty \hat{P}_\mu(A_i, \phi)$.
(iii) *Whenever $\rho(A, B) > 0$, and S is finite,*

$$\hat{P}_\mu(A \cup B, \phi) = \hat{P}_\mu(A, \phi) + \hat{P}_\mu(B, \phi).$$

(iv) *If $\mathscr{H} = \sigma(\mathscr{C})$ is the σ-field generated by the cylinder sets, then $\hat{P}_\mu(\cdot, \phi)$ is $\mathscr{H}$-regular.*
(v) *For any sequence of sets $A_n \uparrow A$, $\hat{P}_\mu(A_n, \phi) \to \hat{P}(A, \phi)$.*
(vi) $\hat{P}_\mu(A, \phi) = \inf\{\lim_{n\to\infty} P_\mu(A_n, \phi) : A_n \uparrow A\}$.

Proof. (i) and (ii) are true whenever the Munroe construction is applied to a premeasure. (iii) follows from (4.1) and Lemma 3.1(ii). We have been unable to decide whether $\hat{P}_\mu(\cdot, \phi)$ is a metric outer measure whenever the state space S is countably infinite, so we obtain our measurability conditions without this assumption. If $C \in \mathscr{C}$, (4.1) gives

$$\hat{P}_\mu(A, \phi) = \inf\left\{ \sum_{i=1}^{\infty} P_\mu(A_i, \phi) : A \subset \cup A_i \right\},$$

and applying (v) of Lemma (4.1) gives

$$\begin{aligned} &\inf\left\{ \sum_{i=1}^{\infty} P_\mu(A_i \cap C, \phi) + P_\mu(A_i \setminus C, \phi) : A \subset \cup A_i \right\} \\ &\quad \geq \inf\left\{ \sum_{i=1}^{\infty} P_\mu(B_i, \phi) : A \cap C \subset \cup B_i \right\} + \inf\left\{ \Sigma P_\mu(B_i, \phi) : A \setminus C \subset \cup B_i \right\} \\ &\quad = \hat{P}_\mu(A \cap C, \phi) + \hat{P}_\mu(A \setminus C, \phi). \end{aligned}$$

Using (ii) now shows that C is measurable with respect to the outer measure $\hat{P}_\mu(\cdot, \phi)$. Hence each set in $\mathscr{H} = \sigma(\mathscr{C})$ is also measurable. Applying Lemma 3.1 (vi) to the definition (4.1), we see that each set A has a cover $H \in \mathscr{H}$ with the same $\hat{P}_\mu$-measure.

The proof of (v) and (vi) is now the same as that of Lemma 5.1(v) and (vii) in [10], so we omit it.

It is clear that we can use the family of outer measures $\hat{P}_\mu^\alpha$, $\alpha > 0$ to define

$$\operatorname{Dim}_\mu(A) = \inf\left\{\alpha : \hat{P}_\mu^\alpha(A) = 0\right\}. \tag{4.2}$$

If $\operatorname{Dim}_\mu(A) = \alpha_0 > 0$, then

$$0 < \alpha < \alpha_0 \Rightarrow \hat{P}_\mu^\alpha(A) = +\infty,$$

so that

$$\operatorname{Dim}_\mu(A) = \sup\left\{\alpha : \hat{P}_\mu^\alpha(A) = +\infty\right\}.$$

Tricot [13] used $\hat{}$ to denote the operation on a dimension index defined as follows:

$$\hat{\Delta}_\mu(A) = \inf\left\{ \sup_n \{\Delta_\mu(A_n)\} : A = \cup A_n \right\}. \tag{4.3}$$

We can prove analogous relations to those holding in the Euclidean case.

THEOREM 4.2. *For each $A \subset \Omega$,*

$$0 \le \dim_\mu(A) \le \operatorname{Dim}_\mu(A) = \hat{\Delta}_\mu(A) \le \Delta_\mu(A) \le 1.$$

Proof. Lemma 3.4(ii) already gives us $0 \le \dim_\mu(A) \le \Delta_\mu(A) \le 1$. If $A = \cup A_n$, then

$$\dim_\mu(A) = \sup_n \{\dim_\mu(A_n)\} \le \sup_n \{\Delta_\mu(A_n)\},$$

so that

$$\dim_\mu(A) \le \hat{\Delta}_\mu(A) \le \Delta_\mu(A).$$

Now suppose $\operatorname{Dim}_\mu(A) = \alpha$, and $\beta > \alpha$. Then $\hat{P}_\mu^\beta(A) = 0$ and Theorem 4.1(vi) ensures there is a sequence of sets $A_n \uparrow A$ such that $P_\mu^\beta(A_n) \le 1$ for all n which implies $\Delta_\mu(A_n) \le \beta$ for all n, and (4.3) now gives $\hat{\Delta}_\mu(A) \le \beta$. This gives

$$\hat{\Delta}_\mu(A) \le \operatorname{Dim}_\mu(A). \tag{4.4}$$

In the other direction, suppose $\hat{\Delta}_\mu(A) = \alpha$ and $\beta > \alpha$. There is a sequence $\{A_n\}$ such that $A = \cup A_n$ and $\Delta_\mu(A_n) < \frac{1}{2}(\alpha + \beta)$ so that $P_\mu^\beta(A) = 0$ for all n. For this sequence $\hat{P}_\mu^\beta(A_n) = 0$ so that $\hat{P}_\mu^\beta(A) = 0$, and $\operatorname{Dim}_\mu(A) \le \beta$. Together with (4.4) we have

$$\hat{\Delta}_\mu(A) = \operatorname{Dim}_\mu(A).$$

5. Density theorems

Billingsley [2] uses a density result for comparing two probability measures and for computing $\dim_\mu(A)$. We prefer to obtain more exact theorems for $L_\mu(\cdot, \phi)$ from which the Billingsley theorems can be deduced as corollaries. We also obtain the corresponding results for $\hat{P}_\mu(\cdot, \phi)$; these are analogous to those in [10], but the nested structure of $\mathscr{C}$ simplifies many arguments. Each of our density theorems require two preliminary lemmas.

We do not wish to restrict ourselves by the assumption that $\mu(u_n(\omega)) > 0$ for all n, ω. To avoid ambiguity let us assume that

$$\lim_{n\to\infty} \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} = +\infty$$

if ω is such that $\mu(u_n(\omega)) = 0$ for $n \ge n_0$.

LEMMA 5.1. *Suppose ν is a finite measure on $(\Omega, \mathscr{F})$, ϕ is a measure function, and $\lambda \geq 0$, and $A \in \mathscr{F}$. Let*

$$A_\lambda = \left\{ \omega \in A \colon \limsup_{n \to \infty} \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} \leq \lambda \right\}. \tag{5.1}$$

Then

$$L_\mu(A_\lambda, \phi) \geq \frac{1}{\lambda} \nu(A_\lambda). \tag{5.2}$$

Note. The case $\lambda = 0$ is important if $\nu(A_0) > 0$; (5.2) is then interpreted to mean

$$L_\mu(A_0, \phi) = +\infty.$$

Proof. Since μ is $\mathscr{H}$-continuous,

$$\limsup_{n \to \infty} \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} = \lim_{\delta \downarrow 0} H(\delta, \omega)$$

where

$$H(\delta, \omega) = \sup \left\{ \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} : \mu(u_n(\omega)) \leq \delta \right\}.$$

For fixed $\varepsilon > 0$, put

$$B_{\lambda+\varepsilon, \delta} = \{ \omega \in A_\lambda \colon H(\delta, \omega) \leq \lambda + \varepsilon \}.$$

Clearly $B_{\lambda+\varepsilon,\delta} \uparrow A_\lambda$ as $\delta \downarrow 0$, and we can cover $B_{\lambda+\varepsilon,\delta}$ by sets $C_i = u_n(\omega)$ such that $\mu(C_i) < \delta$ and $\omega \in B_{\lambda+\varepsilon,\delta}$. For any such covering,

$$\sum_i \phi(\mu(C_i)) \geq \frac{1}{\lambda + \varepsilon} \sum_i \nu(C_i) \geq \frac{1}{\lambda + \varepsilon} \nu(B_{\lambda+\varepsilon,\delta}).$$

Now any covering of A_λ by sets C_i with $\mu(C_i) < \delta$ is also a covering of $B_{\lambda+\varepsilon,\delta}$, so after letting $\delta \downarrow 0$,

$$L_\mu(A_\lambda, \phi) \geq \frac{1}{\lambda + \varepsilon} \nu(A_\lambda),$$

and (5.2) follows since ε is arbitrary.

LEMMA 5.2. *Suppose ν is a finite measure on $(\Omega, \mathscr{F})$, ϕ is a measure function, and $0 \leq \lambda \leq +\infty$. Let*

$$A_\lambda = \left\{\omega: \limsup_{n\to\infty} \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} \geq \lambda\right\}. \tag{5.3}$$

Then

$$L_\mu(A_\lambda, \phi) \leq \frac{1}{\lambda}\nu(\Omega). \tag{5.4}$$

Note. The case $\lambda = +\infty$ in (5.4) is taken to mean that

$$L_\mu(A_{+\infty}, \phi) = 0.$$

Proof. If $\lambda = 0$ there is noting to prove. Assume $0 < \lambda < +\infty$, since the $\lambda = +\infty$ case will follow if we can show (5.4) for each finite λ. Let $\delta > 0$, $0 < \varepsilon < 1$. Then, for each $\omega \in A_\lambda$, there are arbitrarily large integers $n = n(\omega)$ such that, for $C = u_n(\omega)$,

$$\mu(C) < \delta, \quad \phi(\mu(C)) \leq \frac{1}{\lambda(1-\varepsilon)}\nu(C). \tag{5.5}$$

The collection of such C covers A_λ, so we can find (since $\mathscr{C}$ is nested) a countable disjoint subcollection $\{C_i\}$, each of which satisfies (5.5), which covers A_λ. For this covering,

$$L_\mu(A_\lambda, \phi, \delta) \leq \sum_i \phi(\mu(C_i)) \leq \frac{1}{\lambda(1-\varepsilon)} \sum_i \nu(C_i) \leq \frac{1}{\lambda(1-\varepsilon)}\nu(\Omega).$$

First let $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$ to give (5.4).

THEOREM 5.3. *Suppose ν is a probability measure on $(\Omega, \mathscr{F})$, ϕ is any measure function, and $A \in \mathscr{F}$. Then*

$$\nu(A) \inf_{\omega\in A}\left\{\liminf_{n\to\infty} \frac{\phi(\mu(u_n(\omega)))}{\nu(u_n(\omega))}\right\} \leq L_\mu(A, \phi)$$
$$\leq \sup_{\omega\in A}\left\{\liminf_{n\to\infty} \frac{\phi(\mu(u_n(\omega)))}{\nu(u_n(\omega))}\right\}. \tag{5.6}$$

Proof. Note that condition (5.1) is equivalent to

$$A_\lambda = \left\{ \omega : \liminf_{n\to\infty} \frac{\phi(\mu(u_n(\omega)))}{\nu(u_n(\omega))} \geq \frac{1}{\lambda} \right\}.$$

If we replace $1/\lambda$ by

$$\inf_{\omega\in A} \left\{ \liminf_{n\to\infty} \frac{\phi(\mu(u_n(\omega)))}{\nu(u_n(\omega))} \right\},$$

the left inequality in (5.6) follows from (5.2). A similar argument using (5.4) establishes the right inequality of (5.6).

We now look for corresponding results for packing measure.

LEMMA 5.4. *Suppose ν is a finite measure on $(\Omega, \mathscr{F})$, ϕ is a measure function, and $0 \leq \lambda \leq +\infty$ and $A \in \mathscr{F}$. Let*

$$A_\lambda = \left\{ \omega \in A : \liminf_{n\to\infty} \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} \leq \lambda \right\}. \tag{5.7}$$

Then

$$\hat{P}_\mu(A_\lambda, \phi) \geq \frac{1}{\lambda}\nu(A_\lambda). \tag{5.8}$$

Proof. If $\lambda = +\infty$ there is nothing to prove. Suppose $0 \leq \lambda < +\infty$ and $\varepsilon > 0$, $\delta > 0$. For each $\omega \in A_\lambda$, we can find arbitrarily large integers $n = n(\omega)$ such that

$$\mu(u_n(\omega)) < \delta, \phi(\mu(u_n(\omega))) \geq \frac{1}{\lambda+\varepsilon}\nu(u_n(\omega)). \tag{5.9}$$

If $\mathscr{E}$ denotes the collection of all cylinder sets satisfying (5.9) with $\omega \in A_\lambda$, then we can find a countable disjoint sequence $\{C_i\}$ of sets from $\mathscr{E}$ which covers A_λ. Since this collection is also a packing, we deduce that

$$P_\mu(A_\lambda, \phi, \delta) \geq \sum_i \phi(\mu(C_i)) \geq \frac{1}{\lambda+\varepsilon}\sum_i \nu(C_i) \geq \frac{1}{\lambda+\varepsilon}\nu(A_\lambda).$$

This is true for all $\varepsilon > 0$, so that

$$P_\mu(A_\lambda, \phi) \geq \frac{1}{\lambda}\nu(A_\lambda). \tag{5.10}$$

We can apply (5.10) to any sequence $\{B_n\}$ of sets in $\mathscr{F}$ such that $B_n \uparrow A_\lambda$, and use Theorem 4.1(vi) to obtain (5.8).

LEMMA 5.5. *Suppose ν is a finite Borel measure on $(\Omega, \mathscr{F})$, ϕ is a measure function, and $0 \leq \lambda \leq +\infty$. Define*

$$A_\lambda = \left\{\omega \colon \liminf_{n\to\infty} \frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} \geq \lambda\right\}. \tag{5.11}$$

Then

$$\hat{P}_\mu(A_\lambda, \phi) \leq \frac{1}{\lambda}\nu(\Omega). \tag{5.12}$$

Proof. If $\lambda = 0$ there is nothing to prove; the case $\lambda = +\infty$ will follow if we can prove (5.12) for finite λ. Suppose $0 < \lambda < +\infty$, $\delta > 0$, $0 < \varepsilon < 1$. Put $\lambda' = \lambda(1-\varepsilon)$ and define

$$B_{\lambda',\delta} = \{\omega \in A_\lambda \colon K(\delta, \omega) \geq \lambda'\},$$

where

$$K(\delta, \omega) = \inf\left\{\frac{\nu(u_n(\omega))}{\phi(\mu(u_n(\omega)))} \colon \mu(u_n(\omega)) \leq \delta\right\}.$$

Then, if $\{C_i\}$ is any packing of $B_{\lambda',s}$ by cylinder sets with $\mu(C_i) < \delta$, we have

$$\sum_i \phi(\mu(C_i)) \leq \frac{1}{\lambda'}\sum_i \nu(C_i) \leq \frac{1}{\lambda'}\nu(\Omega) \tag{5.13}$$

If we note that $B_{\lambda',\delta} \uparrow A_\lambda$ as $\delta \downarrow 0$ and apply Theorem 4.1(vi) to this sequence, we get

$$\hat{P}_\mu(A_\lambda, \phi) \leq \lim_{\delta\downarrow 0} P_\mu(B_{\lambda',\delta}, \phi) \leq \frac{1}{\lambda'}\nu(\Omega)$$

by (5.13). (5.12) now follows by letting $\varepsilon \downarrow 0$.

THEOREM 5.6. *Suppose ν is a probability measure on $(\Omega, \mathscr{F})$, ϕ is any measure function, and $A \in \mathscr{F}$. Then*

$$\nu(A) \inf_{\omega\in A}\left\{\limsup_{n\to\infty} \frac{\phi(\mu(u_n(\omega)))}{\nu(u_n(\omega))}\right\} \leq \hat{P}_\mu(A, \phi) \leq \sup_{\omega\in A}\left\{\limsup_{n\to\infty} \frac{\phi(\mu(u_n(\omega)))}{\nu(u_n(\omega))}\right\}. \tag{5.14}$$

Proof. This theorem can be deduced from Lemmas 5.4, 5.5 in the same way that (5.6) was deduced from Lemmas 5.1, 5.2.

Remark 5.7. The reader will notice the similarities between these density results and those for Hausdorff and packing measures in $\mathbf{R}^d$. The main difference is that we have not required a smoothness condition on ϕ and there are no constants depending on ϕ in (5.6) and (5.14). This follows from the fact that $\mathscr{C}$ is nested.

6. Fractal subsets of Ω

Tricot [14] introduced the notion of a regularity index for subsets of $\mathbf{R}^d$, and Taylor [11] suggested that only regular sets should be called fractals. We use this idea to provide the basis for defining fractals in $(\Omega, \mathscr{F}, \mu)$.

DEFINITION 6.1. A subset $A \subset \Omega$ is said to be a fractal with respect to μ if $\mu(A) = 0$ and

$$\dim_\mu(A) = \mathrm{Dim}_\mu(A).$$

The standard method for finding Hausdorff and packing dimensions of a subset $E \subset \mathbf{R}^d$ starts with the construction of a measure ν concentrated on E but evenly spread over E. Some version of density theorem is then applied to ν. We now make that method explicit using the ideas of Billingsley [1, 2] and techniques developed by Cutler [4].

THEOREM 6.2. *Suppose $A \in \mathscr{F}$ and there is a finite measure ν on $(\Omega, \mathscr{F})$ with $\nu(A) > 0$ and such that*

$$\liminf_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \le c \text{ for all } \omega \in A, \tag{6.1}$$

and, for each $\varepsilon > 0$, $\nu(A_\varepsilon) > 0$ where

$$A_\varepsilon = \left\{\omega \in A\colon \liminf_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} > c - \varepsilon\right\}. \tag{6.2}$$

Then

$$\dim_\mu(A) = c.$$

Proof. (i) If $\omega \in A_\varepsilon$, then

$$\limsup_{n\to\infty} \frac{\nu(u_n(\omega))}{[\mu(u_n(\omega))]^{c-\varepsilon}} = 0.$$

By Lemma 5.1, this implies that $L_\mu^{c-\varepsilon}(A_\varepsilon) = +\infty$, so that

$$\dim_\mu(A) \geq \dim_\mu(A_\varepsilon) \geq c - \varepsilon.$$

Since ε is arbitrary, it follows that

$$\dim_\mu(A) \geq c. \tag{6.3}$$

(ii) For $\delta > 0$, (6.1) implies that, for all $\omega \in A$,

$$\limsup_{n\to\infty} \frac{\nu(u_n(\omega))}{[\mu(u_n(\omega))]^{c+\delta}} = +\infty,$$

and an application of Lemma 5.2 gives $L_\mu^{c+\delta}(A) = 0$, so that $\dim_\mu(A) \leq c + \delta$. Again δ is arbitrary, so $\dim_\mu(A) \leq c$ which, with (6.3), establishes the theorem.

Very similar arguments give the next result.

THEOREM 6.3. *Suppose $A \in \mathscr{F}$ and there is a finite measure ν on $(\Omega, \mathscr{F})$ with $\nu(A) > 0$ and such that*

$$\limsup_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \leq c \text{ for all } \omega \in A, \tag{6.4}$$

and, for each $\varepsilon > 0$, $\nu(A_\varepsilon) > 0$ where

$$A_\varepsilon = \left\{\omega \in A : \limsup_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} > c - \varepsilon\right\}. \tag{6.5}$$

Then

$$\operatorname{Dim}_\mu(A) = c.$$

Combining these results gives a technique for showing $A \in \mathscr{F}$ is a fractal with respect to μ.

COROLLARY 6.4. *Suppose $A \in \mathscr{F}$, $\mu(A) = 0$, and there is a finite measure ν on $(\Omega, \mathscr{F})$ with $\nu(A) > 0$ and satisfying* (6.2) *and* (6.4). *Then A is a fractal* *and*

$$\dim_\mu(A) = \operatorname{Dim}_\mu(A) = c.$$

Remark 6.5. The definition 6.1 really only examines the thickest part of the set A, and requires $\dim_\mu$ to equal Dim_μ on that part. Stronger conditions are needed if we want the set to *look the same* near each of its points. We could call A a fractal with uniform dimension c if $\mu(A) = 0$ and there is some finite measure ν on $(\Omega, \mathscr{F})$ such that $\nu(A) > 0$ and

$$\lim_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} = c \quad \text{for all } \omega \in A. \tag{6.6}$$

7. Fractal measures in $(\Omega, \mathscr{F}, \mu)$

The condition (6.6) can be relaxed by allowing the limit to depend on ω. This gives the relevant version of a definition proposed in [12].

DEFINITION 7.1. A finite measure ν on $(\Omega, \mathscr{F})$ is said to be a fractal with respect to μ if

$$\hat{\nu}_\mu(\omega) = \lim_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \text{ exists } \nu \text{ a.s.} \tag{7.1}$$

DEFINITION 7.2. If ν is a fractal with respect to μ and there is a constant α such that

$$\hat{\nu}_\mu(\omega) = \alpha, \nu \text{ a.s.}, \tag{7.2}$$

we say that ν has exact dimension α; otherwise we say that ν is a multifractal measure.

We remark that our proposed definition of a multifractal measure is not analogous to those appearing in the physics literature. We are requiring a local regularity condition to be satisfied by ν at ν a.a. points of Ω.

Note that, if ν is a fractal with exact dimension α, then Corollary 6.4 tells us that the set

$$A = \{\omega : \hat{\nu}_\mu(\omega) = \alpha\}$$

is a fractal with uniform dimension α. More generally, for any fractal measure ν we can define its upper dimension by

$$\dim_\mu(\nu) = \inf\{\beta > 0: \hat{\nu}_\mu(\omega) \le \beta, \nu \text{ a.s.}\}.$$

The results of Section 6 lead immediately to the following:

THEOREM 7.3. *Suppose ν is a fractal measure with respect to μ and* $\dim_\mu \nu = \beta, 0 \le \beta \le 1$. *Then*

$$\dim_\mu(A) = \mathrm{Dim}_\mu(A) = \beta,$$

where

$$A = \{\omega: 0 \le \hat{\nu}_\mu(\omega) \le \beta\}.$$

This means that, if $\beta < 1$, or at least $\mu(A) = 0$, fractal measures are concentrated on fractal sets. In general, the converse is false; a uniformity condition like (6.6) is needed if we are to start with a set and construct a fractal measure concentrated on it. Otherwise, the thinner parts of the set (that is subsets of points ω where $\hat{\nu}_\mu(\omega) \le \beta' < \beta$) may cause a problem.

8. The Lebesgue case

It is clear that, just as Billingsley [1] related his definition of Hausdorff measure and dimension in $(\Omega, \mathscr{F}, \mu)$ to the classical definitions on the real line, we can compare our definition of packing measure and dimension to those defined in [10]. We take $\Omega = [0, 1]$, $\mathscr{F}$ to be the class of Borel subsets, and μ to be Lebesgue measure. For a fixed integer $s \ge 2$, $\omega \in \Omega$, let

$$\omega = \sum_{i=1}^{\infty} X_i(\omega) s^{-i} \tag{8.1}$$

be the nonterminating expansion of ω to base s. Then $\{X_1, X_2, \dots\}$ becomes a stochastic process taking values in $S = \{0, 1, 2, \dots, s-1\}$, and $u_n(\omega)$, defined by (2.1), becomes a half-open interval of length (or Lebesgue measure) s^{-n}. If $\phi - m$ denotes Hausdorff ϕ-measure as defined in Rogers [9], then a modification of the standard arguments for dyadic covers (corresponding to the case $s = 2$), shows that there is a finite constant c such that, for all $A \subset \Omega$,

$$\phi - m(A) \le L_\mu(A, \phi) \le c\phi - m(A), \tag{8.2}$$

so that $\dim_\mu(A)$ is identical with the usual definition of Hausdorff dimension.

In [10] several definitions of ϕ-packing are explored, and the preferred one, denoted $\phi - p(A)$ uses balls with center in the set A. Because $u_n(\omega)$ is a closed ball of radius 2^{-n} in the ρ- metric we can interpret $\hat{P}_\mu(\cdot, \phi)$ as equivalent to $\phi - p(\cdot)$ in the modified metric. However, it is easy to check that our definition of $\hat{P}_\mu(\cdot, \phi)$ reduces to $\phi - p^*(\cdot)$ in the Euclidean metric in the case $s = 2$. For all values of the integer s, its relation to $\phi - p$ and $\phi - q$ will in general by similar to that of $\phi - p^*$. There will be some sets A for which $\hat{P}_\mu(A, \phi) = +\infty$ but $\phi - p(A) < \infty$, but we can show that there is a positive constant c such that

$$c\phi - p(A) \leq \hat{P}_\mu(A, \phi) \leq \phi - q(A). \tag{8.3}$$

It was proved in [10] that $\phi - p$ and $\phi - q$ define the same dimension index; that is,

$$\mathrm{Dim}(A) = \inf\{\alpha > 0: s^\alpha - q(A) = 0\}.$$

Hence (8.3) implies that our index $\mathrm{Dim}_\mu(A)$ is identical with $\mathrm{Dim}(A)$ as defined in [14] or [10].

It follows that our definition of a fractal with respect to μ, given in Section 6, is identical with the suggested definition in [11], and the fractal measures, defined in Section 7, correspond to those defined in [12].

Because of the above connections we can translate any known results about subsets of [0, 1] to the probability space ([0, 1], $\mathscr{B}$, μ). Eggleston [5, 6] considered sets obtained by asymptotic conditions on the expansion of a real number of x in [0, 1] to base s. Now packing dimension had not been defined at the time of these papers, so Eggleston only considered Hausdorff dimension. However, an examination of his argument shows that, in calculating an upper bound for the Hausdorff dimension of

$$E = \bigcup_{n=1}^{\infty} E_n,$$

he counted the number $N_r(E_n)$ of intervals of length s^{-r} which the set E_n intersects, and showed that, for all $\varepsilon > 0$, whenever r is large,

$$N_r(E_n) < (s^r)^{\alpha+\varepsilon}. \tag{8.4}$$

Eggleston's definitions of E, E_n are given in terms of the sequence X_i of (8.1). We will use several distinct cases which arise as translations of the structures of this paper.

The case $s = 2$ of the following result is due to Tricot [13].

LEMMA 8.1. *If $A \subset \mathbf{R}$ and $N_r(A)$ denotes the number of intervals of the form $[js^{-r}, (j+1)s^{-r})$ which contain a point of A, then if*

$$N_r(A) \le s^{r\alpha} \quad \text{for } r \ge r_0,$$

we have

$$\Delta_\mu(A) \le \alpha.$$

Proof. Consider any packing of A by cylinder sets of rank at least r_0. For $\varepsilon > 0, \phi(x) = x^{\alpha+\varepsilon}$,

$$\sum_i \phi(\mu(C_i)) \le \sum_{r=r_0}^{\infty} N_r(A)(s^{-r})^{\alpha+\varepsilon}$$

(this ignores the possible overlaps between cylinders of distinct ranks because all candidates are counted)

$$\le \sum_{r=r_0}^{\infty} s^{-r\varepsilon} \to 0 \text{ as } r_0 \to +\infty.$$

Thus $P_\mu^{\alpha+\varepsilon}(A) = 0$, so that

$$\Delta_\mu(A) \le \alpha + \varepsilon$$

for each $\varepsilon > 0$.

Since Eggleston's arguments use (8.4) to cover E, he is actually proving $\Delta(E_n) \le \alpha + \varepsilon$, which implies $\hat{\Delta}(E) \le \alpha + \varepsilon$ for each $\varepsilon > 0$. This means that the method by which he obtains the Hausdorff dimension includes a proof that $\hat{\Delta}_\mu(E) = \dim_\mu(E)$, and his subsets are fractals in the sense of [11]. In the context of this paper, his subsets of $[0,1]$ are fractals with respect to Lebesgue measure.

The reader can easily translate many of the results in [5, 6]; we illustrate by considering some sample path properties of simple random walk on the integer lattice. Define independent random variables

$$Y_i(\omega) = \begin{cases} +1 & \text{with probability } \frac{1}{2}; \\ -1 & \text{with probability } \frac{1}{2}, \end{cases}$$

and put $S_0 = 0$,

$$S_n(\omega) = \sum_{i=1}^{n} Y_i(\omega), \qquad n = 1, 2, \ldots.$$

Then $\{S_n\}$ is called a simple random walk on the integer lattice. The strong law of large numbers implies that

$$\frac{1}{n}S_n(\omega) \to 0 \text{ a.s.,} \tag{8.5}$$

so that any subset of

$$A = \left\{\omega: \frac{1}{n}S_n(\omega) \nrightarrow 0\right\}$$

has measure zero and can be analysed using the methods of this paper. For all ω, it is clear that

$$-1 \le \frac{1}{n}S_n(\omega) \le 1,$$

and, if $-1 < c < 0$, each of the following has probability zero since it is a subset of A:

$$B_c = \left\{\omega: \frac{1}{n}S_n(\omega) \to c \text{ as } n \to \infty\right\},$$

$$D_c = \left\{\omega: \limsup_{n\to\infty} \frac{1}{n}S_n(\omega) \le c\right\},$$

$$E_c = \left\{\omega: \liminf_{n\to\infty} \frac{1}{n}S_n(\omega) \le c\right\}.$$

Each of these sets has an easy geometrical interpretation in terms of the long-term behaviour of the graph $\{n, S_n(\omega)\}$ relative to the lines $y = (c \pm \varepsilon)x$.

THEOREM 8.2. *Each of the subsets B_c, D_c, E_c of Ω is a fractal of dimension $\alpha = f(c)$, where*

$$f(c) = 1 - \tfrac{1}{2}(1+c)\log_2(1+c) - \tfrac{1}{2}(1-c)\log_2(1-c).$$

Proof. Clearly

$$B_c \subset D_c \subset E_c \subset A, \tag{8.6}$$

so it is sufficient to show that

$$\dim_\mu(B_c) \ge f(c), \mathrm{Dim}_\mu(E_c) \le f(c).$$

If μ is the probability measure of the random walk, the mapping

$$X_i = \tfrac{1}{2} + \tfrac{1}{2}Y_i$$

gives a stochastic process $\{X_i\}$ taking values 0, 1, so that the dyadic expansion ($s = 2$)

$$\omega \leftrightarrow \sum_{i=1}^{\infty} X_i(\omega)2^{-i}$$

is a measure preserving isomorphism between $(\Omega, \mathscr{F}, \mu)$ and $([0,1], \mathscr{B}, \lambda)$ with λ for Lebesgue measure. Now let $N(r, x)$ be the number of times 1 occurs in the first r places of the dyadic expansion of $x \in [0,1]$. $B_c \supset \Omega$ translates to

$$B'_c = \left\{x \in [0,1]: \lim_{r\to\infty} \frac{N(r,s)}{r} = \frac{1}{2} + \frac{1}{2}c\right\}.$$

The main theorem of [5] asserts that the Hausdorff dimension $\alpha = \dim(B'_c)$ satisfies

$$2^{-\alpha} = \left(\tfrac{1}{2} + \tfrac{1}{2}c\right)^{\frac{1}{2}+\frac{1}{2}c}\left(\tfrac{1}{2} - \tfrac{1}{2}c\right)^{\frac{1}{2}-\frac{1}{2}c},$$

or

$$-\alpha = \left(\tfrac{1}{2} + \tfrac{1}{2}c\right)\log_2\left(\tfrac{1}{2} + \tfrac{1}{2}c\right) + \left(\tfrac{1}{2} - \tfrac{1}{2}c\right)\log_2\left(\tfrac{1}{2} - \tfrac{1}{2}c\right),$$

or

$$\alpha = 1 - \tfrac{1}{2}(1 + c)\log_2(1 + c) - \tfrac{1}{2}(1 - c)\log_2(1 - c).$$

Hence

$$\dim(B'_c) = \dim_\mu(B_c) = f(c). \tag{8.7}$$

Theorem 14 of [6] asserts that $\dim(E'_c) = f(c)$. However, as explained above this is done by counting the number of dyadic intervals of length 2^{-n} which meet subsets E_k with $E'_c = \bigcup_k E_k$. That is, the proof on pages 78, 79 of [6] actually yields

$$\hat{\Delta}(E'_c) = \mathrm{Dim}(E'_c) \le f(c).$$

By translation we get

$$\mathrm{Dim}_\mu(E_c) \le f(c).$$

Putting this together with (8.6) and (8.7) we have shown that each of the sets B_c, D_c, E_c is a fractal with dimension index $f(c)$.

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