ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 3, Fall 1994

INHOMOGENEOUS INEQUALITIES OVER NUMBER FIELDS

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1. Introduction

In the classical theory of diophantine approximation, Kronecker, in 1884, was the first to investigate inhomogeneous approximation to real linear forms which were, in some sense, independent over **Z**. In a slightly different direction, Khintchine, in 1936, proved that if a homogeneous system of real linear forms was not approximated well by integers (i.e., it was a badly approximable system), then this implied the existence of an excellent integer approximation to any associated inhomogenous system (see, for example, [9]). Here we study related inhomogeneous problems in the setting of an arbitrary number field. In particular, we examine these issues in the context of the ring of S-integers and over the associated adèle ring of the number field. Diophantine approximation over the adèle ring was first studied by Cantor in 1965 [4], then later by Sweet [12] and more recently by the author [2].

In Section 2 we precisely describe all our notation, but briefly, let k be a number field and S a finite collection of places of k containing all archimedean places. We write $k_S = \prod_{v \in S} k_v$ for the topological product of the completions k_v. Let $\{A_v\}_{v \in S}$ be a collection of $M \times N$ matrices such that for each $v \in S$, A_v has its entries over k_v. The S-system $\{A_v\}_{v \in S}$ is said to be a *badly approximable S-system of linear forms* if there exists a constant $\tau > 0$ such that

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v \in S} |A_v \vec{x} - \vec{y}|_v^M$$

for all S-integer column vectors $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{x} \neq \vec{0}$ and $\vec{y} \in (\mathscr{O}_S)^M$, where h_S is a suitably normalized S-height. Our first result is a generalization of Khintchine's theorem to number fields.

THEOREM 1. *Let $\{A_v\}_{v \in S}$ be a badly approximable S-system of dimension $M \times N$. For each $v \in S$ suppose that $\varepsilon_v \in k_v$ satisfies $0 < \|\varepsilon_v\|_v < 1$. Then for any $\vec{\beta} = (\vec{\beta}_v) \in (k_S)^M$, there exist vectors $\vec{x} \in (\mathscr{O}_S)^N$ and $\vec{y} \in (\mathscr{O}_S)^M$ such*

Received June 26, 1992.
1991 Mathematics Subject Classification. Primary 11J20, 11J61; Secondary 11R56.

that

$$\|A_v\vec{x} - \vec{y} - \vec{\beta}_v\|_v \leq C_1(k, v, \{A_v\})\|\varepsilon_v\|_v^N$$

and

$$\|\vec{x}\|_v \leq C_2(k, v, \{A_v\})\|\varepsilon_v\|_v^{-M}$$

for all places $v \in S$, where $\| \quad \|_v$ is a supremum norm.

The constants C_1 and C_2 are explicitly given in Theorem 8. As the constants are independent of $\vec{\beta}$, given the S-system $\{A_v\}_{v \in S}$ and $(\varepsilon_v) \in (k_S)^\times$, there exists an explicitly computable finite subcollection $\mathscr{F} \subseteq (\mathscr{O}_S)^N$ so that for any $\vec{\beta} \in (k_S)^M$, the S-integer vector $\vec{x}$ of Theorem 1 may be selected from the finite set $\mathscr{F}$. As an aside, we remark that Mahler considered homogeneous inequalities involving both real and p-adic linear systems over $\mathbf{Q}$ (for example, [10], Chapter 3, Section 6, Theorem 3) and in some sense, one may view Theorem 1 as an inhomogeneous analogue of Mahler's result over k.

With respect to independent systems, Cantor proved ([4], Theorem 3.2) a number field generalization of Kronecker's theorem in the context of the ring of S-integers. We state this result in our present notation below. For an $M \times N$ matrix B, we define an associated $(M+N) \times N$ matrix $\mathfrak{A}(B)$ by

$$\mathfrak{A}(B) = \left(\frac{B}{\mathbf{1}_N}\right),$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. Let $\{A_v\}_{v \in S}$ be an S-system and $\vec{\beta} = (\vec{\beta}_v) \in (k_S)^M$. Cantor proved that the following two statements are equivalent:

(i) *If $\vec{u} \in (\mathscr{O}_S)^M$ is a vector for which there exists a vector $\vec{w} \in (\mathscr{O}_S)^N$ so that*

$$\begin{pmatrix} \vec{u} \\ \vec{w} \end{pmatrix}^T \mathfrak{A}(A_v) = \vec{0}^T$$

for all $v \in S$, then there exists an $\eta \in \mathscr{O}_S$ such that

$$\begin{pmatrix} \vec{u} \\ \eta \end{pmatrix}^T \mathfrak{A}(\vec{\beta}_v) = 0$$

for all $v \in S$.

(ii) *Given* $\varepsilon = (\varepsilon_v) \in (k_S)^{\times}$, *there exist vectors* $\vec{x} \in (\mathscr{O}_S)^N$ *and* $\vec{y} \in (\mathscr{O}_S)^M$ *so that*

$$\|A_v\vec{x} - \vec{y} - \vec{\beta}_v\|_v \leq \|\varepsilon_v\|_v$$

for all $v \in S$.

The result is not quantitative in the sense that no explicit bound on the size of the S-integer vector $\vec{x}$ is given. The following result is formulated purely over the adèles and may be viewed as a quantitative number field analogue of Kronecker's theorem.

THEOREM 2. *Let* $A = (A_v)$ *be an* $M \times N$ *matrix over the adèle ring* $k_{\mathbf{A}}$. *Suppose there exist idèles* $\varepsilon = (\varepsilon_v)$ *and* $\delta = (\delta_v)$, *with volume* $V(\delta) > 1$, *so that the system of inequalities*

$$\left\|A_v^T\vec{u} - \vec{w}\right\|_v \leq \|\delta_v\|_v^{-1} \quad \textit{for each place } v \textit{ of } k,$$

where A_v^T *is the transpose of* A_v, *is not solvable with* $\vec{u} \in k^M$, $\vec{w} \in k^N$ *and*

$$0 < \|\vec{u}\|_v \leq \|\varepsilon_v\|_v^{-1} \quad \textit{for each place } v \textit{ of } k.$$

Then for any $\vec{\beta} = (\vec{\beta}_v) \in (k_{\mathbf{A}})^M$, *there exist vectors* $\vec{x} \in k^N$ *and* $\vec{y} \in k^M$ *so that for each place* v *of* k,

$$\|A_v\vec{x} - \vec{y} - \vec{\beta}_v\|_v \leq C(k, v, M, N)\|\varepsilon_v\|_v$$

and

$$\|\vec{x}\|_v \leq C(k, v, M, N)\|\delta_v\|_v.$$

The constant C, which is independent of the matrix A, is given in Theorem 9. In particular, for the nonarchimedean places, $C(k, v, M, N) = 1$. Just as in Theorem 1, given the initial data, one is able to construct a finite collection $\mathscr{F} \subseteq k^N$ so that for any $\vec{\beta} \in (k_{\mathbf{A}})^M$, the vector $\vec{x}$ of Theorem 2 may be selected from the set $\mathscr{F}$.

By selecting the matrices $A_v = (0)$ for all $v \notin S$, one is naturally lead to an S-integer formulation of Theorem 2 which is more compatible with Cantor's result. We say that the S-system $\{A_v\}_{v \in S}$ and $\mathbf{1}_N$ are *independent over* $\mathscr{O}_S$ if the only vector $\vec{z} \in (\mathscr{O}_S)^{(M+N)}$ for which

$$\vec{z}^T\mathfrak{A}(A_v) = \vec{0}^T$$

for all $v \in S$ is $\vec{z} = \vec{0}$. It follows that if the S-system $\{A_v\}_{v \in S}$ and $\mathbf{1}_N$ are independent over $\mathcal{O}_S$, then for any $\varepsilon = (\varepsilon_v) \in (k_{\mathbf{A}})^{\times}$, we may find $\delta = (\delta_v) \in (k_{\mathbf{A}})^{\times}$ so that the hypotheses of Theorem 2 are satisfied. This produces an even closer analogue to Kronecker's theorem (see, for example, [5], Section 51, Theorem 1).

The proofs of Theorems 1 and 2 involve techniques in the geometry of numbers over the adèle space. In particular, for Theorem 1 we given an adelic analogue of an inequality due to Hlawka and for Theorem 2 we use a recent result of O'Leary and Vaaler.

The author wishes to thank the referee for several valuable comments and suggestions.

2. Notation and normalizations

Let k be an algebraic number field of degree d over $\mathbf{Q}$. If v is a place of k we write k_v for the completion of k at v and let $d_v = [k_v: \mathbf{Q}_v]$ denote the local degree. If v is an infinite place we write $\| \quad \|_v$ for the usual Euclidean absolute value on k_v. If v is a finite place then $\| \quad \|_v$ denotes the unique absolute value on k_v which extends the usual p-adic absolute value on $\mathbf{Q}_p$, where $v|p$. We normalize a second absolute value $| \quad |_v$ at each place v by setting $| \quad |_v = \| \quad \|_v^{d_v/d}$. It follows that these absolute values satisfy the *product formula*: $\prod_v |\alpha|_v = 1$ for all $\alpha \in k$, $\alpha \neq 0$. If $v \nmid \infty$ we write $\mathcal{O}_v$ for the ring of v-adic integers in k_v.

Let S be a finite collection of places of k containing all places lying over infinity. We write

$$\mathcal{O}_S = \{x \in k\colon x \in \mathcal{O}_v \text{ for all } v \notin S\}$$

for the ring of S-integers in k. Let k_S be the topological product of $\{k_v\}_{v \in S}$, that is, $k_S = \prod_{v \in S} k_v$. If we identify $\mathcal{O}_S$ with its image under the canonical injection in k_S then by the product formula $\mathcal{O}_S$ is discrete in k_S and the quotient $k_S/\mathcal{O}_S$ is compact.

Let

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

be a column vector in $(k_v)^N$. We extend the absolute values $\| \quad \|_v$ and $| \quad |_v$ to $(k_v)^N$ by defining

$$\|\vec{x}\|_v = \max_{1 \le n \le N} \{\|x_n\|_v\}$$

and $|\vec{x}|_v = \|\vec{x}\|_v^{d_v/d}$ for all v. We define the absolute value $|||\ \ |||_v$ by

$$|||\vec{x}|||_v = \begin{cases} \sum_{n=1}^{N} \|x_n\|_v & \text{for } v \mid \infty \\ \|\vec{x}\|_v & \text{for } v \nmid \infty. \end{cases}$$

If $A = (a_{mn})$ is an $M \times N$ matrix over k_v, we define $|A|_v$ by

$$|A|_v = \max_{\substack{1 \le m \le M \\ 1 \le n \le N}} \{|a_{mn}|_v\}.$$

Suppose that $\vec{x} \in (\mathscr{O}_S)^N$ and $\vec{y} \in (\mathscr{O}_S)^M$. We define the *S-height* of $\vec{x}$ and $\vec{y}$ by

$$h_S(\vec{x}, \vec{y}) = \prod_{v \in S} \max\{|\vec{x}|_v, |\vec{y}|_v\}.$$

For each place $v \in S$, let A_v be an $M \times N$ matrix over k_v. The S-system $\{A_v\}_{v \in S}$ is said to be a *badly approximable S-system of linear forms* (*of dimension* $M \times N$) if there exists a real constant $\tau = \tau(k, \{A_v\}_{v \in S}) > 0$ such that

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v \in S} |A_v \vec{x} - \vec{y}|_v^M$$

for all vectors $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{x} \neq \vec{0}$ and $\vec{y} \in (\mathscr{O}_S)^M$. See Section 6 of [2] for further details on badly approximable S-systems.

We select a Haar measure β_v on the additive group of k_v by the following normalization:

(i) If $k_v \cong \mathbf{R}$ then β_v is the usual Lebesgue measure on $\mathbf{R}$.

(ii) If $k_v \cong \mathbf{C}$ then β_v is Lebesgue measure on the complex plane multiplied by 2.

(iii) If $v \nmid \infty$ we require that $\beta_v(\mathscr{O}_v) = |\mathscr{D}_v|_v^{d/2}$, where $\mathscr{D}_v$ is the local different of k at v.

We write $k_{\mathbf{A}}$ for the adèle ring of k and β for the normalized Haar measure on $k_{\mathbf{A}}$ which is induced by the product measure $\prod_v \beta_v$. If $(k_{\mathbf{A}})^L$ is the L-fold product of adèle spaces we write V for the product Haar measure β^L on $(k_{\mathbf{A}})^L$. We remark that in the geometry of numbers over the adèles, the Haar measure V plays the rôle of volume in the classical theory.

We may embed $k^L \hookrightarrow (k_{\mathbf{A}})^L$ via the usual diagonal embedding. It follows that k^L is discrete and the quotient $(k_{\mathbf{A}})^L/k^L$ is compact with induced Haar measure

$$\bar{V}\left((k_{\mathbf{A}})^L/k^L\right) = 1$$

(see, for example, [13]). The vector space k^L plays the rôle of the lattice. An overview of recent results in the geometry of numbers over the adèles may be found in [3]. Finally, for an idèle $\delta = (\delta_v)$ we define the *volume* of δ as

$$V(\delta) = \prod_v |\delta_v|_v.$$

3. A result from the geometry of numbers over the adèles

For each place v of k let $R_v \subseteq (k_v)^L$ be a nonempty set. If $v|\infty$ we assume that R_v is convex, symmetric and bounded with nonempty interior. If $v \nmid \infty$ we assume that R_v is a k_v-lattice, that is, a compact open $\mathcal{O}_v$-module. We further assume that for almost all finite v, $R_v = (\mathcal{O}_v)^L$. We define the set

$$\mathscr{R} = \prod_v R_v.$$

From our previous assumptions it is clear the $\mathscr{R} \subseteq (k_{\mathbf{A}})^L$. We call a subset $\mathscr{R}$ *admissible* if it has the form described above. For $\sigma > 0$ we define the dilation $\sigma\mathscr{R}$ by

$$\sigma\mathscr{R} = \prod_{v|\infty} (\sigma R_v) \times \prod_{v \nmid \infty} R_v.$$

We now recall the definition of the successive minima $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_L < \infty$ of $\mathscr{R}$. We define

$$\lambda_l = \inf\{\sigma > 0 \colon \sigma\mathscr{R} \cap k^L \text{ contains } l \text{ linearly independent vectors}\}.$$

The adelic analogue of Minkowski's successive minima theorem (see [1], Theorem 3) states that

$$(\lambda_1\lambda_2 \cdots \lambda_L)^d V(\mathscr{R}) \leq 2^{dL}.$$

Another natural constant associated with the admissible set $\mathscr{R}$ is its inhomogeneous minimum $\mu(\mathscr{R})$ defined by

$$\mu = \mu(\mathscr{R}) = \inf\left\{\sigma > 0 \colon (k_{\mathbf{A}})^L \subseteq \bigcup_{\vec{\xi} \in k^L} \left(\sigma\mathscr{R} + \vec{\xi}\,\right)\right\}.$$

The proof of Theorem 1 will require an adelic version of an inequality of Hlawka [6] which relates the inhomogeneous minimum to the first successive minima. Toward this end we require a lemma which easily follows from a general formulation of Blichfeldt's theorem.

LEMMA 3. *For $N \geq 2$, let $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N$ be Borel measurable subsets of $(k_{\mathbf{A}})^L$ satisfying $\sum_{n=1}^N V(\mathscr{T}_n) > 1$. If none of the sets contain two distinct vectors $\vec{\alpha}$ and $\vec{\beta}$ so that $\vec{\alpha} - \vec{\beta} \in k^L$, then there exist indices i and j, $i \neq j$, and a vector $\vec{\zeta} \in k^L$ so that $\vec{\zeta} \in \mathscr{T}_i - \mathscr{T}_j$.*

Proof. If the sets $\{\mathscr{T}_n\}$ are not pairwise disjoint, then the lemma is trivial and $\vec{\zeta}$ may be taken to be $\vec{0}$. We assume now that the sets $\{\mathscr{T}_n\}$ are pairwise disjoint. Thus we have

$$V\left(\bigcup_{n=1}^{N} \mathscr{T}_n\right) = \sum_{n=1}^{N} V(\mathscr{T}_n) > 1 = \bar{V}\left((k_{\mathbf{A}})^L/k^L\right).$$

Hence by Blichfeldt's theorem in this setting (see [13], Chapter II, Section 4, Lemma 1), there exist distinct vectors $\vec{\alpha}$ and $\vec{\beta}$ in $\bigcup_{n=1}^N \mathscr{T}_n$ so that $\vec{\alpha} - \vec{\beta} \in k^L$. By hypothesis $\vec{\alpha}$ and $\vec{\beta}$ cannot be elements of the same set and therefore there must exist distinct indices i and j so that $\vec{\alpha} \in \mathscr{T}_i$ and $\vec{\beta} \in \mathscr{T}_j$. The vector $\vec{\zeta}$ may now be taken to be $\vec{\alpha} - \vec{\beta}$ which completes the proof.

THEOREM 4. *Let $\mathscr{R}$ be an admissible subset of $(k_{\mathbf{A}})^L$ with λ_1 and μ as its first successive minima and inhomogeneous minimum, respectively. Then*

$$\mu \leq \tfrac{1}{2}\lambda_1([\gamma] + 1),$$

where $\gamma = (2/\lambda_1)^{dL} V(\mathscr{R})^{-1}$ and $[x]$ denotes the integer part of x.

Proof. Let $\vec{\varphi} \in (k_{\mathbf{A}})^L$ be a fixed vector. We define the *inhomogeneous minimum of $\mathscr{R}$ with respect to $\vec{\varphi}$* by

$$\mu_{\vec{\varphi}} = \mu_{\vec{\varphi}}(\mathscr{R}) = \inf\left\{\sigma > 0 : \vec{\varphi} \in \bigcup_{\vec{\xi} \in k^L} \left(\sigma\mathscr{R} + \vec{\xi}\,\right)\right\}.$$

Let $\vec{\zeta}_0 \in k^L$ be a fixed vector so that for all $\sigma > \mu_{\vec{\varphi}}$,

$$\vec{\varphi} \in \sigma\mathscr{R} + \vec{\zeta}_0. \tag{3.1}$$

We remark that $\mu_{\vec{\varphi}}$ is related to the inhomogeneous minimum μ via the identity

$$\mu(\mathscr{R}) = \sup_{\vec{\eta} \in (k_{\mathbf{A}})^L} \mu_{\vec{\eta}}(\mathscr{R}). \tag{3.2}$$

Next we let $\vec{\varphi}_0 = \vec{\varphi} - \vec{\zeta}_0$ and $N = [\gamma] + 1$. By the successive minima theorem we have that $N \geq 2$. We now define the vectors $\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_N$ in $(k_{\mathbf{A}})^L$ by

$$\vec{\varphi}_n = (2nN^{-1})\vec{\varphi}_0,$$

for $n = 1, 2, \ldots, N$. Let $\operatorname{int}(\mathscr{R})$ be the *interior* of $\mathscr{R}$, that is,

$$\operatorname{int}(\mathscr{R}) = \prod_{v \mid \infty} \operatorname{int}(R_v) \times \prod_{v \nmid v} R_v.$$

We define sets $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N$ in $(k_{\mathbf{A}})^L$ by

$$\mathscr{T}_n = \tfrac{1}{2}\lambda_1 \operatorname{int}(\mathscr{R}) + \vec{\varphi}_n,$$

for $n = 1, 2, \ldots, N$ and observe that

$$\sum_{n=1}^{N} V(\mathscr{T}_n) = \sum_{n=1}^{N} V\left(\tfrac{1}{2}\lambda_1 \operatorname{int}(\mathscr{R})\right) = N\left(\tfrac{1}{2}\lambda_1\right)^{dL} V(\mathscr{R}) > 1.$$

Finally we we note that

$$\lambda_1 \operatorname{int}(\mathscr{R}) \cap k^L = \{\vec{0}\}$$

and thus it follows that the sets $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N$ satisfy the hypotheses of Lemma 3. Hence there exist indices i and j with $1 \leq i < j \leq N$ and a vector $\vec{\zeta} \in k^L$ so that $\vec{\zeta} \in \mathscr{T}_i - \mathscr{T}_j$. That is,

$$\vec{\zeta} \in \lambda_1 \mathscr{R} + \left(\vec{\varphi}_i - \vec{\varphi}_j\right)$$

or

$$\vec{\zeta} - 2(i-j)N^{-1}\vec{\varphi}_0 \in \lambda_1 \mathscr{R}. \tag{3.3}$$

Suppose that $\sigma > \mu_{\vec{\varphi}}$. In view of (3.1), (3.3) and the fact that $\mathscr{R}$ is an admissible subset of $(k_{\mathbf{A}})^L$ we have

$$\begin{aligned}
\vec{\varphi} + \left(\vec{\zeta} - \vec{\zeta}_0\right) &= \vec{\varphi}_0 + \vec{\zeta} = (1 + 2(i-j)N^{-1})\vec{\varphi}_0 - 2(i-j)N^{-1}\vec{\varphi}_0 + \vec{\zeta} \\
&= (1 + 2(i-j)N^{-1})\left(\vec{\varphi} - \vec{\zeta}_0\right) + \vec{\zeta} - 2(i-j)N^{-1}\vec{\varphi}_0 \\
&\in \left(|1 + 2(i-j)N^{-1}|\sigma\right)\mathscr{R} + \lambda_1 \mathscr{R} \\
&= \left(N^{-1}|N + 2(i-j)|\sigma\right)\mathscr{R} + \lambda_1 \mathscr{R} \\
&\subseteq \left(N^{-1}(N-2)\sigma + \lambda_1\right)\mathscr{R}.
\end{aligned}$$

Thus

$$\mu_{\vec{\varphi}} \le N^{-1}(N-2)\sigma + \lambda_1.$$

If we let $\sigma \to \mu_{\vec{\varphi}}$ from above, the previous inequality reveals that

$$\mu_{\vec{\varphi}} \le (N/2)\lambda_1.$$

In view of identity (3.2) we conclude that

$$\mu = \sup_{\vec{\varphi} \in (k_{\mathbf{A}})^L} \mu_{\vec{\varphi}} \le \tfrac{1}{2}\lambda_1([\gamma] + 1)$$

which completes the proof.

COROLLARY 5. *Let $\mathscr{R}$, λ_1 and μ be as in Theorem 4. If $\lambda_1 > 1$, then*

$$\mu \le V(\mathscr{R})^{-1/dL}\left(\left[2^{dL}V(\mathscr{R})^{-1}\right] + 1\right).$$

Proof. Since $\lambda_1 > 1$, it follows that $\gamma \le 2^{dL}V(\mathscr{R})^{-1}$ where γ is the constant defined in Theorem 4. By the successive minima theorem,

$$(\lambda_1\lambda_2 \cdots \lambda_L)^d V(\mathscr{R}) \le 2^{dL}$$

and so $\frac{1}{2}\lambda_1 \le V(\mathscr{R})^{-1/(dL)}$. The corollary now follows from Theorem 4.

Remark. In the classical geometry of numbers one is able to construct examples for which Hlawka's inequality is relatively sharp. Similarly, one may construct examples in this setting to indicate that the inequality of Theorem 4 cannot be substantially improved. As an illustration, we consider the adèle ring $\mathbf{Q}_{\mathbf{A}}$ and define the admissible set

$$\mathscr{R} = R_\infty \times \prod_{p \text{ prime}} R_p \subseteq (\mathbf{Q}_{\mathbf{A}})^L$$

as follows. Let $\alpha > 1$ and define the $L \times L$ real matrix A by

$$A = \begin{pmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & 1 & \\ & & & & \alpha \end{pmatrix}.$$

We define

$$R_\infty = \{\vec{x} \in \mathbf{R}^L : |A\vec{x}|_\infty \leq 1\}$$

and $R_p = (\mathbf{Z}_p)^L$ for all primes p, where $\mathbf{Z}_p$ is the ring of p-adic integers. It is a straight-forward calculation to verify that $\lambda_1 = 1$ and $V(\mathscr{R}) = 2^L\alpha$, so in particular,

$$\alpha = (2/\lambda_1)^L V(\mathscr{R})^{-1}.$$

Finally, $\mu(\mathscr{R}) = \frac{1}{2}\alpha = \frac{1}{2}\lambda_1\alpha$ while the inequality of Theorem 4 reveals that

$$\mu \leq \tfrac{1}{2}\lambda_1([\alpha] + 1).$$

4. A transference theorem over number fields

Let B_v be an $L \times L$ nonsingular matrix over k_v for each place v of k. Let $R_v \subseteq (k_v)^L$ be defined by

$$R_v = \{\vec{z} \in (k_v)^L : \|B_v\vec{z}\|_v \leq 1\}.$$

Suppose that for almost all places v, $R_v = (\mathscr{O}_v)^L$. Thus if we let $\mathscr{R} = \prod_v R_v$, then $\mathscr{R}$ is an admissible subset of $(k_{\mathbf{A}})^L$.

LEMMA 6. *Let $\mathscr{R}$ be the admissible set described above with λ_1 as its first successive minima. If $\lambda_1 > 1$, then for any $\vec{\zeta} = (\vec{\zeta}_v) \in (k_{\mathbf{A}})^L$, there exists a vector $\vec{z} \in k^L$ such that*

$$\|B_v\vec{z} - \vec{\zeta}_v\|_v \leq T_v \tag{4.1}$$

for all places v of k, where

$$T_v = \begin{cases} V(\mathscr{R})^{-1/dL}\left(\left[2^{dL}V(\mathscr{R})^{-1}\right] + 1\right) & \text{for } v|\infty \\ 1 & \text{for } v \nmid \infty. \end{cases}$$

Proof. By Corollary 5 we have

$$(k_{\mathbf{A}})^L \subseteq \bigcup_{\vec{\xi} \in k^L} \left\{\left(V(\mathscr{R})^{-1/dL}\left(\left[2^{dL}V(\mathscr{R})^{-1}\right] + 1\right)\right)\mathscr{R} + \vec{\xi}\right\}. \tag{4.2}$$

Define $\vec{\psi} = (\vec{\psi}_v) \in (k_{\mathbf{A}})^L$ by $\vec{\psi}_v = B_v^{-1}\vec{\zeta}_v$ for each place v of k. By the containment of (4.2), there exists a vector $\vec{z} \in k^L$ such that

$$\vec{z} - \vec{\psi} \in \left(V(\mathscr{R})^{-1/dL}\left(\left[2^{dL}V(\mathscr{R})^{-1}\right] + 1\right)\right)\mathscr{R}. \tag{4.3}$$

In view of the definition of $\mathscr{R}$ and $\vec{\psi}$, (4.3) is identical to the inequalities of (4.1).

THEOREM 7. *For each $v \in S$, let A_v be an $M \times N$ matrix over k_v and ε_v, δ_v be two nonzero elements of k_v with $\|\varepsilon_v\|_v < 1$. Let*

$$\Phi = \Phi(\{\varepsilon_v\}_{v\in S}, \{\delta_v\}_{v\in S})$$

be defined by

$$\Phi = \prod_{v\in S}\left(\|\varepsilon_v\|_v^M\|\delta_v\|_v^N\right)^{d/d_v}.$$

Suppose that there are no vectors $\vec{x} \in (\mathscr{O}_S)^N \setminus \{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$ satisfying

$$\|A_v\vec{x} - \vec{y}\|_v \le \|\varepsilon_v\|_v \quad \text{and} \quad \|\vec{x}\|_v \le \|\delta_v\|_v$$

for all $v \in S$. Then for any vector $\vec{\beta} = (\vec{\beta}_v) \in (k_S)^M$, there exist $\vec{x} \in (\mathscr{O}_S)^N$ and $\vec{y} \in (\mathscr{O}_S)^M$ such that

$$\|A_v\vec{x} - \vec{y} - \vec{\beta}_v\|_v \le \rho'_v\|\varepsilon_v\|_v \quad \text{and} \quad \|\vec{x}\|_v \le \rho'_v\|\delta_v\|_v$$

for each $v \in S$ where

$$\rho'_v = \begin{cases} \frac{1}{2}(c_k)(\Phi)^{-1/(M+N)}\left(\left[(c_k)^{d(M+N)}(\Phi)^{-d}\right] + 1\right) & \text{for } v\mid\infty \\ 1 & \text{for } v \nmid \infty, \end{cases}$$

$$c_k = \left(\left(\frac{2}{\pi}\right)^s|\Delta_k|^{1/2}\right)^{1/d}$$

and s is the number of complex places of k and Δ_k is the discriminant of k.

Proof. Set $L = M + N$ and for each $v \in S$ define the $L \times L$ nonsingular matrix B_v by blocks as

$$B_v = \left(\begin{array}{c|c} \delta_v^{-1}\mathbf{1}_N & 0 \\ \hline \varepsilon_v^{-1}A_v & \varepsilon_v^{-1}\mathbf{1}_M \end{array}\right)$$

and for $v \notin S$ set $B_v = \mathbf{1}_L = \mathbf{1}_{(M+N)}$ where $\mathbf{1}_L$ is the $L \times L$ identity matrix. Next define $R_v \subseteq (k_v)^L$ by

$$R_v = \left\{ \vec{z} \in (k_v)^L : \|B_v \vec{z}\|_v \leq 1 \right\}$$

and write $\mathscr{R} = \prod_v R_v \subseteq (k_{\mathbf{A}})^L$. Clearly

$$|\det B_v|_v = \begin{cases} |\varepsilon_v|_v^{-M} |\delta_v|_v^{-N} & \text{for } v \in S \\ 1 & \text{for } v \notin S \end{cases}$$

and thus (see identity (2.2) in [2]),

$$V(\mathscr{R}) = 2^{dL} \left(\frac{\pi}{2} \right)^{sL} |\Delta_k|^{-L/2} (\Phi)^d.$$

We now demonstrate that the first successive minima, λ_1, of $\mathscr{R}$ is greater than one. Suppose instead that $\lambda_1 \leq 1$. Then there must exist vectors $\vec{x} \in k^N$ and $\vec{y} \in k^M$, not *both* identically zero, such that

$$\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathscr{R}.$$

That is, $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{y} \in (\mathscr{O}_S)^M$ and

$$\|A_v \vec{x} - \vec{y}\|_v \leq \|\varepsilon_v\|_v \quad \text{and} \quad \|\vec{x}\|_v \leq \|\delta_v\|_v \tag{4.4}$$

for all $v \in S$. Since we are given that there are no S-integer solutions to (4.4) with $\vec{x} \neq \vec{0}$, it follows that $\vec{x} = \vec{0}$ and so $\vec{y} \neq \vec{0}$. From (4.4) we conclude that

$$\prod_v |\vec{y}|_v \leq \prod_{v \in S} |\varepsilon_v|_v < 1.$$

By the product formula this implies $\vec{y} = \vec{0}$ which is a contradiction. Thus it must be the case that $\lambda_1 > 1$ and hence $\mathscr{R}$ satisfies the hypothesis of Lemma 6.

We now define the vector $\vec{\zeta} = (\vec{\zeta}_v) \in (k_{\mathbf{A}})^L$ by

$$\vec{\zeta}_v = \begin{cases} \begin{pmatrix} \vec{0} \\ \varepsilon_v^{-1} \vec{\beta}_v \end{pmatrix} & \text{for } v \in S \\ \vec{0} & \text{for } v \notin S. \end{cases}$$

By Lemma 7, there exists a vector

$$\vec{z} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in k^L,$$

where $\vec{x} \in k^N$ and $\vec{y} \in k^M$, such that

$$\left\| B_v \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} - \vec{\zeta}_v \right\|_v \leq T_v$$

for all places v of k. Thus for $v \notin S$,

$$\left\| \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \right\|_v \leq 1,$$

so $\vec{x} \in (\mathscr{O}_S)^N$ and $\vec{y} \in (\mathscr{O}_S)^M$. For $v \in S$,

$$\|A_v \vec{x} - \vec{y} - \vec{\beta}_v\|_v \leq T_v \|\varepsilon_v\|_v$$

and

$$\|\vec{x}\|_v \leq T_v \|\delta_v\|_v.$$

This completes the proof of the theorem in view of the definition of T_v and the calculation for $V(\mathscr{R})$.

5. Badly approximable S-systems

Let $\{A_v\}_{v \in S}$ be a badly approximable S-system of dimension $M \times N$. Let $\tau = \tau(k, \{A_v\}_{v \in S}) > 0$ be a real constant such that $\tau \leq 1$ and

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v \in S} |A_v \vec{x} - \vec{y}|_v^M$$

for all $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{x} \neq \vec{0}$ and $\vec{y} \in (\mathscr{O}_S)^M$. Next we define the constant $C_\tau = C_\tau(\{A_v\}_{v \in S})$ by

$$C_\tau = \left(2N\Omega(\{A_v\}_{v \in S})\right)^{-N/M} \tau^{1/M},$$

where $\Omega(\{A_v\}_{v\in S}) = \prod_{v\in S}\max\{1, |A_v|_v\}$. We trivially remark that $0 < C_\tau \leq 1$. For each place $v \in S$, we select $\tau_v \in k_v$ such that $0 < |\tau_v|_v \leq 1$ and

$$\prod_{v\in S} |\tau_v|_v = C_\tau.$$

Finally we define the constant $\rho_v = \rho_v(k, C_\tau) \in k_v$ for $v \in S$ by

$$\rho_v = \begin{cases} \frac{1}{2}(c_k)(C_\tau)^{-M/(M+N)}\Big(\Big[(c_k)^{d(M+N)}(C_\tau)^{-dM}\Big] + 1\Big) & \text{for } v|\infty \\ 1 & \text{for } v \nmid \infty. \end{cases}$$

We now give a proof of Theorem 1 which we restate here with explicit constants.

THEOREM 8. *Let $\{A_v\}_{v\in S}$ be a badly approximable S-system of dimension $M \times N$. Let τ, $\{\tau_v\}_{v\in S}$ be as defined above. For each $v \in S$ suppose that $\varepsilon_v \in k_v$ satisfies $0 < \|\varepsilon_v\|_v < 1$. Then for any $\vec{\beta} = (\vec{\beta}_v) \in (k_S)^M$, there exist vectors $\vec{x} \in (\mathscr{O}_S)^N$ and $\vec{y} \in (\mathscr{O}_S)^M$ such that*

$$\|A_v\vec{x} - \vec{y} - \vec{\beta}_v\|_v \leq \rho_v\|\tau_v\|_v\|\varepsilon_v\|_v^N$$

and

$$\|\vec{x}\|_v \leq \rho_v\|\varepsilon_v\|_v^{-M}$$

for all $v \in S$.

Proof. For each $v \in S$ set

$$\varepsilon'_v = \varepsilon_v^N\tau_v \quad \text{and} \quad \delta_v = \varepsilon_v^{-M}.$$

Suppose now that there exist vectors $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{x} \neq \vec{0}$ and $\vec{y} \in (\mathscr{O}_S)^M$ so that

$$\|A_v\vec{x} - \vec{y}\|_v \leq \|\varepsilon'_v\|_v \quad \text{and} \quad \|\vec{x}\|_v \leq \|\delta_v\|_v$$

for all $v \in S$. From the inequalities of (4.4) of [2] we have

$$\prod_{v\in S}|A_v\vec{x} - \vec{y}|_v \leq \prod_{v\in S}|\varepsilon'_v|_v$$

and

$$h_S(\vec{x}, \vec{y}) \leq 2N\Omega(\{A_v\}_{v\in S})\prod_{v\in S}\max\{|\varepsilon'_v|_v, |\delta_v|_v\}.$$

Therefore

$$\begin{aligned}
\tau &< h_S(\vec{x}, \vec{y})^N \prod_{v \in S} |A_v \vec{x} - \vec{y}|_v^M \\
&\leq \left(2N\Omega(\{A_v\}_{v \in S})\right)^N \prod_{v \in S} \max\{|\varepsilon'_v|_v, |\delta_v|_v\}^N \prod_{v \in S} |\varepsilon'_v|_v^M \\
&= \left(2N\Omega(\{A_v\}_{v \in S})\right)^N \prod_{v \in S} |\delta_v|_v^N |\varepsilon'_v|_v^M \\
&= \left(2N\Omega(\{A_v\}_{v \in S})\right)^N \prod_{v \in S} |\tau_v|_v^M \\
&= \tau
\end{aligned}$$

which is impossible. Hence there are *no* vectors $\vec{x} \in (\mathscr{O}_S)^N$, $\vec{x} \neq \vec{0}$, $\vec{y} \in (\mathscr{O}_S)^M$ satisfying

$$\|A_v \vec{x} - \vec{y}\|_v \leq \|\varepsilon'_v\|_v \quad \text{and} \quad \|\vec{x}\|_v \leq \|\delta_v\|_v$$

for all $v \in S$. We now apply Theorem 7 with $\{A_v\}_{v \in S}$, $\{\varepsilon'_v\}_{v \in S}$ and $\{\delta_v\}_{v \in S}$ and observe that the ρ'_v's of Theorem 7 are equal to the ρ_v's defined here, which completes the proof.

6. A quantitative formulation of Kronecker's theorem over k

Again we write $\mu = \mu(\mathscr{R})$ for the inhomogeneous minimum of $\mathscr{R}$ and $\lambda_1, \lambda_2, \ldots, \lambda_L$ for its successive minima. Recently O'Leary and Vaaler formulated an adelic version of an inequality due to Jarnik. In particular they proved ([11], Theorem 5) that

$$\mu \leq \nu(k)(\lambda_1 + \lambda_2 + \cdots + \lambda_L), \tag{6.1}$$

where the field constant $\nu(k)$ is defined as follows. Let

$$\mathscr{S}_k = \prod_v \mathscr{O}_v \subseteq k_{\mathbf{A}}$$

be the admissible subset defined by

$$\mathscr{O}_v = \begin{cases} \{x \in k_v : \|x\|_v < 1\} & \text{for } v \mid \infty \\ \{x \in k_v : \|x\|_v \leq 1\} & \text{for } v \nmid \infty, \end{cases}$$

and define $\nu(k) = \mu(\mathscr{S}_k)$. So, for example, $\nu(\mathbf{Q}) = \frac{1}{2}$. General estimates for $\nu(k)$ in terms of classical field constants are given in Section 7 of [11]. Finally, we define the constant $c_k(M, N)$ by

$$c_k(M, N) = \nu(k)(M + N)\{c'_k(M, N)\}^{1/d},$$

where

$$c'_k(M, N) = \frac{(M + N)!^{2r}[(2(M + N))!]^{2s}|\Delta_k|^{M+N}}{4^{s(M+N)}},$$

and r and s are the number of real and complex places of k, respectively.

We now prove Theorem 2 which we state below with explicit constants. The proof is an adelic adaptation of arguments given by Khintchine [7] and Mahler [8].

THEOREM 9. *Let $A = (A_v)$ be an $M \times N$ matrix over the adèle ring $k_{\mathbf{A}}$. Suppose there exist idèles $\varepsilon = (\varepsilon_v)$ and $\delta = (\delta_v)$, with volume $V(\delta) > 1$, so that the system of inequalities*

$$\|A_v^T \vec{u} - \vec{w}\|_v \le \|\delta_v\|_v^{-1} \quad \textit{for each place } v \textit{ of } k,$$

is not solvable with $\vec{u} \in k^M$, $\vec{w} \in k^N$ and

$$0 < \|\vec{u}\|_v \le \|\varepsilon_v\|_v^{-1} \quad \textit{for each place } v \textit{ of } k.$$

Then for any $\vec{\beta} = (\vec{\beta}_v) \in (k_{\mathbf{A}})^M$, there exist vectors $\vec{x} \in k^N$ and $\vec{y} \in k^M$ so that for each place v of k,

$$\|A_v \vec{x} - \vec{y} - \vec{\beta}_v\|_v \le C(k, v, M, N)\|\varepsilon_v\|_v$$

and

$$\|\vec{x}\|_v \le C(k, v, M, N)\|\delta_v\|_v,$$

where $C(k, v, M, N) = c_k(M, N)$ if $v|\infty$ and $C(k, v, M, N) = 1$ otherwise.

Proof. If $L = M + N$, we define the $L \times L$ matrix B_v over k_v by

$$B_v = \left(\begin{array}{c|c} \delta_v^{-1}\mathbf{1}_N & 0 \\ \hline \varepsilon_v^{-1}A_v & \varepsilon_v^{-1}\mathbf{1}_M \end{array}\right),$$

and define $R_v \subseteq (k_v)^L$ by

$$R_v = \left\{\vec{z} \in (k_v)^L : |||B_v \vec{z}|||_v \le 1\right\}.$$

We note that for each place v, the polar body, R_v^*, is given by

$$R_v^* = \left\{\vec{z} \in (k_v)^L : \left\|(B_v^T)^{-1}\vec{z}\right\|_v \le 1\right\}$$

where

$$(B_v^T)^{-1} = \left(\begin{array}{c|c} \delta_v \mathbf{1}_N & -\delta_v A_v^T \\ \hline 0 & \varepsilon_v \mathbf{1}_M \end{array}\right)$$

(for an analysis of polar bodies in the nonarchimedean setting, see [2], Section 3). Let $\mathscr{R} = \prod_v R_v$ and $\mathscr{R}^* = \prod_v R_v^*$. We now show that the first successive minima, λ_1^*, of $\mathscr{R}^*$ satisfies $\lambda_1^* \ge 1$. If $\lambda_1^* < 1$, then there must exist vectors $\vec{u} \in k^M$ and $\vec{w} \in k^N$ so that

$$\begin{pmatrix} \vec{w} \\ \vec{u} \end{pmatrix} \ne \vec{0} \quad \text{and} \quad \begin{pmatrix} \vec{w} \\ \vec{u} \end{pmatrix} \in \mathscr{R}^*.$$

Thus, for each place v of k,

$$\|A_v^T\vec{u} - \vec{w}\|_v \le \|\delta_v\|_v^{-1} \quad \text{and} \quad \|\vec{u}\|_v \le \|\varepsilon_v\|_v^{-1}.$$

If $\vec{u} = \vec{0}$ then

$$\prod_v |\vec{w}|_v \le \prod_v |\delta_v|_v^{-1} < 1.$$

By the product formula, this implies that $\vec{w} = \vec{0}$ which is impossible. Therefore; it must be the case that $\vec{u} \ne \vec{0}$. Hence, for all places v of k, we have

$$0 < \|\vec{u}\|_v \le \|\varepsilon_v\|_v^{-1} \quad \text{and} \quad \|A_v^T\vec{u} - \vec{w}\|_v \le \|\delta_v\|_v^{-1}.$$

But this contradicts the hypothesis which states that the previous system is unsolvable. This contradiction implies that $\lambda_1^* > 1$.

By inequality (6.1) we have

$$\mu(\mathscr{R}) \le \nu(k) L \lambda_L. \tag{6.2}$$

We now recall the adelic analogue of an inequality due to Mahler (Theorem 3.7 of [2]):

$$(\lambda_1^* \lambda_L)^d \le c_k'(M, N). \tag{6.3}$$

Inequalities (6.2) and (6.3) along with $\lambda_1^* > 1$ combine to give

$$\mu(\mathscr{R}) \leq c_k(M, N).$$

Thus we have

$$(k_{\mathbf{A}})^L \subseteq \bigcup_{\vec{\xi} \in k^L} \left\{ c_k(M, N)\mathscr{R} + \vec{\xi} \right\}.$$

Let $\vec{\psi} = (\vec{\psi}_v) \in (k_{\mathbf{A}})^L$ be defined by

$$\vec{\psi}_v = \begin{pmatrix} \vec{0} \\ \varepsilon_v^{-1} B_v^{-1} \vec{\beta}_v \end{pmatrix}.$$

Then there exists a vector $\vec{\xi} \in k^L$ so that

$$\vec{\xi} - \vec{\psi} \in c_k(M, N)\mathscr{R}.$$

Alternatively, we have

$$\left|\left|\left| B_v\left(\vec{\xi} - \vec{\psi}_v\right) \right|\right|\right|_v \leq \begin{cases} c_k(M, N) & \text{for } v|\infty \\ 1 & \text{for } v \nmid \infty. \end{cases}$$

The theorem now follows by partitioning $\vec{\xi}$ as

$$\vec{\xi} = \begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix}.$$

In fact we have produced slightly stronger inequalities over the archimedean places than required since plainly

$$\|\vec{\zeta}_v\|_v \leq |||\vec{\zeta}_v|||_v$$

for all $v|\infty$.

References

1. E. Bombieri and J. Vaaler, *On Siegel's lemma*, Invent. Math. **73** (1983), 11–32.
2. E.B. Burger, *Homogeneous diophantine approximation in S-integers*, Pacific J. Math. **152** (1992), 211–253.
3. ______, "Bad approximable systems and inhomogeneous approximation over number fields" in *Number theory with an emphasis on the Markoff spectrum*, A. Pollington and W. Moran, eds., New York, Marcel Dekker, 1993.
4. D.G. Cantor, *On the elementary theory of diophantine approximation over the ring of adeles I*, Illinois J. Math. **9** (1965), 677–700.

5. P.M. Gruber and C.G. Lekkerkerker, *Geometry of numbers*, North-Holland Mathematical Library, Vol. 37, North-Holland, Amsterdam, 1987.
6. E. Hlawka, *Zur Theorie des Figurengitters*, Math. An. **125** (1952), 183–207.
7. A.Ya. Khintchine, *On a limiting case of Kronecker's approximation theorem*, Doklady Akad. Nauk SSSR (N.S.) **56** (1947), 563–565.
8. K. Mahler, *A remark on Kronecker's theorem*, Enseignement Math. (2) **12** (1966), 183–189.
9. ______, *A Theorem on inhomogeneous Diophantine Inequalities*, Proc. Kon. Ned. Akad. Wet. **41** (1938), 634–637.
10. ______, *Lectures on diophantine approximations, Part 1: g-adic numbers and Roth's Theorem*, University of Notre Dame Press, South Bend, Indiana, 1961.
11. R. O'Leary and J.D. Vaaler, *Small solutions to inhomogeneous linear equations over number fields*, Trans. Amer. Math. Soc., to appear.
12. M.M. Sweet, *An asymptotic formula in adele diophantine approximations*, Trans. Amer. Math. Soc. **195** (1974), 81–96.
13. A. Weil, *Basic number theory*, Springer-Verlag, New York, 1974.

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