ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 3, Fall 1994

FORMALIZABILITY OF DG MODULES AND MORPHISMS OF CDG ALGEBRAS

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The aim of this paper is to study homotopy types, more precisely, formality, of certain local systems over compact Kähler manifolds, following works of J. Morgan [Mor] and V. Navarro Aznar [Na_1], [Na_2]. Global sections of these local systems can be seen as dg modules over cdg algebras. Following [Sul], we prove that formalizability of such dg modules does not depend on the ground field (Theorem 2.2). Results and proofs are easily translated for cdg algebra morphisms, so we develope the case of dg modules in detail and confine ourselves to state them for cdg algebra morphisms, remarking differences whenever they can arise. For both situations, our principal tool is the minimal model.

D. Sullivan's theory of the minimal model says that, for a rational space X, the $\mathbf{Q}$-homotopy type is determined by a minimal model of the $\mathbf{Q}$-cdg algebra $A_{PL}(X)$ (see [Sul]). For certain spaces, this minimal model, and so its homotopy type, is a formal consequence of its rational cohomology algebra $H^*(X;\mathbf{Q})$. They are called *formal* spaces. Among them one can find Lie groups, classifying spaces, compact Kähler manifolds... Formality of these latter ones was proved in [D-G-M-S] over the real numbers. The descent of formality from $\mathbf{R}$ to $\mathbf{Q}$ is proved in [Sul]. This is done in two steps: first, one gives a caracterization of formality in terms of the lifting property of automorphism from the cohomology algebra to the algebra [Sul, Theorem 12.7]. Second, one sees that this property does not depend on the ground field [Sul, Theorem 12.1]. For morphisms of cdg algebras, the first one of these results is due to Y. Felix and D. Tanré (see [F-T]). The second one can be found in [ViP_1]. In this paper, we give independent proofs of both results, which, in the line of Sullivan's, do not depend on the choice of a particular construction of models with extra structure (namely, filtrations), but rather on abstract properties of formalizability and minimality (Theorems 3.1 and 3.2).

The paper is organized as follows: in § 1, we give some preliminaries which will allow us to translate Sullivan's results from cdg algebras to dg modules and cdg algebra morphisms. Particularly, we define the notions of formalizability and minimality in abstract terms in such a way that they include the cases of cdg algebras, dg modules over cdg algebras, morphisms of cdg

Received June 25, 1992
1991 Mathematics Subject Classification. Primary S5P62; Secondary S5U35.
[1]Partially supported by CICYT PS90-0069.

algebras and others. We check that these notions agree with the known ones in the examples. § 2 is dedicated to the statement and proof of the two theorems of formalizability for dg modules. Finally, in § 3 we translate the results and the idea of the proofs for cdg algebra morphisms.

I am indebted to Vicente Navarro Aznar for showing me the interest of the subject and for many helpful discussions. I am also grateful to Francisco Guillén and Pere Pascual-Gainza who have read this paper and made valuable suggestions.

1. Some preliminaries

We begin with some categorical definitions of formalizability and minimality which will allow us the mentioned generalization. Our definition of formalizability is done in terms of the *homotopy category* (see [Quil]): given a category $\mathscr{C}$ and a class of morphisms $S \subset \operatorname{mor}\mathscr{C}$, the homotopy category, $\operatorname{Ho}\mathscr{C}$, is the category obtained by adjoining to $\mathscr{C}$ the inverses of the morphisms of S. We are interested in classes of morphisms S obtained in the following way: let

$$\mathscr{D} \xrightarrow{\iota} \mathscr{C} \xrightarrow{H} \mathscr{D}$$

be a couple of functors such that $H\iota = 1_{\mathscr{D}}$. Take

$$S = \{s \in \operatorname{mor}\mathscr{C} \mid Hs \text{ is an isomorphism}\} \tag{1}$$

DEFINITION 1.1. An object x of $\mathscr{C}$ is said to be *(H, ι)-formalizable* (or simply *formalizable*) if it is isomorphic to ιHx in the homotopy category $\operatorname{Ho}\mathscr{C}$.

When $\mathscr{C}$ is a model category with all objects being fibrant, and S is the class of its weak equivalences, by [Qui], Corollary 2 of Theorem 1, Chapter I, § 1, S admits a *calculus of right fractions* "up to homotopy"; i.e., we have such a calculus in the category $\pi\mathscr{C}$, whose objects are the same as those of $\mathscr{C}$ and whose morphisms are the homotopy classes of morphisms of $\mathscr{C}$. This implies that, by taking representatives of homotopy classes, we may think of morphisms of the homotopy category $\operatorname{Ho}\mathscr{C}$ as being diagrams of $\mathscr{C}$ like

$$a \xleftarrow{s} \cdot \xrightarrow{f} b$$

with $s \in S$. If, in addition, $\mathscr{C}$ is a *closed* model category, by [Qui], Proposition 1, Chapter 1, § 5, isomorphisms of $\operatorname{Ho}\mathscr{C}$ are then exactly those diagrams with $s, f \in S$; otherwise, their description is more complicate. So, in this case, x is formalizable if and only if there exists a diagram in $\mathscr{C}$,

$$x \xleftarrow{s} \cdot \xrightarrow{f} \iota Hx \tag{2}$$

with $s, f \in S$. We will call such a diagram a *formalization* of x.

We will also need an abstract notion of minimality, which is inspired by the definition of *minimal* (R, r)*-algebras* of [H-T]:

DEFINITION 1.2. Let $\mathscr{C}$ be a category and $S \subset \operatorname{mor} \mathscr{C}$ a class of morphisms. An object m of $\mathscr{C}$ is *S-left minimal* (or *minimal* for short) if for all s: $x \to m \in S$, there exists s': $m \to x \in \operatorname{mor} \mathscr{C}$ such that $ss' = 1_m$.

From now on we will assume that isomorphism of $\mathscr{C}$ are in S, and that if in the diagram

$$x \xrightarrow{f} y \xrightarrow{g} z$$

of $\mathscr{C}$ two of the morphisms $\{f, g, gf\}$ are in S then so is the third (this is true if we choose S as in (1) or if it is the class of weak equivalences of a model category). Then, it is an easy exercise (see [Roig_3]) to prove:

PROPOSITION 1.3. *If m_1 and m_2 are minimal objects and s: $m_1 \to m_2 \in S$, then s is an isomorphism.*

Example 1.4. If $\mathbf{k}$ is a field of characteristic zero, $\mathscr{C} = \mathbf{CDGA}_{hc}(\mathbf{k})$ the *category of homologically connected* $\mathbf{k}$*-cdg algebras* and S is the class of *quasi-isomorphisms* (*quis*), i.e., morphisms which induce isomorphisms in cohomology, then the minimal objects of $\mathscr{C}$, in the sense of definition 1.2, are exactly the minimal cdg algebras of [Sul].

Example 1.5. For $A \in \operatorname{obj} \mathrm{CDGA}_{hc}(\mathbf{k})$ let $\mathbf{CDGA}_{hc}(A)$ be the *category* of *A-cdg algebras*; i.e., the category whose objects are morphisms of $\mathbf{k}$-cdg homologically connected algebras like $A \to B$ and whose morphisms are the commutative triangles of $\mathbf{CDGA}_{hc}(\mathbf{k})$ like

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B & \xrightarrow[f]{} & C \end{array}$$

For this category, take S the class of such triangles with f a *quis*. Then the minimal objects are the *KS-minimal extensions* of [Hal].

Example 1.6. Let $\mathbf{CDGA}_{hc}(\mathbf{k})^2$ be the *category of morphisms of* $\mathbf{CDGA}_{hc}(\mathbf{k})$. Precisely, it is the category whose objects are morphisms of the latter. A morphism between two objects λ: $A \to B$ and μ: $C \to D$ is a commutative diagram of $\mathbf{CDGA}_{hc}(\mathbf{k})$ like

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & D \\ {\scriptstyle\lambda}\uparrow & & \uparrow{\scriptstyle\mu} \\ A & \xrightarrow{f} & C \end{array}$$

We take for S the class of such diagrams with f and φ quis. Then the minimal objects of $\mathbf{CDGA}_{hc}(\mathbf{k})^2$ are the so-called *Λ-minimal Λ-extensions* of [Hal]; i.e., those λ: $A \to B$ with A a minimal $\mathbf{k}$-cdg algebra and f a KS-minimal extension.

Let $A \in$ obj $\mathbf{CDGA}_{hc}(\mathbf{k})$ and $\mathbf{DGM}(A)$ be the *category of non-negatively graded A-dg modules*. Results of this paper will apply to the following

Example 1.7. Let X be a topological space and $\mathscr{L}$ a local system over X. Taking the *Thom-Whitney derived functors* [Na_1] of the global sections of $\mathbf{C}$-valued functions and $\mathscr{L}$-valued functions on X, one obtains a cdg algebra $\mathbf{R}_{TW}\Gamma(X, \mathbf{C})$ and a $\mathbf{R}_{TW}\Gamma(X, \mathbf{C})$-dg module: $\mathbf{R}_{TW}\Gamma(X, \mathscr{L})$.

Let S be the class of *quis* of A-dg modules. The minimal objects of $\mathbf{DGM}(A)$ admit a caracterization analogous to the *KS*-minimal extensions of $\mathbf{k}$-cdg algebras. Precisely:

DEFINITION 1.8. Let M be a A-dg module and n a non negative integer. A *degree n Hirsch extension* of M is an inclusion of A-dg modules

$$M \to M \oplus_d (A \otimes_{\mathbf{k}} (V)_n)$$

in which

(1) V is a homogeneous $\mathbf{k}$-vector space of degree n,
(2) $A \otimes_{\mathbf{k}} (V)_n$ is the free A-dg module over V, and
(3) the differential of $M \oplus_d (A \otimes_{\mathbf{k}} (V)_n)$ sends V into M.

DEFINITION 1.9 (cf. [Hal], [Nav_2]). A *minimal KS-module* is an A-dg module M together with an exhaustive filtration $\{M(n,q)\}_{(n,q)\in I}$, indexed by $I = \{(n,q) \in \mathbf{Z} \times \mathbf{Z} \mid q \geq 0\}$ with lexicographical order, such that:

(1) $M(0,0)$ is the zero A-dg module,
(2) for $q > 0$, $M(n,q)$ is a degree n Hirsch extension of $M(n, q-1)$, and
(3) $M(n+1, 0) = \varinjlim_q M(n,q)$.

Minimal KS-modules are the minimal objects of $\mathbf{DGM}(A)$, in the sense of definition 1.2 (see [Roig_1] and [Roig_3], cf. [H-T]).

Let **DGM** be the *category of dg modules over all cdg algebras*. Objects in this category are couples (A, M) with A a cdg algebra and M an A-dg module. Morphisms are couples

$$(f, \varphi): (A, M) \to (B, N),$$

where $f: A \to B$ is a cdg algebra morphism and $\varphi: M \to f^*N$ is an A-dg module morphism; i.e., a morphism of dg modules *f-equivalent* (or an *f-morphism*). For this category we take S to be the class of couples (f, φ) where both f and φ are *quis*. Then the minimal objects are those couples (A,M) with A a minimal cdg algebra and M a minimal A-dg module.

DEFINITION 1.10. Let $x \in \mathrm{obj}\mathscr{C}$. An *S-left model* (or simply, a *model*) for x is a morphism $s: m \to x \in S$. We will say that it is an *S-left minimal model* (or simply, a *minimal model*) if m is a minimal object of $\mathscr{C}$.

Remark 1.11. In the preceding examples, the hypothesis about connected cohomology ensures us the existence of a minimal model for every object in the category (see [Hal] for $\mathbf{CDGA}_{hc}(\mathbf{k})^2$ and [Roig_1] for **DGM**). From next proposition it follows also that there are no more minimal objects, in the sense of Definition 1.2, that the ones we have made an explicit description.

PROPOSITION 1.12. *If $\mathscr{C}$ admits a class of objects $\mathscr{M}$ such that*
(a) *every object in $\mathscr{C}$ has a model in $\mathscr{M}$, and*
(b) *every morphism in S between objects of $\mathscr{M}$ is an isomorphism,*
then
(1) *objects of $\mathscr{M}$ are minimal, and*
(2) *every minimal object of $\mathscr{C}$ is isomorphic to an object of $\mathscr{M}$.*

Proof. Let $m \in \mathscr{M}$. Let us see that it is a minimal object: take $s: x \to m \in S$. Because of (a), there exists $t: m' \to x \in S$ with $m' \in \mathscr{M}$. Because of (b), $st \in S$ is an isomorphism: let u be its inverse. Then, taking $s' = tu$, we have $ss' = 1_m$ and so m is a minimal object.

Let m be a minimal object. Let us see that it is isomorphic to an object of $\mathscr{M}$: because of (a), there exists $m' \in \mathscr{M}$ and $s: m' \to m \in S$. Because of (1) and Proposition 1.3, s is an isomorphism. ∎

We will also need two more properties of minimal objects. The first one says that a minimal object does not admit non-trivial quasi-isomorphic subobjects.

PROPOSITION 1.13. *Let $s: a \to m$ be a representative of a subobject of m. If $s \in S$ and m is minimal, then s is an isomorphism.*

Proof. Because m is minimal, there exists $s'\colon m \to a$ such that $ss' = 1_m$. Then $s(s's) = s$ and so $s's = 1_a$ because s is a monomorphism. That is to say, s is an isomorphism. ■

The second one is a lifting property.

PROPOSITION 1.14. *Let $\mathscr{C}$ be a model category with all objects fibrant and S the class its of weak equivalences. Given a diagram in $\mathscr{C}$*

$$\begin{array}{ccc} & & x \\ & & {\scriptstyle s}\big\downarrow{\scriptstyle \sim} \\ m & \xrightarrow{f} & y \end{array}$$

in which $s \in S$ and m is a minimal object, then there exists a morphism $\tilde{f}$: $m \to x$ such that $s\tilde{f} \sim f$. This $\tilde{f}$ is unique up to homotopy.

Proof. By the properties of the calculus of right fractions (see [G-Z]) that we have in $\pi\mathscr{C}$, there exists a homotopy commutative diagram

$$\begin{array}{ccc} z & \xrightarrow{g} & x \\ {\scriptstyle r}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle s}\big\downarrow{\scriptstyle \sim} \\ m & \xrightarrow{f} & y \end{array}$$

in which $s \in S$. Since m is minimal, we have a section $r'\colon m \to z$ of r. Then, we take $\tilde{f} = gr'$ and have $s\tilde{f} = sgr' \sim frr' = f$.

The unicity is verified as follows: Let $h\colon m \to x$ be another morphism of $\mathscr{C}$ such that $sh \sim f$. So $s\tilde{f} = sh$ in $\pi\mathscr{C}$. Then, by the calculus of fractions, there exists $t\colon a \to m \in S$ such that $\tilde{f}t = ht$ in $\pi\mathscr{C}$. But m is minimal, so t has a section $t'\colon m \to a$. Thus $h = htt' \sim \tilde{f}tt' = \tilde{f}$. ■

COROLLARY. *Let $\mathscr{C}$ be a model category with S the class of its weak equivalences and let $s\colon m \to x$ and $s'\colon m' \to x$ be two minimal models of x. Then there exists an isomorphism $u\colon m \to m'$ such that $s'u \sim s$. This isomorphism is unique up to homotopy.*

In the preceding examples, categories have a closed model category structure taking weak equivalences to be the *quis* and with all objects being fibrant (see [Roig_1] and [Roig_2]). So we can take diagram (2) as a definition of formalizability. Also, we have a minimal model for every object. Hence, for these categories, the fact that x is formalizable is equivalent to x and ιHx having the same minimal model. That is to say, x is formalizable if and only if there exists a minimal object m and a diagram of $\mathscr{C}$,

$$x \xleftarrow{s} m \xrightarrow{f} \iota Hx,$$

with s and f weak equivalences. Moreover, we can choose f in such a way that $Hf = Hs$, by replacing it, if necessary, by $f' = (\iota Hs)(\iota Hf)^{-1}f$. For instance, if $m \in \operatorname{obj}\mathscr{C}$ is a minimal object, then m is formalizable if and only if there is a morphism $s\colon m \to \iota Hm \in S$, and we can choose s such that $Hs = 1_{Hm}$.

Example 1.15. The formalizable objects of $\mathscr{C} = \mathbf{CDGA}_{hc}(\mathbf{k})$ are the *formal* **k**-cdg algebras of [Sul]: $A \in \mathbf{CDGA}_{hc}(\mathbf{k})$ is formal if and only if there is a *quis* $M \to HA$, where M is a minimal model of A.

Example 1.16. For morphisms of $\mathscr{C} = \mathbf{CDGA}_{hc}(\mathbf{k})$, let us compare our definition with the one we find in [Tho], [L-S], or [F-T]. To this end, notice that the cohomology functor induces in an obvious way a functor from $\mathscr{C}^2 = \mathbf{Adgc}_{hc}(\mathbf{k})^2$ to the category of morphisms of cdg algebras with zero differential and that the morphisms of S in Example 1.6 are precisely those morphisms made invertible by this functor. Then our definition for a morphism of cohomologically connected **k**-cdg algebras says that $f\colon A \to B$ is formalizable if and only if there is a commutative diagram of $\mathscr{C}$,

$$\begin{array}{ccccc} B & \xleftarrow{\sigma} & M_{f\rho} & \xrightarrow{\sigma'} & HB \\ {\scriptstyle f}\uparrow & & {\scriptstyle f'}\uparrow & & {\scriptstyle Hf}\uparrow \\ A & \xleftarrow{\rho} & M_A & \xrightarrow{\rho'} & HA \end{array} \tag{3}$$

in which we have omitted the functor ι, M_A is a minimal model for A and for HA and so is f' for $f\rho$ and for $(Hf)\rho'$. The authors above mentioned ask the diagram only to be homotopy commutative (notice that, when talking about morphisms, [F-T] use the term "minimal model" in a different sense to the one we have defined). Nevertheless, both notions agree because, if in (3) squares commute only up to homotopy, we can replace σ and σ' by homotopic *quis* which make both rectangles commute.

Now we can prove in this general setting, the result that every automorphism of Hm lifts to an automorphism of m (see [Sul]). Precisely, let $\operatorname{Aut}(m)$ denote the set of automorphism of m in $\mathscr{C}$ and $\operatorname{Aut}(Hm)$ the corresponding set in $\mathscr{D}$.

PROPOSITION 1.17. *Let $\mathscr{C}$ be a closed model category in which weak equivalences are the class of morphism made invertible by some given functor H.*

Let $m \in \mathscr{C}$ be a formalizable minimal object. Then

$$H\colon \mathrm{Aut}(m) \to \mathrm{Aut}(Hm)$$
$$\varphi \mapsto H\varphi$$

is a surjective map.

Proof. Let $\phi \in \mathrm{Aut}\,(Hm)$ and $s\colon m \to \iota Hm \in S$ such that $Hs = 1_{Hm}$. Then $(\iota\phi)s\colon m \to \iota Hm$ is a minimal model of ιHm and by Proposition 1.14, there exists $\varphi \in \mathrm{Aut}(m)$ such that $s\varphi \sim (\iota\phi)s$; i.e., $H\varphi = H(s\varphi) = H\iota\phi = \phi$. ∎

Remark 1.18. If $\mathscr{C}$, ι and H are as in definition 1.1, $x \in \mathrm{obj}\mathscr{C}$ is a formalizable object and the functor $F\colon \mathscr{C} \to \mathscr{E}$ admits a left derived function in the sense of Quillen ([Qui]) $\mathbf{L}F\colon \mathrm{Ho}\mathscr{C} \to \mathscr{E}$, then, obviously, $(\mathbf{L}F)(x) = (\mathbf{L}F)(\iota Hx)$. This can be seen as a generalization of Deligne's criterion for the collapse of spectral sequences [De] for categories of complexes of an abelian category, which, in our terminology, says that, if a complex is formalizable, then the hiperhomology spectral sequence of any functor collapses at the E_2-term. Another example: the differential torsion product of dg-modules can be seen as the left derived functor, in the sense of Quillen, of the tensor product

$$\otimes : \mathbf{2DGM} \to \mathbf{DGM}$$

Here **2DGM** stands for the category of triples (M, A, N) in which A is a cdg algebra and M and N are A-dg modules. So, if (A, M) is formalizable, then

$$\mathrm{Tor}_A(M, N) = \mathrm{Tor}_{HA}(HM, HN)$$

where $\mathrm{Tor}_{HA}(HM, HN)$ means the classical (non-differential) torsion product. Classically this last isomorphism was deduced from the collapse of the Eilenberg-Moore spectral sequence (see [ViP_2]). In our presentation, it follows directly from the definition of formalizability. The only point which needs some work is the verification that the differential torsion product is actually (the cohomology of) the left derived functor of the tensor product, *in the sense of Quillen* (see [$\mathbf{Roig}_3$]).

2. Formalizability of DG modules

Let $\mathbf{k}$ be a field, $\mathbf{Q} \subset \mathbf{k}$, A a $\mathbf{k}$-cdg algebra and M an A-dg module, both minimals. Hence (A, M) is a minimal object of **DGM**. So (A, M) is formalizable if and only if there is a *quis* of **DGM**

$$(g, \psi): (A, M) \to (HA, HM)$$

That is to say, $g: A \to HA$ is a *quis* of **k**-cdg algebras and $\psi: M \to g^*HM$ is a *quis* of A-dg modules and we can choose them in such a way that $Hg = 1_{HA}$ and $H\psi = 1_{g^*HM}$. As we have seen in Proposition 1.17, this implies that every automorphism of (HA, HM) lifts to an automorphism of (A, M). We are going to show that the converse is also true. In particular:

THEOREM 2.1. *Let (A, M) be a minimal object of* **DGM**, *A a finitely generated* **k**-*cdg and M a finitely generated A-dg module. If every automorphism $(\tilde{f}, \tilde{\varphi})$ of (HA, HM) lifts to an automorphism (f, φ) of (A, M), then (A, M) is formalizable.*

That is to say, if for every $\tilde{f} \in \mathrm{Aut}(HA)$, there exists $f \in \mathrm{Aut}(A)$ such that $Hf = \tilde{f}$ and for every $\tilde{f}$-morphism $\tilde{\varphi} \in \mathrm{Aut}(HM)$ there exists an f-morphism $\varphi \in \mathrm{Aut}(M)$ such that $H\varphi = \tilde{\varphi}$, then (A, M) is formalizable.

Using the caracterization of formalizability of theorem 2.1, we will be able to prove that formalizability does not depend on the ground field: if $\mathbf{k} \subset \mathbf{K}$ is a field extension, A a **k**-cdg algebra and M an A-dg module, put

$$A_{\mathbf{K}} = A \otimes_{\mathbf{k}} \mathbf{K} \quad \text{and} \quad M_{\mathbf{K}} = M \otimes_{\mathbf{k}} \mathbf{K}$$

THEOREM 2.2. *Let A be a finitely generated* **k**-*cdg algebra and M a finitely generated A-dg module. Then $(A_{\mathbf{K}}, M_{\mathbf{K}})$ is formalizable if and only if (A, M) is formalizable.*

Remark 2.3. In what follows, we may assume that **K** is algebraically closed, because if Theorem 2.2 is true in this case, then it is true for every field.

To prove Theorem 2.2 we will also need some results about algebraic groups, for which our reference will be [Bo]. The role of algebraic groups in this context is, on one side, to show that, under the lifting hypothesis of Theorem 2.1, there is a subobject of (A, M) quasi-isomorphic to the total object for which formalizability is quite obvious. By minimality and Proposition 1.13, this subobject coincides with (A, M). On the other hand, once we have shown that formalizability is equivalent to the lifting property of automorphisms, one proves that this fact is independent on the ground field by making use of rationality results of algebraic groups, which we state here.

PROPOSITION 2.4. *Let A be a finitely generated* **k**-*cdg algebra and M a finitely generated A-dg module. Then*:

(1) $H_{\mathbf{K}}$: $\mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}}) \to \mathrm{Aut}(HA_{\mathbf{K}}, HM_{\mathbf{K}})$ *is a* **k**-*morphism of* **k**-*groups*;

(2) $H_{\mathbf{K}}(\mathbf{k}) = H$: $\mathrm{Aut}(A, M) \to \mathrm{Aut}(HA, HM)$;

(3) *if (A, M) is minimal*,

$$N = \ker(H_{\mathbf{K}}\colon \mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}}) \to \mathrm{Aut}(HA_{\mathbf{K}}, HM_{\mathbf{K}}))$$

is an unipotent group.

Remark 2.5. It is clear that $\mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}}) \to \mathrm{Aut}(A_{\mathbf{K}}) \times \mathrm{Aut}(M_{\mathbf{K}})$ is also a morphism of algebraic groups. So it preserves semi-simple and unipotent parts of the Jordan multiplicative decompositions. Hence, for every $(f, \varphi) \in \mathrm{Aut}(A, M)$, the morphisms appearing in this decomposition $(f, \varphi) = (f, \varphi)_s (f, \varphi)_u$ are morphisms of **DGM**. Moreover, $(f, \varphi)_s = (f_s, \varphi_s)$ and $(f, \varphi)_u = (f_u, \varphi_u)$. Finally, by [Bo], Theorem (4.4), if $(f, \varphi) \in \mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}})(\mathbf{k}) = \mathrm{Aut}(A, M)$, then $(f, \varphi)_s$ and $(f, \varphi)_u \in \mathrm{Aut}(A, M)$.

Proof of Theorem 2.2 (cf. [Mor], proof of Lemma 10.2). The fact that (A, M) formalizable implies $(A_{\mathbf{K}}, M_{\mathbf{K}})$ formalizable is trivial. Let us prove the converse. We can suppose that (A, M) is a minimal object, because if we assume that the theorem is true in this case, given (A, M) not necessarily minimal, we take a minimal model

$$(f, \varphi)\colon (A', M') \to (A, M).$$

Then

$$(f_{\mathbf{K}}, \varphi_{\mathbf{K}})\colon (A'_{\mathbf{K}} = A' \otimes_{\mathbf{k}} \mathbf{K}, M'_{\mathbf{K}} = M' \otimes_{\mathbf{k}} \mathbf{K}) \to (A_{\mathbf{K}}, M_{\mathbf{K}})$$

is also a minimal model and if $(A_{\mathbf{K}}, M_{\mathbf{K}})$ is formalizable, then also $(A'_{\mathbf{K}}, M'_{\mathbf{K}})$ is, and because this is a minimal object, our assumption says that also (A', M') is formalizable; hence so is (A, M).

So, let (A, M) be a minimal object. If $(A_{\mathbf{K}}, M_{\mathbf{K}})$ is formalizable, because of Propositions 1.17 and 2.4(1), one has an exact sequence of **k**-morphisms of **k**-groups

$$1 \to N \to \mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}}) \to \mathrm{Aut}(HA_{\mathbf{K}}, HM_{\mathbf{K}}) \to 1$$

where N is a connected solvable **k**-split subgroup, because it is unipotent by Proposition 2.4(3). Then by [Bo], Corollary (15.7), and Proposition 2.4(2).

$$H_{\mathbf{K}}(\mathbf{k}) = H\colon \mathrm{Aut}(A, M) \to \mathrm{Aut}(HA, HM)$$

is surjective and so, by Theorem 2.1, (A, M) is formalizable. ∎

Following [Sul] (cf. [F-T]), we will prove Theorem 2.1 by showing that, in fact, it suffices that one *grading automorphism* lifts.

DEFINITION 2.6. Let $\alpha \in \mathbf{k}$ be different from zero and not a root of unity. The *grading automorphism* of (HA, HM) defined by α is the pair of maps $(\tilde{f}_\alpha, \tilde{\varphi}_\alpha)$: $(HA, HM) \to (HA, HM)$ defined by

$$\tilde{f}_\alpha(a) = \alpha^{|a|}a \quad \text{and} \quad \tilde{\varphi}_\alpha(x) = \alpha^{|x|}x$$

In what follows, we will fix α and write simply $(\tilde{f}, \tilde{\varphi})$ for $(\tilde{f}_\alpha, \tilde{\varphi}_\alpha)$.

Proof of Theorem 2.1. The proof is a consequence of some lemmas which will be proved at the end.

LEMMA 2.1.1. $(\tilde{f}, \tilde{\varphi}) \in \operatorname{Aut}(HA, HM)$.

Let $(f, \varphi) \in \operatorname{Aut}(A, M)$ be a lifting of the grading automorphism $(\tilde{f}, \tilde{\varphi})$. Consider the semi-simple parts of f and φ. Then we have the decompositions as vector spaces:

$$A = \left(\bigoplus_{j \geq 0} A_j \right) \oplus B \quad \text{and} \quad M = \left(\bigoplus_{j \geq 0} M_j \right) \oplus N \tag{4}$$

where $A_j = \ker(f_s - \alpha^j I)$, $M_j = \ker(\varphi_s - \alpha^j I)$ and B and N are the complementary invariant subspaces of the previous ones.

LEMMA 2.1.2. *For every* i, j,
(i) $dA_j \subset A_j$, $dM_j \subset M_j$,
(ii) $A_i \cdot A_j \subset A_{i+j}$, $A_i \cdot M_j \subset M_{i+j}$,
(iii) $dB \subset B$ *and* $dN \subset N$.

This implies that $\oplus_j A_j$ is a $\mathbf{k}$-cdg subalgebra of A and $\oplus_j M_j$ a A-dg submodule of M. The fact that A and M are minimal, together with Proposition 1.13, imply:

LEMMA 2.1.3. $A = \oplus_j A_j$ *and* $M = \oplus_j M_j$.

Now, for every i, j, let $A_j^i = A_j \cap A^i$ and $M_j^i = M_j \cap M^i$ and

$$A' = \left(\bigoplus_{i<j} A_j^i \right) \oplus \left(\bigoplus_j A_j \cap Z^j A \right)$$

and

$$M' = \left(\bigoplus_{i<j} M_j^i \right) \oplus \left(\bigoplus_j M_j \cap Z^j M \right) \tag{5}$$

Again, minimality and Proposition 1.13 give:

LEMMA 2.1.4. *$A = A'$ and $M = M'$.*

Let

$$J = \left(\bigoplus_{i<j} A_j^i \right) \oplus \left(\bigoplus_j dA_j^{j-1} \right) \quad \text{and} \quad K = \left(\bigoplus_{i<j} M_j^i \right) \oplus \left(\bigoplus_j dM_j^{j-1} \right) \tag{6}$$

LEMMA 2.1.5. *J is a dg ideal of A, K is an A-dg submodule of M and $J \cdot M \subset K$.*

So M/K is an A/J-dg module. Also, because of lemma 2.1.4 and definitions of J and K, we have

$$A/J = \bigoplus_j H^j A_j \quad \text{and} \quad M/K = \bigoplus_j H^j M_j$$

Let $g\colon A \to A/J$ and $\psi\colon M \to M/K$ be the natural projections. Then

$$(g, \psi)\colon (A, M) \to (A/J, M/K)$$

is a morphism of **DGM**. Then, taking cohomology and applying lemmas 2.1.2(i), 2.1.3 and 2.1.4, we have

$$\begin{aligned}(HA, HM) &= \left(\bigoplus_j HA_j, \bigoplus_j HM_j \right) = \left(\bigoplus_j H^j A_j, \bigoplus_j H^j M_j \right) \\ &= (A/J, M/K).\end{aligned}$$

So $(Hg, H\psi)\colon (HA, HM) \to (A/J, M/K) = (HA, HM)$ is an isomorphism and (A, M) is formalizable. ■

Proof of Lemma 2.1.1. Obviously, $\tilde{f}$ is a morphism of **k**-graded algebras and $\tilde{\varphi}$ a morphism of **k**-graded modules. For instance,

$$\tilde{f}(ab) = \alpha^{|ab|} ab = (\alpha^{|a|} a)(\alpha^{|b|} b) = \tilde{f}a \cdot \tilde{f}b.$$

Let us show that $\tilde{\varphi}$ is an $\tilde{f}$-morphism:

$$\tilde{\varphi}(ax) = \alpha^{|ax|}(ax) = (\alpha^{|a|}a)(\alpha^{|x|}x) = \tilde{f}(a)\tilde{\varphi}(x). \quad \blacksquare$$

Proof of Lemma 2.1.2. Let us prove (i) for A: if $a \in A_j$, then

$$f_s(da) = d(f_s a) = d(\alpha^j a) = \alpha^j(da).$$

Let us prove (ii) for M: if $a \in A_i$ and $x \in M_j$, then

$$\varphi_s(ax) = (f_s a)(\varphi_s x) = (\alpha^i a)(\alpha^j x) = \alpha^{i+j}(ax).$$

Here we have used Remark 2.5.

Finally, $B_{\mathbf{K}} = B \otimes_k \mathbf{K}$ is invariant under $d \otimes 1$, since $\mathbf{K}$ is an algebraically closed field, and so $B_{\mathbf{K}} = \oplus_{\lambda \in K} \ker(f_s - \lambda I)$, for some $\lambda \neq \alpha^j$ and then one applies the same proof as in (i). But if $(d \otimes 1)B_{\mathbf{K}} \subset B_{\mathbf{K}}$, then $dB \subset B$. $\blacksquare$

Proof of Lemma 2.1.3. Lemma 2.1.2 implies that $\oplus_j A_j$ is a cdg subalgebra of A and, since B is a dg submodule, $HA = (\oplus_{j \geq 0} HA_j) \oplus HB$. But HA has nothing but the eigenvector subspaces of the eigenvalues α^j. Hence $HB = 0$ and $\oplus_j A_j \hookrightarrow A$ is a *quis*. Because A is minimal, it can not have non-trivial quasi-isomorphic subobjects, by Proposition 1.13. So the first equality follows.

Lemma 2.1.2 implies also that $\oplus_j M_j$ is a $\oplus_j A_j$-dg module and by the previous result, an A-dg module too. One sees in an analogous way, that the inclusion $\oplus_j M_j \mapsto M$ is a *quis*. Again, the minimality of M implies $M = \oplus_j M_j$. $\blacksquare$

Proof of Lemma 2.1.4. Firstly, A' is a cdg subalgebra of A, by Lemma 2.1.2. Secondly, the inclusion $A' \hookrightarrow A$ is a *quis* because, by definition, all elements in H^iA are eigenvectors of eigenvalue α^i and so $A_i \cap Z^iA \to H^iA$ is surjective. Injectivity in cohomology follows from Lemma 2.1.2. Equality is again a consequence of A being minimal. An analogous reasoning shows that $M = M'$. $\blacksquare$

Proof of Lemma 2.1.5. All the statements are alike: one has to apply Lemma 2.1.2 again, taking into account that by Lemma 2.1.4 $A = A'$, in the first two statements. Explicitly, for the second one: let $a + b \in A$ and $x + dy \in K$, $a \in A_j^i$, $b \in A_k \cap Z^kA$ and $x \in M_m^l$, $y \in M_n^{n-1}$ with $i < j$ and $l < m$. Then $(a + b)(x + dy) = ax + a \cdot dy + bx + b \cdot dy$ belongs to K since, by Lemma 2.1.2, $ax \in M^{l+i}_{m+j}$, $a \cdot dy \in M^{n+i}_{n+j}$ and $bx \in M^{l+k}_{m+k}$, with $l + i < m + j$, $n + i < n + j$ and $l + k < m + k$. Finally: $b \cdot dy = \pm d(by) \in dM^{i+k-1}_{i+k}$, because b is a cocycle.

The proof of the third one proceeds in an analogous fashion: use Lemma 2.1.2, now taking into account that by Lemma 2.1.4, $M = M'$. ■

Now we return to the proof of proposition 2.4.

Proof of Proposition 2.4. Since $\mathbf{Q} \subset \mathbf{k}$, by [Bo], page 47, the proof that $\mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}})$ is defined over $\mathbf{k}$ and that its $\mathbf{k}$-points are $\mathrm{Aut}(A, M)$ reduces to verify that it has some set of defining equations with coefficients in $\mathbf{k}$. Also the proof that the group structure is defined over $\mathbf{k}$ reduces to the verification that the coordinate functions of the product and unity have its coefficients in $\mathbf{k}$. In the same way, we may see that $H_{\mathbf{K}}$ is defined over $\mathbf{k}$ and that $H_{\mathbf{K}}(\mathbf{k}) = H$.

Let us assume that A and M are generated in degrees $\leq n - 1$ and $\leq m - 1$. Then $\mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}})$ is a subgroup of

$$\bigoplus_{i \leq n} GL\left(A^i_{\mathbf{K}}\right) \times \bigoplus_{j \leq m} GL\left(M^j_{\mathbf{K}}\right).$$

An element

$$(f, \varphi) \in \bigoplus_{i \leq n} GL\left(A^i_{\mathbf{K}}\right) \times \bigoplus_{j \leq m} GL\left(M^j_{\mathbf{K}}\right)$$

induces a morphism of **DGM** if and only if

(i) $dfa = fda$, $d\varphi x = \varphi\, dx$,

(ii) $f(aa') = fa \cdot fa'$ and $\varphi(ax) = fa \cdot \varphi x$

for $a, a' \in A_{\mathbf{K}}$ and $x \in M_{\mathbf{K}}$ such that $|a|$, $|x| \leq n$ in (i) and $|aa'|$, $|ax| \leq n$ in (ii). Differentials and products of $(A_{\mathbf{K}}, M_{\mathbf{K}})$ come from differentials and products of (A, M) after tensoring with $1_{\mathbf{K}}$. So, we may choose basis of A and M as $\mathbf{k}$-vector spaces and we see that these morphisms have coefficients in $\mathbf{k}$. So the coordinates of f and φ in these basis satisfies equations which are linear (i) or second order equations (ii) with coefficients in $\mathbf{k}$. Hence $\mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}})$ is defined over $\mathbf{k}$, also its group structure and $\mathrm{Aut}(A_{\mathbf{K}}, M_{\mathbf{K}})(\mathbf{k}) = \mathrm{Aut}(A, M)$. Analogously for $\mathrm{Aut}(HA_{\mathbf{K}}, HM_{\mathbf{K}})$. To end with (1) and (2), by choosing basis accordingly with decompositions

$$A = HA \oplus BA \oplus BA[+1] \quad \text{and} \quad M = HM \oplus BM \oplus BM[+1]$$

we may see that the coefficients of the coordinate functions of $H_{\mathbf{K}}$ reduce to 1 or 0.

Finally, take $\omega = (f, \varphi) \in N$ and decompose it in its semi-simple and unipotent parts: $\omega = \omega_s \cdot \omega_u$. We have to show that $\omega_s = 1$. Since $H\omega = 1$ and an algebraic group morphism preserves semi-simple and unipotent parts, $1 = H\omega = (H\omega_s). (H\omega_u) = (H\omega)_s \cdot (H\omega)_u$. So, by the uniqueness of this decomposition, $H\omega_s = H\omega_u = 1$. Thus the only eigenvalue of $H\omega_s$ is 1. Let us see that the same holds for ω_s. By Remark 2.5, $\omega_s = (f_s, \varphi_s)$. Take $A_1 = \ker(f_s - I)$ and $M_1 = \ker(\varphi_s - I)$ and decompose A and M into invariant subspaces: $A = A_1 \oplus B$ and $M = M_1 \oplus N$. Since A and M are minimals, one concludes that $A = A_1$ and $M = M_1$, by analogous reasonings that we have used in proving lemmas 2.1.3 and 2.1.4. Hence $\omega_s = 1$ as we wanted to prove. ∎

3. Formalizability of morphisms

Let us translate the previous results for the category $\mathbf{CDGA(k)}^2$ (cf. [F-T] for Theorem 3.1, and [ViP$_1$] for theorem 3.2). Let $m\colon A \to B$ be a minimal object of $\mathbf{CDGA(k)}^2$. By Example 1.6 this means that A is a minimal $\mathbf{k}$-cdg algebra and m is a minimal KS-extension.

THEOREM 3.1. *Let $m\colon A \to B$ be a minimal object of $\mathbf{CDGA(k)}^2$ and A and B finitely generated $\mathbf{k}$-cdg algebras. If every automorphism $(\tilde{f}, \tilde{\varphi})$ of Hm lifts into an automorphism (f, φ) of m, then m is formalizable.*

Let $\mathbf{k} \subset \mathbf{K}$ be as in § 2. Put

$$m_{\mathbf{K}} = m \oplus_k \mathbf{K}\colon A_{\mathbf{K}} \to B_{\mathbf{K}}$$

THEOREM 3.2. *Let $m\colon A \to B$ be a morphism of finitely generated $\mathbf{k}$-cdg algebras. Then $m_{\mathbf{K}}$ is formalizable if and only if m is formalizable.*

If m is formalizable, one also says that the induced map on homotopy types *is a formal consequence* of the map $m_*\colon HA \to HB$. It is proved in [D-G-M-S], main theorem (ii), § 6, that the map induced on real De Rham homotopy types by a holomorphic map between compact Kähler manifolds is a formal consequence of the induced map on real cohomology. This result, together with Theorem 3.2 implies:

COROLLARY. *Let $f\colon M \to N$ be a holomorphic map between compact Kähler manifolds. Then the induced map on rational De Rham homotopy types is a formal consequence of the induced map on rational cohomology.*

Proofs of both theorems follow the same steps of § 2.

PROPOSITION 3.3. *Let* m: $A \to B$ *be a morphism of finitely generated* **k**-*cdg algebras. Then*:
(1) H_K: $\mathrm{Aut}(m_{\mathbf{K}}) \to \mathrm{Aut}(Hm_{\mathbf{K}})$ *is a* **k**-*morphism of* **k**-*groups*;
(2) $H_{\mathbf{K}}(\mathbf{k}) = H$: $\mathrm{Aut}(m) \to \mathrm{Aut}(Hm)$;
(3) *if* m *is minimal*,

$$N = \ker(H_{\mathbf{K}}: \mathrm{Aut}(m_{\mathbf{K}}) \to \mathrm{Aut}(Hm_{\mathbf{K}}))$$

is an unipotent group.

Proof. Let $(f, \varphi) \in \mathrm{Aut}(m_{\mathbf{K}})$; i.e., $f \in \mathrm{Aut}(A_{\mathbf{K}})$, $\varphi \in \mathrm{Aut}(B_{\mathbf{K}})$ and $m_{\mathbf{K}} f = \varphi m_{\mathbf{K}}$. It is the same thing as an element of $\oplus_{i \le n} GL(A^i_{\mathbf{K}}) \times \otimes_{j \le m} GL(B^j_{\mathbf{K}})$ satisfying the following conditions:
(1) $dfa = fda$, $d\varphi\, b = \varphi\, db$,
(2) $f(aa') = fa \cdot fa'$, $\varphi(bb') = \varphi b \cdot \varphi b'$, and $m_{\mathbf{K}} fa = \varphi m_{\mathbf{K}} a$.
and so on ... ∎

One also has analogous grading automorphisms:

DEFINITION 3.4. Let $\alpha \in \mathbf{k}$ be different from zero and not a root of unity. The *grading automorphism* of Hm defined by α is the pair of maps $(\tilde{f}_\alpha, \tilde{\varphi}_\alpha)$: $Hm \to Hm$, defined by

$$\tilde{f}_\alpha(a) = \alpha^{|a|} a \quad \text{and} \quad \tilde{\varphi}_\alpha(b) = \alpha^{|b|} b$$

We fix α and denote $(\tilde{f}_\alpha, \tilde{\varphi}_\alpha)$ by $(\tilde{f}, \tilde{\varphi})$. In the same way as in Lemma 2.1.1 we have $(\tilde{f}, \tilde{\varphi}) \in \mathrm{Aut}(Hm)$. Considering semi-simple and unipotent parts of a lifting of one of these grading automorphism, we get decompositions for A and B:

$$A = \Big(\bigoplus_{j \ge 0} A_j\Big) \oplus C \quad \text{and} \quad B = \Big(\bigoplus_{j \ge 0} B_j\Big) \oplus D$$

in which (see Lemma 2.1.2) the subspaces A_i, B_j, C and D are invariant under differentials, $A_i \cdot A_j \subset A_{i+j}$, $B_i \cdot B_j \subset B_{i+j}$ and $m(A_i) \subset B_i$, for all $i, j \ge 0$. Then:

LEMMA 3.5. $m = m|_{\oplus_j A_j}$: $\oplus_j A_j \to \oplus_j B_j$.

Proof. As in Lemma 2.1.3, $A = \oplus_j A_j$. Now the subobject we have to consider is $m|_{\oplus_j A_j}$:

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ \bigoplus_j B_j & \hookrightarrow & B \end{array}$$

in the category $A \setminus$ **CDGA(k)**. Since m is minimal there and the inclusion $\oplus_j B_j \hookrightarrow B$ is a *quis*, $m = m|_{\oplus_j A_j}$. ■

For all i, j, define A^i_j, B^i_j, A' and B' as in (5). Then $m(A') \subset B'$. Let

$$m' = m|_{A'}: A' \to B'$$

As in Lemma 2.1.4, we have $m = m'$. We define also J and K in the same way as in (6) and we have:

LEMMA 3.6. *J and K are dg ideals of A and B, respectively and $m(J) \subset K$.*

Thus m induces $\tilde{m}: A/J \to B/K$ that makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ \downarrow g & & \downarrow \psi \\ A/J & \xrightarrow{\tilde{m}} & B/K \end{array}$$

commutative. So the pair (g, ψ) defines a morphism of **CDGA(k)**^{2} that becomes a *quis* between m and $Hm = \tilde{m}$. Finally, Proposition 3.3 allows us, as in the proof of Theorem 2.2, to lift automorphism of Hm assuming so for the automorphisms of $Hm_{\mathbf{K}}$.

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