ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 3, Fall 1994

LORENTZ-IMPROVING MEASURES

RAYMOND J. GRINNELL AND KATHRYN E. HARE[1]

Introduction

Throughout this paper G will denote an infinite compact abelian group, λ its normalized Haar measure, and Γ its discrete dual group. The space of bounded regular Borel measures on G will be denoted by $M(G)$. Measures, which acting by convolution map L^p to $L^{p+\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$ and $1 < p < \infty$ (or equivalently for all $1 < p < \infty$), are called L^p-improving measures and have been investigated in a number of recent papers (cf. [5] and the papers cited therein). Examples of such measures include all $L^q(G)$ functions for $q > 1$ (by Young's inequality), Riesz products [15], and the Cantor-Lebesgue measure [4].

In this paper we study measures which act by convolution on the Lorentz spaces $L(p,q)$.

DEFINITION. A measure μ is called *Lorentz-improving* if there exists p, q and r, where $1 < p < \infty$ and $1 \le q < r \le \infty$, such that $\mu * L(p,r) \subseteq L(p,q)$.

The Lorentz spaces are function spaces intermediate to the L^p spaces in the sense that whenever $1 \le q < p < r \le \infty$,

$$L^\infty \subseteq \bigcup_{t>p} L^t \subseteq L(p,q) \subseteq L^p \subseteq L(p,r) \subseteq \bigcap_{s<p} L^s \subseteq L^1. \tag{1}$$

We show that the class of Lorentz-improving measures properly contains the class of L^p-improving measures. In fact there are Lorentz-improving measures that are not even a limit, in the total variation norm, of L^p-improving measures. Lorentz-improving measures are characterized in terms of the size of the sets $\{\gamma \in \Gamma: |\hat{\mu}(\gamma)| > \varepsilon\}$. This characterization is analogous to a known characterization of L^p-improving measures [8] and requires the introduction of a new type of "thin" set, which generalizes the notion of a $\Lambda(p)$ set. Further estimates of the size of Lorentz-improving measures are made in Section 4. In particular we prove that all such measures are continuous. In Sections 5 and 6 we focus on Lorentz-improving measures on the circle group

Received December 30, 1991.
1991 Mathematics Subject Classification. Primary 43A05; Secondary 43A25, 43A15.

[1]Research partially supported by the NSERC.

T. These measures satisfy certain summability conditions, however, unlike L^p-improving measures, they need not be Lipschitz. Lastly we study random Lorentz-improving Cantor measures and characterize almost surely those which are L^p-improving. We also answer an open problem in [5].

1. Lorentz-spaces

We begin by briefly reviewing for the reader the definition and basic properties of the Lorentz spaces. Let f be a complex-valued measurable function on G which is finite λ a.e.. The distribution function of f is defined by

$$\lambda_f(y) = \lambda\{x \in G: |f(x)| > y\} \quad \text{for } y \geq 0.$$

The non-increasing rearrangement of f is the function f^* defined by

$$f^*(t) = \inf\{y > 0: \lambda_f(y) \leq t\} \quad \text{for each } t \geq 0.$$

The Lorentz space $L(p, q)$ is defined as the set of equivalence classes of functions f as above such that $\|f\|_{p,q}^* < \infty$, where

$$\|f\|_{p,q}^* = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^1 \left(t^{1/p} f^*(t)\right)^q \dfrac{dt}{t}\right)^{1/q} & \text{if } 1 \leq p, q < \infty \\ \displaystyle\sup_{t \in (0,\infty)} t^{1/p} f^*(t) & \text{if } 1 \leq p \leq \infty,\ q = \infty. \end{cases}$$

Since f^* and f have the same distribution function, it follows that $\|f\|_{p,p}^* = \|f\|_p$, so the Lorentz space $L(p, p)$ is equal to L^p.

The function $\| \ \|_{p,q}^*$ is a quasi-norm, but is not in general a norm. For this reason it is useful to define the function f^{**} by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds, \quad \text{for } t > 0,$$

and then set

$$\|f\|_{(p,q)} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \dfrac{dt}{t}\right)^{1/q} & \text{for } 1 \leq p, q < \infty \\ \displaystyle\sup_{t \in (0,\infty)} t^{1/p} f^{**}(t) & \text{for } 1 \leq p \leq \infty,\ q = \infty. \end{cases}$$

If $1 < p$, $q < \infty$ or if $p = q \in \{1, \infty\}$ then $L(p, q)$ is a Banach space with the norm $\| \ \|_{(p,q)}$. Hardy's inequality can be used to prove that the quasi-norm and norm are related by

$$\left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}^* \le \|f\|_{(p,q)} \le p' \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}^* \tag{2}$$

(where $(p/q)^{1/q} = 1$ if $q = \infty$). These facts can essentially be found in [11].

We have already mentioned that $L(p, q) \subseteq L(p, r)$ if $q < p < r$. In fact this inclusion holds whenever $q < r$. Moreover (see [11] and [12])

$$\|f\|_{p,r}^* \le \|f\|_{p,q}^* \tag{3}$$

and

$$\|f\|_{(p,r)} \le \left(\frac{q}{p}\right)^{(q^{-1}-r^{-1})} \|f\|_{(p,q)}. \tag{4}$$

Notice also that if $1 < p_1 < p_2 < \infty$ and $1 \le q < \infty$ then

$$\begin{aligned}\left(\|f\|_{p_1,q}^*\right)^q &\le \left(\frac{q}{p_1}\right) \sup_{t\in(0,1)} \left(t^{1/p_2} f^*(t)\right)^q \int_0^1 t^{q(p_1^{-1}-p_2^{-1})} \frac{dt}{t} \\ &= \frac{p_2}{p_2 - p_1} \left(\|f\|_{p_2,\infty}^*\right)^q. \end{aligned} \tag{5}$$

It follows from this that if $1 < p_1 < p_2 < \infty$ and $1 \le q_1, q_2 \le \infty$ then

$$L(p_2, q_2) \subseteq L(p_1, q_1). \tag{6}$$

If we define a total ordering on $(1, \infty) \times [1, \infty]$ by $(r, s) > (p, q)$ if $r > p$ or if $r = p$ and $s < q$, then inclusions (1) and (6) can be combined as

$$L(r, s) \subseteq L(p, q) \quad \text{if } (r, s) > (p, q). \tag{7}$$

Moreover this inclusion is proper [21, 2.7].

A final inequality [11, 4.6] we mention is that if $h \in L^1$ and if $g \in L(p, \infty)$ then

$$\|h * g\|_{(p,\infty)} \le \frac{p}{p-1} \|h\|_1 \|g\|_{(p,\infty)}. \tag{8}$$

The next fact will be useful later.

LEMMA 1.1. *If $1 < p < \infty$ and if $f \in L(p, r)$ for all $r > 1$ then $\|f\|_{(p,r)} \to \|f\|_{(p,1)}$ as $r \to 1$.*

Proof. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence decreasing to 1 and let $A = \{t: t^{1/p}f^{**}(t) \geq 1\}$. Since

$$\left(t^{1/p}f^{**}(t)\right)^{r_n}\chi_A \leq \left(t^{1/p}f^{**}(t)\right)^{r_1}\chi_A \in L^1\left(\frac{dt}{t}\right)$$

and $\{(t^{1/p}f^{**}(t))^{r_n}\chi_{A^c}\}$ is an increasing sequence, it follows by the dominated and monotone convergence theorems that

$$\int_0^{\infty}\left(t^{1/p}f^{**}(t)\right)^{r_n}\frac{dt}{t} \to \int_0^{\infty} t^{1/p}f^{**}(t)\frac{dt}{t}$$

as $r_n \to 1$. This clearly suffices to prove the lemma. □

As with the classical L^p spaces, the simple functions, and hence the trigonometric polynomials $T(G)$, are dense in $L(p, q)$ whenever $1 < p < \infty$ and $1 \leq q < \infty$. Moreover the dual of $L(p, q)$ is $L(p', q')$ (where $1/p' + 1/p = 1$, $1/q' + 1/q = 1$) [11, 2.4 and 2.7]. A duality argument [7, 11.6] proves that $\mu * L(p_1, q_1) \subseteq L(p_2, q_2)$ precisely when $\mu * L(p_2', q_2') \subseteq L(p_1', q_1')$ if $1 < p_1, p_2 < \infty$, $1 \leq q_1, q_2 \leq \infty$.

A Riesz-Thorin like interpolation theorem applies to operators on Lorentz spaces. As we make extensive use of this result, we will state it here for the convenience of the reader.

Notation. For $j \in \{0, 1\}$, let p_j, q_j, r_j, s_j satisfy either $1 < p_j, r_j, q_j, s_j < \infty$ or $p_j = r_j \in \{1, \infty\}$ or $q_j = s_j \in \{1, \infty\}$. If $0 \leq \Theta \leq 1$, define p, q, r, s by

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$$
$$\frac{1}{r} = \frac{1-\Theta}{r_0} + \frac{\Theta}{r_1}, \quad \frac{1}{s} = \frac{1-\Theta}{s_0} + \frac{\Theta}{s_1}.$$

We use this notation in the theorem below.

THEOREM 1.2. [11, *p.* 266] *For $j \in \{0, 1\}$ let T: $L(p_j, q_j) \to L(r_j, s_j)$ be a bounded linear operator satisfying $\|Tf\|_{r_j, s_j}^* \leq M_j\|f\|_{p_j, q_j}^*$ for all $f \in L(p_j, q_j)$. Then*

$$\|Tf\|_{r,s}^* \leq C(p_j, q_j, \Theta)M_0^{1-\Theta}M_1^{\Theta}\|f\|_{p,q}^*$$

for all $f \in L(p, q)$.

This theorem has an important consequence for measures acting by convolution on the Lorentz spaces.

THEOREM 1.3. *If $\mu \in M(G)$ then $\mu * L(p,q) \subseteq L(p,q)$ for all $1 < p < \infty$, $1 \leq q \leq \infty$.*

Proof. Since $\mu * L^1 \subseteq L^1$ and $\mu * L^\infty \subseteq L^\infty$ this follows from the interpolation theorem. □

THEOREM 1.4. *Let μ be a Lorentz-improving measure.*

(a) *For every $1 < p < \infty$ there exist $1 \leq q_1 < q_2 \leq \infty$ (depending on p) such that $\mu * L(p, q_2) \subseteq L(p, q_1)$.*

(b) *For every $1 < p < \infty$ and $1 < q < \infty$ there exists $r < q$ such that*

$$\mu * L(p,q) \subseteq L(p,r).$$

(c) *For every $1 < p < \infty$ and $1 < s < \infty$ there exists $t > s$ such that*

$$\mu * L(p,t) \subseteq L(p,s).$$

Proof. (a) Since μ is Lorentz-improving there exist $1 < r < \infty$ and $1 \leq s_1 < s_2 \leq \infty$ satisfying

$$\mu * L(r, s_2) \subseteq L(r, s_1).$$

If $p = r$ the result is proved. If $p > r$ interpolate using the fact that $\mu * L^\infty \subseteq L^\infty$, otherwise interpolate using the fact that $\mu * L^1 \subseteq L^1$.

(b) and (c) These are similar but use (a) and Theorem 1.3. □

Remark. The ordering of the Lorentz spaces (7) might suggest calling a measure μ Lorentz-improving if for some $(r,s) > (p,q)$ we have $\mu * L(p,q) \subseteq L(r,s)$. The inclusions show that this definition is actually the same as the one we gave in the introduction. Moreover the set of measures μ for which there exists some $p < r$ with $\mu * L(p,q) \subseteq L(r,s)$ is easily seen by (6) to coincide with the set of L^p-improving measures.

2. Examples of Lorentz-improving measures

Our first result yields numerous examples of Lorentz-improving measures.

THEOREM 2.1. *If μ is L^p-improving then $\mu * L(p,\infty) \subseteq L(p,1)$ for all $1 < p < \infty$.*

Proof. Fix $p > 1$ and choose $1 < r < p$. As μ is L^p-improving there exists some $q > r$ such that $\mu * L^r \subseteq L^q$. By setting $\varepsilon = p(q-r)/(q+r)$

and noting that $\mu * L^\infty \subseteq L^\infty$, we interpolate and conclude that $\mu * L^{p-\varepsilon} \subseteq L^{p+\varepsilon}$. By (1), $L(p,\infty) \subseteq L^{p-\varepsilon}$ and $L(p,1) \supseteq L^{p+\varepsilon}$ so we obtain $\mu * L(p,\infty) \subseteq L(p,1)$ for all p. □

The main objective of this section is to construct, on any infinite compact abelian group, an example of a non-L^p-improving measure which maps $L(p,\infty)$ to $L(p,1)$ for all $1 < p < \infty$. First we need some easy results on convolution powers of μ. These were motivated by [8].

Notation. μ^n will denote the convolution of μ with itself n times.

PROPOSITION 2.2. *Let $\mu \in M(G)$ and suppose there are indices $1 < p < \infty$ and $1 \le q_1 < q_2 \le \infty$ such that $\mu * L(p,q_2) \subseteq L(p,q_1)$. Then for any $1 \le r < \infty$ there is some positive integer m such that $\mu^m * L(p,r) \subseteq L(p,q_1)$.*

Proof. There is nothing to prove unless $q_2 < r < \infty$. For each positive integer n set $q_{n+1} = q_2(q_2/q_1)^{n-1}$. Since $\mu * L(p,\infty) \subseteq L(p,\infty)$, if one assumes inductively that

$$\mu * L(p,q_{n+1}) \subseteq L(p,q_n),$$

then by interpolation we obtain $\mu * L(p,q_{n+2}) \subseteq L(p,q_{n+1})$. It is now easy to see that

$$\mu^n * L(p,q_{n+1}) \subseteq L(p,q_1) \quad \text{for all } n,$$

and as $q_n \to \infty$ as $n \to \infty$, the proof is complete. □

COROLLARY 2.3. *Let $\mu \in M(G)$ and suppose $\mu * L(p,q_2) \subseteq L(p,q_1)$ for some $1 < p < \infty$ and $1 \le q_1 < q_2 \le \infty$. Then given any $1 < r \le \infty$ there is some positive integer m such that $\mu^m * L(p,q_2) \subseteq L(p,r)$.*

Proof. By duality $\mu * L(p',q_1') \subseteq L(p',q_2')$. Since $1 \le r' < \infty$, by Proposition 2.2 there exists a positive integer m such that $\mu^m * L(p',r') \subseteq L(p',q_2')$. Dualizing again gives the result. □

PROPOSITION 2.4. *Let $\mu \in M(G)$ and suppose for some $1 < p < \infty$ and $1 \le q < \infty$ we have $\mu * L(p,\infty) \subseteq L(p,q)$. Then there exists a positive integer m such that $\mu^m * L(p,\infty) \subseteq L(p,1)$.*

Proof. By duality $\mu * L(p',q') \subseteq L(p',1)$, thus if $p = 2$, $\mu * L(2,q') \subseteq L(2,1)$. If $p \ne 2$ then taking $s = 1$ if $p < 2$ or $s = 2p$ if $p > 2$ and using the fact that $\mu * L(s,1) \subseteq L(s,1)$, interpolating yields $\mu * L(p,t) \subseteq L(p,1)$ for some $t > 1$. Thus, in either case, applying Proposition 2.2 with $q_2 = t$, $q_1 = 1$

and $r = q$ we can choose m satisfying $\mu^m * L(p,q) \subseteq L(p,1)$. Hence

$$\mu^{m+1} * L(p,\infty) \subseteq \mu^m * L(p,q) \subseteq L(p,q). \quad \square$$

COROLLARY 2.5. *Let $\mu \in M(G)$ and suppose $\mu * L(p,q) \subseteq L(p,1)$ for some $1 < p < \infty$ and $1 < q \leq \infty$. Then there exists a positive integer m such that $\mu^m * L(p,\infty) \subseteq L(p,1)$.*

Notation. For a linear operator $T\colon L(p_1,q_1) \to L(p_2,q_2)$ denote the operator quasi-norm of T by

$$\|T\|^*_{(p_1,q_1;p_2,q_2)} \equiv \sup\{\|Tf\|^*_{p_2,q_2} : \|f\|^*_{p_1,q_1} \leq 1\}.$$

(When $p_1 = q_1$ and $p_2 = q_2$ we will simply write $\|T\|_{p_1,p_2}$.)

LEMMA 2.6. *If $p < q$ and $T : L^p \to L^q$ is a bounded linear operator then*

$$\|T\|^*_{(p,p;p,1)} \leq \frac{q}{q-p}\|T\|^*_{(p,p;q,q)}.$$

Proof. If $f \in L^p$ then inequalities (5) and (3) yield

$$\|Tf\|^*_{p,1} \leq \frac{q}{q-p}\|Tf\|^*_{q,\infty} \leq \frac{q}{q-p}\|Tf\|^*_{q,q}. \quad \square$$

We are now ready for our construction of a Lorentz-improving measure which is not L^p-improving.

THEOREM 2.7. *Let G be an infinite compact abelian group. There is an absolutely continuous measure μ on G for which $\mu * L(p,\infty) \subseteq L(p,1)$ for all $1 < p < \infty$, but μ is not L^p-improving.*

Proof. For $n \geq 3$ let $r_n = 1 + n^{-1}$ and define s_n by $1/s_n = 1/r_n - 1/2$. Note that s_n decreases to 2 as n tends to infinity.

Using the sharp form of Young's inequality [13] it is possible to choose a sequence of positive functions $\{\phi_n\}_{n=3}^{\infty}$ having the following properties:

(i) $\phi_n \in L^{r_n}(G)$;
(ii) $\|\phi_n\|_1 = 1$;
(iii) There exists some $h_n \in L^2$ such that $\phi_n * h_n \notin L^{s_n + n^{-1}}$.

By Young's inequality $\phi_n * L^2 \subseteq L^{s_n}$ and of course $\phi_n * L^\infty \subseteq L^\infty$.

For $m > 2$, set $\Theta_m = 2/m$ and $q_{n,m} = ms_n/2$. Since

$$\frac{1}{m} = \frac{\Theta_m}{2} + \frac{1-\Theta_m}{\infty} \quad \text{and} \quad \frac{1}{q_{n,m}} = \frac{\Theta_m}{s_n} + \frac{1-\Theta_m}{\infty},$$

by the Riesz-Thorin interpolation theorem we see that $\phi_n * L^m \to L^{q_{n,m}}$ and

$$\|\phi_n\|_{m,q_{n,m}} \le \|\phi_n\|_{2,s_n}^{\Theta_m} \|\phi_n\|_{\infty,\infty}^{1-\Theta_m} \le \|\phi_n\|_{2,s_n}.$$

Lastly, set

$$D_n = \max\left\{\frac{s_n}{s_n - 2}, \|\phi_n\|_{2,s_n}\right\}.$$

Consider the function

$$w = \sum_{n=3}^{\infty} \frac{\phi_n}{n^2 D_n^2}.$$

Since

$$\|w\|_1 \le \sum_{n=3}^{\infty} \frac{\|\phi_n\|_1}{n^2 D_n^2} \le \sum_{n=3}^{\infty} \frac{1}{n^2} < \infty,$$

$w \in L^1(G)$.

Fix $p > 2$ and choose n_0 so that $s_n + n^{-1} < p$ for all $n > n_0$. Since the functions ϕ_n are positive

$$\|w\|_{2,p} \ge \|w\|_{2,s_n+n^{-1}} \ge \|\phi_n\|_{2,s_n+n^{-1}}.$$

However, by (iii), the final operator norm in the inequality above is infinite and so w does not map L^2 into L^p. As $p > 2$ was arbitrary w is not L^p-improving.

The operator quasi-norms we are working with do not satisfy the triangle inequality, however, because $\| \ \|_{(p,q)}$ is a norm and the relationship (4) holds, one can see that for $m > 2$,

$$\|w\|^*_{(m,m;m,1)} \le m \sum_{n=3}^{\infty} \frac{\|\phi_n\|^*_{(m,m;m,1)}}{n^2 D_n^2}.$$

By Lemma 2.6 and the fact that

$$\frac{q_{n,m}}{q_{n,m} - m} = \frac{s_n}{s_n - 2},$$

we see that

$$\begin{aligned} \|w\|^*_{(m,m;m,1)} &\le m \sum_{n=3}^{\infty} \frac{s_n}{s_n - 2} \frac{\|\phi_n\|_{m,q_{n,m}}}{n^2 D_n^2} \\ &\le m \sum_{n=3}^{\infty} \frac{\|\phi_n\|_{2,s_n}}{n^2 D_n} < \infty. \end{aligned}$$

Hence $w * L(m,m) \subseteq L(m,1)$ for all positive integers m and by interpolating this holds for all $1 < m < \infty$. By duality $\mu \equiv w * w$ maps $L(m,\infty)$ to $L(m,1)$ for all $1 < m < \infty$. Furthermore μ is not L^p-improving since w is not [15]. □

Remark. This measure was constructed as a norm limit of L^p-improving measures. In Section 6 we study random Cantor measures and prove that there is a Lorentz-improving measure on T which is not a norm limit of L^p-improving measures.

3. A characterization theorem

L^p-improving measures have been characterized in terms of the size of their Fourier transform: a measure μ is L^p-improving if and only if the sets

$$E(\mu,\varepsilon) \equiv \{\gamma : |\hat{\mu}(\gamma)| > \varepsilon\}$$

are $\Lambda(p)$ sets for some $p > 2$, with $\Lambda(p)$ constant $0(\varepsilon^{-1})$ [8]. We will give a similar theorem for Lorentz-improving measures, but first it is necessary to generalize the notion of a $\Lambda(p)$ set to the Lorentz space setting.

Notation. For a function space X and $E \subseteq \Gamma$, X_E will denote $\{f \in X : \hat{f}(\gamma) = 0$ for all $\gamma \notin E\}$.

DEFINITION [16]. Let $0 < p < \infty$. A subset E of Γ is called a $\Lambda(p)$ set if $L^p_E = L^r_E$ for some $r < p$ (or equivalently, for all $r < p$).

This definition and the twofold inclusion structure for Lorentz spaces suggests the following.

DEFINITION. Let $E \subseteq \Gamma$, $1 < p < \infty$, and $1 \le q < \infty$. We call E a $\Lambda_1(p,q)$ set if there exists some $1 < s < p$ such that $L_E(p,q) = L_E(s,q)$. The set E is called a $\Lambda_2(p,q)$ set if there is some $r > q$ such that $L_E(p,q) = L_E(p,r)$.

The group Γ is not a $\Lambda_1(p,q)$ set or a $\Lambda_2(p,q)$ set for any p,q since $L(p,q) \ne L(r,s)$ if $(p,q) \ne (r,s)$. If E is a finite set then E is both a $\Lambda_1(p,q)$ set and a $\Lambda_2(p,q)$ set for all $1 < p < \infty$ and $1 \le q < \infty$.

PROPOSITION 3.1. (a) *If E is a $\Lambda(p)$ set then E is a $\Lambda_1(p,p)$ set and a $\Lambda_1(q,1)$ set for all $1 < q < p$.*
(b) *If E is a $\Lambda_1(p,q)$ set then E is a $\Lambda_2(p,q)$ set. Moreover E is a $\Lambda(r)$ set for all $r < p$, and it is a $\Lambda(p)$ set if $q \le p$.*

Proof. This is evident from the inclusions in (1) and (6). □

There are many natural questions concerning $\Lambda_1(p,q)$ and $\Lambda_2(p,q)$ sets which should be pursued. A number of these are obviously implied by the theory of $\Lambda(p)$ sets. Our purpose for introducing $\Lambda_2(p,q)$ sets in this paper is to use them to prove a characterization theorem for Lorentz-improving measures (Theorem 3.4 below). In order to do this, we shall need the next two results which give properties of a set which are equivalent to the definition of a $\Lambda_2(p,q)$ set.

THEOREM 3.2. *Let $E \subset \Gamma$, $1 < p < \infty$, and $1 \leq q < \infty$. The following are equivalent*:

(i) *E is a $\Lambda_2(p,q)$ set*;
(ii) *There is a constant k such that $\|f\|_{(p,q)} \leq k\|f\|_{(p,\infty)}$ for all $f \in T_E(G)$*;
(iii) $L_E(p,q) = L_E(p,\infty)$.

Proof. (i $\Rightarrow$ ii) From the definition of a $\Lambda_2(p,q)$ set and the closed graph theorem we know there exists some $r > q$ and constant k_1 such that $\|f\|_{(p,q)} \leq k_1\|f\|_{(p,r)}$ for all $f \in T_E(G)$.

Now

$$\|f\|_{(p,r)}^r \leq \sup_{t\in(0,\infty)} \left(t^{1/p}f^{**}(t)\right)^{r-q} \int_0^\infty \left(t^{1/p}f^{**}(t)\right)^q \frac{dt}{t}$$
$$= \|f\|_{(p,\infty)}^{r-q}\|f\|_{(p,q)}^q.$$

Combining these inequalities and simplifying gives (ii).

(ii $\Rightarrow$ iii) We need to prove $L_E(p,\infty) \subseteq L_E(p,q)$, so let $f \in L_E(p,\infty)$. Let $\{K_\alpha\}$ be a bounded approximate identity in $T(G)$.

First assume $q \neq 1$. Then

$$\|K_\alpha * f\|_{(p,q)} \leq k\|K_\alpha * f\|_{(p,\infty)} \leq \frac{kp}{p-1}\|f\|_{(p,\infty)},$$

and as $L(p,q)$ has a weak $*$ topology, a subnet of $\{K_\alpha * f\}$ converges weak $*$. But $\Gamma \subseteq L(p',q')$ and so $K_\alpha * f$ converges weak $*$ to f. Thus $f \in L_E(p,q)$, indeed,

$$\|f\|_{(p,q)} \leq \liminf \|K_\alpha * f\|_{(p,q)} \leq \frac{kp}{p-1}\|f\|_{(p,\infty)}.$$

To handle the case $q = 1$ we note that for $s > 1$ inequalities (4) and (7) and assumption (ii) yield

$$\|K_\alpha * f\|_{(p,s)} \leq \left(\frac{1}{p}\right)^{1/s'}\|K_\alpha * f\|_{(p,1)} \leq k\left(\frac{1}{p}\right)^{1/s'}\|K_\alpha * f\|_{(p,\infty)}$$
$$\leq k\frac{p}{p-1}\left(\frac{1}{p}\right)^{1/s'}\|f\|_{(p,\infty)}.$$

As in the first case we can conclude that $f \in L(p,s)$ and that

$$\|f\|_{(p,s)} \le \frac{kp}{p-1}\left(\frac{1}{p}\right)^{1/s'} \|f\|_{(p,\infty)}.$$

Letting $s \to 1$ and applying Lemma 1.1 completes the proof.

(iii $\Rightarrow$ i) is obvious. $\square$

THEOREM 3.3. *Let $E \subset \Gamma$ and let $1 \le q < 2$. The following are equivalent*:

(i) *E is a $\Lambda_2(2,q)$ set*;

(ii) *There exists a constant k such that for each $g \in L(2,q')$ there is some $h \in L^2_E$ with $\hat{g}|_E = \hat{h}|_E$ and $\|h\|_2 \le k\|g\|_{(2,q')}$*;

(iii) *There exists a constant k such that for all $g \in L(2,q')$*

$$\left(\sum_{\gamma\in E} |\hat{g}(\gamma)|^2\right)^{1/2} \le k\|g\|_{(2,q')}.$$

Proof. (i $\Rightarrow$ ii) Applying the previous theorem (iii) it follows that $L^2_E = L(2,q)$. Thus the inclusion map $I: L^2_E \to L(2,q)$ is bounded and hence so is its adjoint, the quotient map $Q: L(2,q') \to L^2_E$. Define h by $\hat{h} = \hat{g}|_E$.

(ii $\Rightarrow$ iii). Obvious.

(iii $\Rightarrow$ i) Property (iii) can be restated as saying the quotient map $Q: L(2,q') \to L^2_E$ is bounded.

Assume first that $q \ne 1$. Taking adjoints it follows that the inclusion map $I: L^2_E \to L(2,q)$ is bounded, which proves (i).

Suppose $q = 1$. For $2 < p < \infty$ consider the quotient map $Q_p: L(2,p) \to L^2_E$. By (ii) and (4),

$$\|Q_p(g)\|_2 = \left(\sum_{\gamma\in E} |\hat{g}(\gamma)|^2\right)^{1/2} \le k\|g\|_{(2,\infty)}$$
$$\le k\left(\frac{p}{2}\right)^{1/p}\|g\|_{(2,p)}.$$

Hence its adjoint $I_p: L^2_E \to L(2,p')$ has norm at most $k(p/2)^{1/p}$. Thus for all $f \in L^2_E$

$$\|f\|_{(2,1)} = \lim_{p'\to 1}\|f\|_{(2,p')} \le \limsup_{p'\to 1} k\left(\frac{p}{2}\right)^{1/p}\|f\|_2 < \infty$$

which proves E is a $\Lambda_2(2,1)$ set. $\square$

Of course, similar results could be obtained for $\Lambda_1(p,q)$ sets.

Notation. For $E \subseteq \Gamma$ and $1 \leq q < 2$ let $\Lambda_2(2, q; E) \equiv \sup\{\|f\|_{(2,q)}: f \in L^2_E, \|f\|_2 \leq 1\}$.

THEOREM 3.4. *For $\mu \in M(G)$ the following are equivalent:*
(i) *μ is Lorentz-improving;*
(ii) *There exists $1 \leq q < 2$ and $\alpha \geq 1$ such that for each $\varepsilon > 0$ the sets $E(\varepsilon)$ are $\Lambda_2(2, q)$ sets and $\Lambda_2(2, q; E(\varepsilon)) = O(\varepsilon^{-\alpha})$;*
(iii) *There exists $1 \leq q < 2$ and a positive integer n such that $\mu^n * L(2, q') \to L^2$.*

Proof. The proof of this theorem is very similar to the equivalence of (1) and (2) in the analogous characterization of L^p-improving measures [8]. There it was observed that if μ^n is L^p-improving for some n then so is μ. This was derived in [15] as a consequence of Stein's analytic interpolation theorem. The same type of arguments, but using the Lorentz space analogue [17] of Stein's analytic interpolation theorem, proves that if μ^n is Lorentz improving then so is μ. We leave the details to the reader. □

An interesting application of this theorem is to prove a sufficient summability condition for Lorentz-improving measures.

COROLLARY 3.5. *Suppose $\mu \in M(G)$ and for some $s < \infty$*

$$\sum_{\gamma \in \Gamma} \left(\exp|\hat{\mu}(\gamma)|^{-s}\right)^{-1} \equiv C < \infty.$$

Then $\mu^{[s]+1} * L(2, \infty) \to L^2$ and μ is Lorentz-improving.

In order to prove this we first need to compute an upper bound for $\Lambda_2(2, q; E)$ when E is a finite set.

LEMMA 3.6. *If $E \subseteq \Gamma$ has cardinality $n \geq 3$ and $1 \leq q < 2$ then $\Lambda_2(2, q; E) \leq 4e(\log n)^{1/q}$.*

Proof. Let $1 \leq q < 2$, and let $f \in T_E(G)$. Define $r > 2$ by

$$\frac{1}{2} - \frac{1}{r} = \frac{1}{q \log n}.$$

Since $\|f\|_r \leq n^{1/2 - 1/r}\|f\|_2$ it follows that

$$\begin{aligned}\|f\|_{(2,q)} &\leq 2\left(\frac{2}{q}\right)^{1/q}\left(\frac{r}{r-2}\right)^{1/q}\|f\|_r \\ &\leq 2\left(\frac{1}{q}\right)^{1/q}\left(\frac{1}{2} - \frac{1}{r}\right)^{-1/q} n^{1/2-1/r}\|f\|_2 \\ &\leq 4e(\log n)^{1/q}\|f\|_2. \quad \square\end{aligned}$$

Proof of Corollary 3.5. Observe that

$$\frac{\operatorname{Card} E(\varepsilon)}{\exp \varepsilon^{-s}} \leq \sum_{\gamma \in E(\varepsilon)} \left(\exp |\mu(\gamma)|^{-s}\right)^{-1} \leq C < \infty.$$

Thus $E(\varepsilon)$ is a set of cardinality at most $C \exp \varepsilon^{-s}$, and so by the lemma $\Lambda_2(2, 1; E(\varepsilon)) \leq 4e \log(C \exp \varepsilon^{-s}) = O(\varepsilon^{-s})$. Now apply the theorem. □

4. The size of Lorentz-improving measures

Recall that a measure μ on G is called strongly continuous if for all closed subgroups H of infinite order in G, and for all $x \in G$, $|\mu|(x + H) = 0$. Obviously strongly continuous measures are continuous. L^p-improving measures are always strongly continuous [5, 3.2]. Although the same result is true for Lorentz-improving measures, a different method of proof is needed. Our method could also give a new proof for L^p-improving measures.

LEMMA 4.1. *Let $\{\nu_\alpha\}_{\alpha \in I}$ be a net in $M(G)$ and let $1 < p < \infty$, $1 \leq q_1 \leq q_2 < \infty$. Suppose for all $\alpha \in I$, $\nu_\alpha * L(p, q_2) \subseteq L(p, q_1)$ with uniformly bounded operator norm and suppose $\lim_\alpha \hat{\nu}_\alpha(\gamma) \equiv \phi(\gamma)$ exists for all $\gamma \in \Gamma$. Then the operator M_ϕ defined by $\widehat{M_\phi f}(\gamma) = \phi(\gamma)\hat{f}(\gamma)$ maps $L(p, q_2)$ to $L(p, q_1)$.*

Proof. Let $f \in T(G)$ and choose $\alpha \in I$ so that for all $\gamma \in \operatorname{supp} \hat{f}$,

$$|\hat{\nu}_\alpha(\gamma) - \phi(\gamma)| < \frac{1}{\operatorname{Card}(\operatorname{supp} \hat{f})}.$$

For this choice of α,

$$\begin{aligned}\|\nu_\alpha * f - M_\phi(f)\|^*_{p, q_1} &\leq \|\nu_\alpha * f - M_\phi(f)\|_\infty \\ &\leq \sum_\gamma |\hat{\nu}_\alpha(\gamma) - \phi(\gamma)|\,|\hat{f}(\gamma)| \\ &\leq \|f\|_1 \leq \frac{p}{p-1}\|f\|^*_{p, q_2}.\end{aligned}$$

Together with the triangle inequality this gives

$$\begin{aligned}\|M_\phi(f)\|^*_{p, q_1} &\leq p'\left(\|\nu_\alpha * f - M_\phi(f)\|^*_{p, q_1} + \|\nu_\alpha * f\|^*_{p, q_1}\right) \\ &\leq p'\left(\frac{p}{p-1} + \|\nu_\alpha\|^*_{p, q_2; p, q_1}\right)\|f\|^*_{p, q_2}.\end{aligned}$$

Since $\|\nu_\alpha\|^*_{(p,q_2;p,q_1)}$ can be bounded independently of the choice of α, and $T(G)$ is dense in $L(p,q_2)$, it follows that M_ϕ maps $L(p,q_2)$ to $L(p,q_1)$. □

Notation. Let $\gamma\mu$ be the measure given by $\gamma\mu(E) = \int_E \bar{\gamma}\, d\mu$.

PROPOSITION 4.2. *Let $\mu * L(p,q_1) \subseteq L(p,q_2)$ for some $1 < p < \infty$, $1 \le q_2 \le q_1 < \infty$. Assume ϕ belongs to the weak closure in $l^\infty(\Gamma)$ of the convex hull of $\{\gamma\mu : \gamma \in \Gamma\}$. Then M_ϕ maps $L(p,q_1)$ to $L(p,q_2)$.*

Proof. To see this we just need to remark that if $\nu = \sum_{i=1}^N a_i \gamma_i \mu$ where $0 \le a_i \le 1$, $\sum_1^N a_i = 1$ and $\gamma_i \in \Gamma$, then

$$\|\nu\|^*_{(p,q_1;p,q_2)} \le \|\mu\|^*_{(p,q_1;p,q_2)},$$

and that evaluation at $\gamma \in \Gamma$ is a continuous linear functional on $l^\infty(\Gamma)$. □

THEOREM 4.3. *If μ is a Lorentz-improving measure then μ is continuous.*

Proof. If μ is not continuous then the unique constant function ϕ in the weak closure of the convex hull of $\{\gamma\mu * \tilde{\mu} : \gamma \in \Gamma\}$ is not the zero function [14]. Being constant, $\phi = c\hat{\delta}_e$ for some $c \ne 0$. (Here δ_e is the point mass measure at the identity of G.) The measure $\mu * \tilde{\mu}$ is Lorentz improving, hence by the proposition so is $c\delta_e$. But $E(c, c\delta_e) = \Gamma$ which is not a $\Lambda_2(2,q)$ set for any $q < 2$, contradicting the characterization Theorem 3.4. □

COROLLARY 4.4. *If μ is Lorentz-improving then μ is strongly continuous.*

Proof. Suppose μ is not strongly continuous. Since translates of Lorentz-improving measures are Lorentz-improving, without loss of generality we may assume that there exists a closed subgroup H of infinite index such that $|\mu|(H) \ne 0$. Let $\pi\mu$ be the measure defined on G/H by the formula

$$\int_{G/H} f(t)\, d\pi\mu(t) \equiv \int_G f \circ \pi(s)\, d\mu(s)$$

for $f \in C(G/H)$.

Let $1 < p < \infty$ and $1 \le q < \infty$. Since $\mu * (f \circ \pi)(g) = \pi\mu * f(\pi(g))$ for all continuous functions f, the distribution functions of $\pi\mu * f$ and $\mu * (f \circ \pi)$ are equal. Thus

$$\|\pi\mu * f\|_{L(p,q)(G/H)} = \|\mu * (f \circ \pi)\|_{L(p,q)(G)},$$

from which it follows that $\pi\mu$ is also Lorentz-improving. But $\pi\mu$ is not continuous since $|\mu|(H) \ne 0$ which contradicts the theorem. □

If the measure μ maps L^2 to L^p for $p > 2$ then $\limsup |\hat{\mu}(\gamma)| \leq \sqrt{2/p}\,\|\mu\|$ [8]. As there are L^p-improving, norm one measures μ on D^∞ with $\limsup |\hat{\mu}| \geq 1 - \varepsilon$ for any given $\varepsilon > 0$ [9, 2.7], the best one could hope for with Lorentz-improving measures is the inequality $\limsup |\hat{\mu}(\gamma)| < \|\mu\|$. This we have for Lorentz-improving measures on T.

COROLLARY 4.5. *If μ is a Lorentz-improving measure on T then* $\limsup |\hat{\mu}(n)| < \|\mu\|$.

Proof. Suppose first that μ is a Lorentz-improving probability measure on T with $\limsup |\hat{\mu}(n)| = 1$. Then, just as in the proof of Lemma 1 of [2], one can argue that there is a Lorentz-improving discrete probability measure, contradicting Theorem 4.3.

Now assume μ is any Lorentz-improving measure with $\limsup |\hat{\mu}(n)| = 1 = \|\mu\|$. Replacing μ if necessary by $\mu * \tilde{\mu}$ we may assume without loss of generality that there exists a net (n_β) in $\mathbf{Z}$ with $0 \leq \hat{\mu}(n_\beta) \to 1$. Let χ be the weak $*$ limit in $\Delta M(T)$ of the subnet (n_α). Then $\hat{\mu}(\chi) = 1$, and as $|\chi_\mu| \leq 1$ and $\|\mu\| = 1$, this implies that $|\chi_\mu| = 1$ $|\mu|$ a.e.

The measures $\mu_\alpha \equiv n_\alpha \mu$ are norm bounded, and thus have a weak $*$ converging subnet (not renamed) with limit $w \in M(T)$. From Lemma 4.1 we know that w is Lorentz-improving. Furthermore $\hat{w}(n) = \lim_\alpha \hat{\mu}_\alpha(n) = \widehat{\chi\mu}(n)$; thus $w = \chi\mu$. As $\|w\|_{M(T)} = \|\chi\mu\|_{M(T)} = 1$ and $\hat{w}(0) = \hat{\mu}(\chi) = 1$, w is a positive measure and thus $\limsup |\hat{w}| < 1$.

Now $\chi \in \overline{\mathbf{Z}}_\infty(w)$ (see [10, p. 37] for the definition) and thus $|\chi|^2 \in \overline{\mathbf{Z}}_\infty(w)$. This means there exists a net (n_γ) in $\mathbf{Z}$ tending to infinity and converging weak $*$ in $L^\infty(w)$ to $|\chi|^2$. In particular $\hat{w}(n_\gamma) \to \hat{w}(|\chi|^2) = 1$ as $|\chi_w|^2 = 1$ w a.e. This contradicts the fact that $\limsup |\hat{w}| < 1$. □

With the aid of Corollary 4.4 we can characterize Lorentz-improving Riesz products. For the definition we refer the reader to [6, 7.1].

COROLLARY 4.6. *For a Riesz product measure ρ the following are equivalent*:

(i) *ρ is L^p-improving*;
(ii) *ρ is Lorentz-improving*;
(iii) $\operatorname{Lim\,sup} |\hat{\rho}(\gamma)| < 1$.

Proof. (i) $\Rightarrow$ (ii) comes from 2.1. The equivalence of (i) and (iii) is established by Ritter in [15]. However, in proving (i) $\Rightarrow$ (iii) [15, p. 294], Ritter actually proves a stronger result, namely, if ρ is a Riesz product measure, with $\limsup |\hat{\rho}(\gamma)| = 1$, then there is a quotient measure $\pi\rho$, defined on an infinite quotient group, which is discrete. Consequently ρ is not strongly continuous and hence not Lorentz-improving, proving (ii) $\Rightarrow$ (iii). □

5. Lipschitz-like conditions

A measure μ on T is said to belong to Lip(α) for some $0 < \alpha \leq 1$ if its distribution function $F(x) \equiv \mu[0, x)$ satisfies a Lipschitz condition of order α. It was proved in [5, 2.1] that if $\mu * L^p \subseteq L^2$ for some $p < 2$ then $\sum_{|j| \leq n} |\hat{\mu}(j)| = O(n^{1/2-1/p})$, which implies (by [3, p.45]) that $\mu \in \mathrm{Lip}(1/p - 1/2)$. This is not the case for Lorentz-improving measures as our next example shows.

Example 5.1. An absolutely continuous measure μ on T such that $\mu * L(p, \infty) \subseteq L(p, 1)$ for all $1 < p < \infty$, and $\mu \notin \mathrm{Lip}(\alpha)$ for any $\alpha > 0$.

Construction. The function

$$f(x) = \sum_{|n| \geq 2} \frac{e^{inx}}{\log |n|} + 2(1 + e^{ix} + e^{-ix})$$

is known to belong to $L^1(T)$. It is easy to verify that

$$\sum_{|n| \geq 2} \left(\exp |\hat{f}(n)|^{-2}\right)^{-1} < \infty,$$

so by Corollary 3.5 $g \equiv f * f * f$ maps $L(2, \infty)$ to $L(2, 2)$. By interpolating it follows that $g * L(p, \infty) \subseteq L(p, p)$ for all $2 \leq p \leq \infty$. By Proposition 2.4, for each integer $n \geq 2$ there is an integer m_n such that g^{m_n} maps $L(n, \infty)$ to $L(n, 1)$. Redefine m_n, if necessary, so the sequence $\{m_n\}$ is increasing and let C_n be the corresponding operator quasi-norm. Let

$$A_n = \max\{\|g\|_1, \|g\|^*_{(p,1;p,1)} \text{ for } p = 2, 3, \ldots, n\}.$$

Choose a sequence of positive integers $\{N_n\}_{n=2}^{\infty}$ with $\log N_n \geq 2n^2 A_n^{m_n}$ and $n(\log N_n)^{3m_n+1} \leq N_n^{1/n}$. Let K_n denote the $2N_n$-th Fejér kernel, i.e.,

$$K_n(t) = \sum_{j=-2N_n}^{2N_n} \left(1 - \frac{|j|}{2N_n + 1}\right) e^{ijt}$$

and let

$$w = \sum_{n=2}^{\infty} \frac{g^{m_n} * K_n}{n^2 A_n^{m_n}}.$$

Clearly $w \in L^1(T)$. For any integer p with $2 \leq p \leq n$

$$\begin{aligned}\| g^{m_n} * K_n \|^*_{(p,\infty;p,1)} &\leq \sup_{\|h\|^*_{p,\infty} \leq 1} \|g^{m_n}\|^*_{(p,\infty;p,1)} \|K_n * h\|^*_{p,\infty} \\ &\leq \sup_{\|h\|^*_{p,\infty} \leq 1} \frac{p}{p-1} \| g^{m_n - m_p} \|^*_{(p,1;p,1)} \| g^{m_p} \|^*_{(p,\infty;p,1)} \|h\|^*_{p,\infty} \\ &\leq \frac{p}{p-1} A_n^{m_n - m_p} C_p \leq \frac{p}{p-1} A_n^{m_n} C_p .\end{aligned}$$

Thus for any integer p with $2 \leq p \leq n$,

$$\begin{aligned}\|w\|^*_{(p,\infty;p,1)} &\leq p' \sum_{n \geq 2} \frac{\| g^{m_n} * K_n \|^*_{(p,\infty;p,1)}}{n^2 A_n^{m_n}} \\ &\leq p' \left(\sum_{2 \leq n < p} \frac{\| g^{m_n} * K_n \|^*_{(p,\infty;p,1)}}{n^2 A_n^{m_n}} + \frac{p}{p-1} \sum_{n \geq p} \frac{C_p}{n^2} \right).\end{aligned}$$

Since $g^{m_n} * K_n \in T(G)$, it maps $L(p,\infty)$ to $L(p,1)$, so the expression above is finite. Thus $w * L(p,\infty) \subseteq L(p,1)$ for all integers $p \geq 2$ and by interpolating and dualizing we see that $w * L(p,\infty) \subseteq L(p,1)$ for all $1 < p < \infty$.

It remains to show the measure w does not belong to $\mathrm{Lip}(\alpha)$ for any $\alpha > 0$. Since $|\hat{w}(n)|$ decreases monotonically as $n \to \pm\infty$ it suffices to show that $|\hat{w}(n)| \neq O(|n|^{-\alpha})$ [1, p. 216]. But the choice of N_n ensures that

$$\begin{aligned}|\hat{w}(N_n)| &\geq \frac{\widehat{g^{m_n}}(N_n)}{2n^2 A_n^{m_n}} = \frac{1}{2n^2 A_n^{m_n} (\log N_n)^{3m_n}} \\ &\geq \frac{1}{(\log N_n)^{3m_n + 1}} \geq \frac{n}{N_n^{1/n}}\end{aligned}$$

which proves the desired result. □

There are, however, some necessary summation conditions which Lorentz-improving measures on the circle possess.

Proposition 5.2. *Let $\mu \in M(T)$ and suppose $\mu * L^2 \subseteq L(2,q)$ for some $q < 2$. Then there exists a constant C such that*

$$\left(\sum_{n=1}^{N} \left(\sum_{|k|=n}^{N} \frac{|\hat{\mu}(k)|}{|k|^{3/2}} \right)^q n^{q/2-1} \right)^{1/q} \leq C\sqrt{\log N}$$

for all N.

We need to prove two lemmas first.

Notation. For $q \le 2$ let $l_w(q) = \{(x_n)_{n=1}^\infty: (\sum |x_n|^q n^{q/2-1})^{1/q} < \infty\}$.

LEMMA 5.3. *If $f \in L(2,q)$ for $q \le 2$ and $\hat{f}(n) \ge 0$ for all n then* $\{\sum_{|k|=n}^\infty \hat{f}(k)/|k|\}_{n=1}^\infty \in l_w(q)$.

Proof. Let $\{a_k\}_{k=1}^\infty$ be the Fourier cosine coefficients of f. Let

$$\phi(x) = \frac{1}{x}\int_0^x f.$$

Then $a_k \ge 0$ for all k and $|\phi(x)| \le |f|^{**} \in L(2,q)$, so by [18, p. 247],

$$\left\{\sum_{k=n}^\infty \frac{a_k}{k}\right\}_{n=1}^\infty \in l_w(q). \quad \square$$

LEMMA 5.4. *Let $\mu * L^2 \subseteq L(2,q)$ for some $q < 2$. For $f \in L^2$ define*

$$T(f) = \{(Tf)_n\}_{n=1}^\infty \equiv \left\{\sum_{|k|=n}^\infty \frac{\widehat{\mu * f}(k)}{|k|}\right\}_{n=1}^\infty.$$

Then T is a bounded linear operator from L^2 to $l_w(q)$.

Proof. Let $f \in L^2$. Notice that $|(Tf)_n| < (Tg)_n$ where $\hat{g}(k) = \operatorname{sgn} \hat{\mu}(k)|\hat{f}(k)|$. Since $g \in L^2$, $\mu * g \in L(2,q)$, and as $\mu * g(n) \ge 0$ for all n, the previous lemma says $Tg \in l_w(q)$. Thus T maps L^2 to $l_w(q)$. Clearly T is linear.

Suppose $f_k \to f$ in L^2 and $T(f_k) \to y \equiv \{y_n\}_{n=1}^\infty$ in $l_w(q)$. It is easy to check that $y_n = (Tf)_n$ for all n so by the closed graph theorem T is a bounded operator. $\square$

Proof of Proposition 5.2. From Lemma 5.4 we know there exists a constant C so that for all $f \in L^2$

$$\sum_{n=1}\left(\left|\sum_{|k|=n}^\infty \frac{\mu * f(k)}{|k|}\right|^q n^{q/2-1}\right)^{1/q} \le C\|f\|_2.$$

Taking the polynomial f with Fourier coefficients

$$\hat{f}(n) = \begin{cases} \dfrac{\operatorname{sgn} \hat{\mu}(n)}{\sqrt{n}} & \text{if } |n| \le N,\ n \ne 0 \\ 0 & \text{otherwise} \end{cases}$$

gives the result. $\square$

If $\mu \in M(T)$ is L^p-improving and $|\hat{\mu}(n)|$ decreases as $n \to \pm\infty$ then $|\hat{\mu}(n)| = O(|n|^{-\alpha})$ [5, 2.2]. For Lorentz-improving measures we have a similar result.

COROLLARY 5.5. *Suppose $\mu \in M(T)$ and $|\hat{\mu}(n)|$ decreases as $n \to \pm\infty$. Then μ is Lorentz-improving if and only if $|\hat{\mu}(n)| = O(\log|n|)^{-\alpha})$ for some $\alpha \leq \frac{1}{2}$.*

Proof. If μ is Lorentz-improving then there exists some $q < 2$ with $\mu * L^2 \subseteq L(2, q)$. It is a straightforward exercise to verify that Proposition 5.2 combined with the assumption that $|\hat{\mu}(n)|$ decreases implies $|\hat{\mu}(n)| = O((\log|n|)^{1/2-1/q})$.

For the converse, simply apply Corollary 3.5. □

Taking $q = 1$ in Proposition 5.2 and simplifying yields the next corollary.

COROLLARY 5.6. *If $\mu * L^2 \subseteq L(2,1)$ then*

$$\sum_{|k|=1}^{N} \frac{|\hat{\mu}(k)|}{|k|} = O\left(\sqrt{\log N}\right).$$

In [5, 2.3] an example is constructed of an L^1 function which belongs to $\mathrm{Lip}(\alpha)$ for all $0 < \alpha < 1$ but which is not L^p-improving. We modify this example for the Lorentz-improving case.

Example 5.7. A function $f \in L^1(T)$ such that $f\lambda_T \in \bigcap_{0<\alpha<1} \mathrm{Lip}(\alpha)$, but $f\lambda_T$ is not Lorentz-improving.

Construction. Since $\mathbf{Z}$ is not a $\Lambda_2(2, q)$ set for any $1 \leq q < 2$,

$$L_q(n) \equiv \Lambda_2(2, q; \{-n, \ldots, 0, 1, \ldots, n\}) \to \infty \text{ as } n \to \infty.$$

Choose a sequence $\{q_n\}_{n=1}^{\infty}$ with $1 \leq q_n < 2$ and q_n increasing to 2, choose integers m_n such that

$$\frac{L_{q_n}(m_n)}{2n^2} \to \infty \text{ as } n \to \infty,$$

and choose integers N_n such that

$$\sum_{k=1}^{n} \frac{4m_k}{k^2} \leq N_n^{1/n} \quad \text{and} \quad N_{n+1} \geq 2m_n N_n.$$

Let K_n denote the n-th Fejér kernel, and let

$$H_n(x) = K_{2m_n}(N_n x), \quad f = \sum_{n=1}^{\infty} \frac{H_n(x)}{n^2}.$$

Clearly $f \in L^1$.

Since $\operatorname{supp} \hat{H}_j \cap \operatorname{supp} \hat{H}_k = \{0\}$, the Fourier coefficients of f satisfy

$$\hat{f}(j) = \begin{cases} \sum_1^{\infty} \frac{1}{n^2} & \text{if } j = 0 \\ \left(1 - \frac{|k|}{2m_n}\right) \frac{1}{n^2} & \text{if } j = N_n k,\ k \in \{\pm 1, \dots, \pm m_n\} \\ 0 & \text{otherwise.} \end{cases}$$

In particular $|\hat{f}(j)| \leq 1/n^2$ if $j = N_n k$, $k \in \{\pm 1, \dots, \pm m_n\}$ and

$$E\left(\frac{1}{2n^2}\right) \supseteq \{0, \pm N_n, \pm 2N_n, \dots, \pm m_n N_n\}.$$

Thus $\Lambda_2(2, q; E(1/2n^2)) \geq L_q(m_n)$. As $L_q(m_n) \geq L_{q_n}(m_n)$ for all n with $q_n \geq q$, there is no $q < 2$ with

$$\Lambda_2\left(2, q; E\left(\frac{1}{2n^2}\right)\right) = O(n^2)$$

so $f\lambda$ is not Lorentz-improving.

The fact that $f\lambda$ belongs to $\operatorname{Lip}(\alpha)$ for all $\alpha > 0$ is proved using a similar argument to that found in [5]. □

6. Random Cantor measures

The examples constructed in Sections 2 and 5 of Lorentz-improving measures which were not L^p-improving were both L^1 functions, and hence the norm limit of L^p-improving measures (to wit, polynomials). Here we prove the existence of a Lorentz-improving Cantor measure which is not such a limit. We will also characterize, almost surely, the L^p-improving Cantor measures. These results are easy consequences of work of Salem [20].

First we describe what we mean by a random Cantor set and measure. Given a sequence $\{\xi_k\}_{k=1}^{\infty}$ with $0 < \xi_k < \frac{1}{2}$, there is associated a Cantor set with ratios of dissection $\{\xi_k^{-1}\}$. The Cantor measure supported on this set satisfies

$$\hat{\mu}(n) = (-1)^n \prod_{k=1}^{\infty} \cos \pi n \xi_1 \cdots \xi_{k-1}(1 - \xi_k).$$

Now suppose $\{a_k\}$ and $\{b_k\}$ are chosen with $0 < a_k < b_k < \frac{1}{2}$ and $b_k - a_k \geq 1/w_k$ with w_k increasing and $\lim_{k\to\infty}(\log w_k)/k = 0$. Define independent random variables $\xi_k(w)$ uniformly distributed over $[a_k, b_k]$ and let μ_w be the associated Cantor measures. These are the measures we will refer to as random Cantor measures.

PROPOSITION 6.1. *Random Cantor measures satisfying*

$$\liminf k^{1/2-\beta}(a_1 \cdots a_k)^{1/k} > 0$$

for some $\beta > 0$ are Lorentz-improving a.s..

Proof. In the proof of Theorem IV of [20] Salem shows that there exist constants $0 < \Theta < 1$ and $\alpha > 0$ so that all measures μ as above satisfy

$$|\hat{\mu}(n)| = O\left(\frac{1}{|n|^{\alpha/(\log n)^{\Theta}}}\right) \quad \text{a.s.}$$

As $|n|^{\alpha/(\log n)^{\Theta}} \geq (\log|n|)^{1/2}$ for n sufficiently large, it follows from Corollary 3.5 that μ is Lorentz-improving. $\square$

Remark. In contrast it is known that if $\lim_k k^{1/2}(a_1 \cdots a_k)^{1/k} = 0$ then $\limsup|\hat{\mu}(n)| = 1$ [19, p. 326] so μ is not Lorentz-improving by Corollary 4.5.

Example 6.2. Here is an example of a singular Lorentz-improving measure which is not the norm limit of L^p-improving measures. With the notation as above, let $b_k = 2a_k = 2k^{-1/3}$, $w_k = 2^{\sqrt{k}}$. Then if $\beta < \frac{1}{6}$

$$\lim_{k\to\infty} k^{1/2-\beta}(a_1 \cdots a_k)^{1/k} = \lim_{k\to\infty} \frac{k^{1/2-\beta}}{(k!)^{1/3k}} = \infty$$

so there exists a Cantor measure μ with $a_k \leq \xi_k \leq b_k$ for all k and which is Lorentz-improving. Observe that the support of μ is contained in the union of 2^k disjoint intervals of length $\xi_1 \cdots \xi_k$. Call this union E_k.

Suppose there are L^p-improving measures μ_n which converge in measure to μ. Choose N such that $\|\mu_N - \mu\| < \frac{1}{2}$. Being L^p-improving, $\mu_N \in \operatorname{Lip}\alpha$ for some $\alpha > 0$. Thus there is a constant C so that for all k,

$$\begin{aligned} \tfrac{1}{2} &\geq |\mu - \mu_N(E_k)| \geq |\mu(E_k)| - C2^k(\xi_1 \cdots \xi_k)^{\alpha} \\ &\geq 1 - C2^k(2^k k!^{-1/3})^{\alpha} \to 1 \end{aligned}$$

which is a contradiction. $\square$

Remark. In contrast to L^p-improving measures, this Lorentz-improving measure has Hausdorff dimension 0.

Salem's work also enables us to easily characterize L^p-improving random Cantor measures.

PROPOSITION 6.3. *Almost surely random Cantor measures belong to* $\mathrm{Lip}(\alpha)$ *for some* $\alpha > 0$ *if and only if they are* L^p*-improving.*

Proof. Necessity is known for all L^p-improving measures as we remarked previously [5, 2.1].

For sufficiency we note that an argument similar to [22, vol. 1, p. 296-7] shows that a Cantor measure with ratios of dissection ξ_k^{-1} is Lipschitz if and only if

$$\limsup(\xi_1 \cdots \xi_k)^{-1/k} < \infty$$

and Salem has shown that random Cantor measures with this property satisfy $|\hat{\mu}(n)| = O(1/|n|^{\delta})$ for some $\delta > 0$ a.s.. Such measures are L^p-improving [8]. □

Previously only Cantor measures with bounded ratios of dissection were known to be L^p-improving [4]. The previous result clearly shows that this is unnecessary in general, however, if the ratios of dissection are integer valued it is a necessary condition as our final result demonstrates.

PROPOSITION 6.4. *Let* μ *be a Cantor measure with integer ratios of dissection* ξ_k^{-1}. *If* $\inf \xi_k = 0$ *then* $\limsup |\hat{\mu}(n)| = 1$.

Proof. Choose a sequence $\{k_n\}_{n=1}^{\infty}$ with $\xi_{k_n} \leq 2^{-n}$ and let $a_n = \prod_1^{k_n-1} \xi_i^{-1}$. We will prove that $|\hat{\mu}(a_n)| \to 1$. First notice that if $j < k_n$, then

$$(\xi_1 \cdots \xi_{j-1})(1 - \xi_j)a_n \in \mathbf{Z}.$$

Also,

$$(\xi_1 \cdots \xi_{k_n-1})(1 - \xi_{k_n})a_n = (1 - \xi_{k_n}) \geq 1 - 2^{-n};$$

thus

$$\prod_{j=1}^{k_n} \left|\cos \pi a_n(\xi_1 \cdots \xi_{j-1})(1 - \xi_j)\right| \geq |\cos \pi(1 - 2^{-n})|.$$

We remark that as $\xi_i \le \frac{1}{2}$ for all i, if $j > k_n$ then

$$(\xi_1 \cdots \xi_{j-1})(1-\xi_j)a_n = (\xi_{k_n} \cdots \xi_{j-1})(1-\xi_j) \le 2^{-n}2^{-(j-1-k_n)}.$$

Thus

$$\begin{aligned}\prod_{j=k_n+1}^{\infty} \left|\cos \pi a_n(\xi_i \cdots \xi_{j-1})(1-\xi_j)\right| &\ge \prod_{k_n+1}^{\infty} \left|\cos 2^{-(j-1-k_n+n)}\pi\right| \\ &\ge \prod_{k_n+1}^{\infty}\left(1 - \frac{(\pi 2^{-(j-1-k_n+n)})^2}{2}\right) \\ &= \prod_{j=0}^{\infty}\left(1 - \frac{\pi^2}{2}2^{-2(n+j)}\right).\end{aligned}$$

Combining these results we have

$$|\hat{\mu}(a_n)| \ge |\cos \pi(1-2^{-n})| \prod_{j=0}^{\infty}\left(1 - \frac{\pi^2}{2}2^{-2(n+j)}\right) \to 1 \quad \text{as } n \to \infty$$

hence $\limsup |\hat{\mu}(n)| = 1$. □

In [5] it is asked if L^p-improving and Lipschitz are equivalent for Cantor measures. Since we can easily arrange for $\xi_k^{-1} \in \mathbf{N}$ for all k, $\inf \xi_k = 0$ and $\limsup(\xi_1 \cdots \xi_k)^{-1/k} < \infty$ this question is answered negatively; there are Lipschitz Cantor measures which are not L^p-improving.

COROLLARY 6.5. *For a Cantor measure with integer ratios of dissection the following are equivalent*:

(i) *The ratios of dissection are bounded*;
(ii) *The measure is L^p-improving*;
(iii) *The measure is Lorentz-improving*.

Proof. These facts are immediate from [4], Theorem 2.1, Proposition 6.4, and Corollary 4.5. □

REFERENCES

1. N. BARY, *Trigonometric series*, Vol. I, MacMillian, New York, 1964.
2. W. BECKNER, S. JANSON and J. JERISON, *Convolution inequalities on the circle*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, W. Beckner et al., Ed., Wadsworth, Belmont, 1983, pp. 32–43.
3. R.P. BOAS, JR., *Integrability theorems for trigonometric transforms*, Springer-Verlag, New York, 1967.
4. M. CHRIST, *A convolution inequality concerning Cantor-Lebesgue measures*, Rev. Mat. Iberoamericana **1** (1985), 79–83.

5. C. Graham, K. Hare and D. Ritter, *The size of L^p-improving measures*, J. Funct. Anal. **84** (1989), 472–495.
6. C. Graham and O. McGehee, *Essays in commutative harmonic analysis*, Springer-Verlag, New York, 1979.
7. R. Grinnell, *Lorentz-improving measures on compact abelian groups*, Ph.D. Dissertation, Queen's University, 1991.
8. K. Hare, *A characterization of L^p-improving measures*, Proc. Amer. Math. Soc. **102** (1988), 295–299.
9. ______, *The size of* (L^2, L^p) *multipliers*, Colloq. Math., **63** (1992), 249–262.
10. B. Host, J.-F. Mela and F. Parreau, *Analyse harmonique des measures*, Asterique **135–136** (1986).
11. R. Hunt, *On $L(p,q)$ spaces*, Enseign. Math. **12** (1966), 249–276.
12. R. O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. **30** (1963), 129–142.
13. T. Quek and L. Yap, *Sharpness of Young's inequality for convolution*, Math. Scand. **53** (1983), 221–239.
14. T. Ramsey and B. Wells, *Fourier-Stieltjes transforms of strongly continuous measures*, Mich. Math. J. **24** (1977) 13–19.
15. D. Ritter, *Most Riesz products are L^p-improving*, Proc. Amer. Math. Soc. **97** (1986), 291–295.
16. W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227.
17. Y. Sagher, *On analytic families of operators*, Israel J. Math. **7** (1969), 350–356.
18. ______, *Some remarks on interpolation of operators and Fourier coefficients*, Studia Math. **44** (1972), 239–252.
19. R. Salem, *The absolute convergence of trigonometrical series*, Duke Math. J. **8** (1941), 317–334.
20. ______, *On sets of multiplicity for trigonometrical series*, Amer. J. Math. **64** (1942), 531–538.
21. L. Yap, *On the impossibility of representing certain functions by convolutions*, Math. Scand. **26** (1970), 132–140.
22. A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge University Press, New York, 1959.

University of Regina
Regina, Saskatchewan, Canada

University of Waterloo
Waterloo, Ontario, Canada