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THE RIESZ TRANSFORMS OF THE GAUSSIAN

E. KOCHNEFF

1. Introduction

It was shown recently ([1]) that the Hilbert transform of the Gaussian

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad x \in R,$$

is a well-known special function:

$$HG(x) = S(x) = \frac{1}{\pi} e^{-x^2/2} \int_0^x e^{s^2/2}\, ds. \tag{1}$$

For some results about the function $S(x)$ see, for example, [2].

The Riesz transform is the natural generalization of the Hilbert transform to R^n. We show that the Riesz transforms of the Gaussian

$$G(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}, \qquad x \in R^n,$$

are confluent hypergeometric functions having the integral representation:

$$R_j G(x) = \frac{2x_j e^{-|x|^2/2}}{|x|^n (2\pi)^{(n+1)/2}} \int_0^{|x|} e^{s^2/2} \left(|x|^2 - s^2\right)^{(n-1)/2} ds, \qquad j = 1, \dots, n. \tag{2}$$

For $n, j = 1$, equation (2) coincides with equation (1). On the other hand, the method in [1] does not generalize into R^n, so our method is different.

2. The Riesz transforms of the Gaussian

For $f \in L^1 \cap L^2(R^n)$, define the Fourier transform of f by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(t) e^{-ix \cdot t}\, dt.$$

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By the Fourier inversion theorem,

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} \hat{f}(t) e^{ix\cdot t}\, dt.$$

The Gaussian satisfies $\hat{G}(x) = G(x)$.
The Riesz transforms are defined by

$$R_j f(x) = c_n p.v. \int_{R^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)\, dy, \qquad j = 1, 2, \ldots, n,$$

where $c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$. Moreover,

$$(R_j f)\hat{}\,(x) = \frac{-ix_j}{|x|} \hat{f}(x), \qquad j = 1, \ldots, n.$$

Letting

$$F_j(x) = (R_j G)\hat{}\,(x) = \frac{-ix_j}{|x|} G(x), \qquad j = 1, \ldots, n, \tag{3}$$

we have by the Fourier inversion theorem $R_j G(-x) = \hat{F}_j(x)$. For $j = 1, 2, \ldots, n$, $F_j \in L^1 \cap L^2(R^n)$ is the product of a radial function and the first degree solid spherical harmonic x_j. Thus, $\hat{F}_j(x) = x_j F(|x|)$ where

$$F(r) = \frac{-1}{(2\pi r)^{n/2}} \int_0^\infty e^{-s^2/2} J_{n/2}(rs) s^{n/2}\, ds \tag{4}$$

and $J_{n/2}$ is a Bessel function. See [4].

From the representation of the confluent hypergeometric function

$${}_1F_1(\sigma; \nu + 1; -\lambda^2/4z^2) = \frac{2\Gamma(\nu + 1) z^{2\sigma}}{\Gamma(\sigma)(\lambda/2)^\nu} \int_0^\infty e^{-z^2 s^2} J_\nu(\lambda s) s^{2\sigma - \nu - 1}\, ds,$$

$\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(z^2) > 0$ with $\lambda = r$, $z^2 = 1/2$, $\nu = n/2$ and $\sigma = (n+1)/2$, we have

$$\frac{1}{r^{n/2}} \int_0^\infty e^{-s^2/2} J_{n/2}(rs) s^{n/2}\, ds = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+2}{2}\right)} {}_1F_1\left(\frac{n+1}{2}; \frac{n+2}{2}; -\frac{r^2}{2}\right).$$

See [3]. Therefore,

$$R_jG(x) = \frac{x_j\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2}\,(2\pi)^{n/2}\Gamma\left(\frac{n+2}{2}\right)}\,{}_1F_1\left(\frac{n+1}{2};\frac{n+2}{2};-\frac{|x|^2}{2}\right). \quad (5)$$

In particular, since (see [3])

$${}_1F_1(a;c;z) \sim \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}, \qquad \mathrm{Re}(z) \to -\infty,$$

we have

$$R_jG(x) \sim \frac{x_j\Gamma\left(\frac{n+1}{2}\right)}{|x|^{n+1}\pi^{(n+1)/2}}, \qquad |x| \to \infty. \quad (6)$$

Finally, since

$${}_1F_1(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{zs}s^{a-1}(1-s)^{c-a-1}\,ds,$$

$$\mathrm{Re}(c) > \mathrm{Re}(a) > 0,$$

(see [3]), we obtain

$$R_jG(x) = \frac{x_j}{(2\pi)^{(n+1)/2}}\int_0^1 e^{-|x|^2s/2}s^{(n-1)/2}(1-s)^{-1/2}\,ds$$

$$= \frac{2x_je^{-|x|^2/2}}{|x|^n(2\pi)^{(n+1)/2}}\int_0^{|x|} e^{s^2/2}(|x|^2-s^2)^{(n-1)/2}\,ds.$$

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Eastern Washington University
Cheney, Washington