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ON c_0-SATURATED BANACH SPACES

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A Banach space E is c_0-saturated if every closed infinite dimensional subspace of E contains an isomorph of c_0. In [2] and [3], it was asked whether all quotient spaces of c_0-saturated spaces having unconditional bases are also c_0-saturated. In [3], Rosenthal expressed the opinion that the answer should be no. Here, we construct an example which confirms this opinion.

In §1, a simple criterion for c_0-saturation is introduced. A key step in verifying that the space constructed satisfies the criterion is the Decomposition Lemma (Lemma 17), which may be of independent interest.

Standard Banach space terminology, as may be found in [1], is employed. For $1 \leq p \leq \infty$, $\|\cdot\|_p$ denotes the l^p-norm. For $1 < p < \infty$, $l^{p,\infty}$ is the Banach space of all real sequences (a_n) such that

$$\|(a_n)\|_{p,\infty} = \sup a_n^* n^{1/p} < \infty,$$

where (a_n^*) is the decreasing rearrangement of $(|a_n|)$. And c_{00} is the vector space of all finitely nonzero real sequences. Some facts concerning vector lattices will also be required. References for these may be found in [4]. In particular, let us mention that a subset S of a vector lattice is *solid* if $x \in S$ whenever there exists $y \in S$ with $|x| \leq |y|$. The *solid hull* of S is the smallest solid set containing S. Finally, the cardinality of a set A is denoted by $|A|$.

1. A criterion for c_0-saturation

In this section, we prove a simple criterion for c_0-saturation which will be used below.

PROPOSITION 1. *Let (e_i) be a normalized unconditional basis of a Banach space F. If (e_i) has the property*

(∗) *every normalized block basis $(\sum_{i=j_k+1}^{j_{k+1}} a_i e_i)$ of (e_i) such that $a_i \to 0$ has a subsequence equivalent to the c_0-basis,*

then F is c_0-saturated.

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Proof. By [1], Proposition 1.a.11, every closed subspace of F contains a basic sequence equivalent to some block basis of (e_i). Thus it is enough to show that every closed subspace of F generated by a block basis of (e_i) contains a c_0-sequence. Let (x_k) be a normalized block basis of (e_i). Note that the coefficients of expansion of the elements x_k with respect to (e_i) are uniformly bounded since (e_i) is normalized. If (x_k) has a subsequence (x_{k_j}) such that $\sup_n \|\Sigma_{j=1}^n x_{k_j}\| < \infty$, then we are done. Otherwise, by taking long averages of (x_k), we obtain a normalized block basis

$$(y_k) = \left(\frac{\Sigma_{i=j_k+1}^{j_{k+1}} x_i}{\|\Sigma_{i=j_k+1}^{j_{k+1}} x_i\|}\right) = \left(\sum_{i=l_k+1}^{l_k+1} a_i e_i\right)$$

such that $a_i \to 0$. Then, by the assumption, (y_k) has a subsequence equivalent to the c_0-basis. □

The example that we are going to construct in the subsequent sections shows that a Banach space with an unconditional basis satisfying $(*)$ may not have a l^1-saturated dual. However, the following proposition is easy to obtain.

PROPOSITION 2. *Let (e_i) be a normalized unconditional basis of a Banach space F. If (e_i) has property $(*)$, and $\|\Sigma_{i\in A_k} e_i\| \to \infty$ whenever (A_k) is a sequence of subsets of* $\mathbf{N}$ *such that* $\max A_k < \min A_{k+1}$ *and* $|A_k| \to \infty$, *then F' is l^1-saturated.*

2. Definition of the space E and simple properties

Let $D = \{(i,j): i, j \in \mathbf{N}, i \geq j\}$ and let G be the vector lattice of all finitely supported functions $x: D \to \mathbf{R}$. For all $i \in \mathbf{N}$, define $z_i: G \to \mathbf{R}$ by $\langle x, z_i\rangle = i^{-1}\Sigma_{j=1}^i x(i,j)$. We are going to construct a space E, which is the completion of G with respect to some norm, so that the sequence (z_i) is a l^2-sequence in the dual of E. Moreover, this must be done without introducing any sequence biorthogonal to (z_i) whose linear combinations can be "normed" by vectors in span $\{z_i\}$. So we make the following definitions. Let

$$B = \{b = (b_i) \in c_{00} : ib_i \in \mathbf{N} \cup \{0\} \text{ for all } i, \|b\|_2 \leq 1\}.$$

For all $b \in B$, define $x_b \in G$ by

$$x_b(i,j) = \begin{cases} 1 & \text{for } 1 \leq j \leq ib_i, i \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

We use the set $\{x_b : b \in B\}$ to "norm" the vectors in span $\{z_i\}$. Let U be the convex solid hull of $\{x_b : b \in B\}$. Then all the elements of the set U must be in the unit ball of the space E we are trying to construct. (The word "solid" is needed since we want a space with an unconditional basis.) Define a seminorm ρ on G by

$$\rho(x) = \left\| \left(\frac{1}{i} \sum_{j=1}^{i} |x(i,j)| \right)_{i=1}^{\infty} \right\|_2 .$$

Two elements $x, y \in G$ are *row disjoint* if

$$\sum_{j=1}^{i} |x(i,j)| \cdot \sum_{j=1}^{i} |y(i,j)| = 0$$

for all i. To try to obtain c_0-saturation, we admit into the unit ball of E *some* elements of the form $y_1 + \cdots + y_m$, where $y_1, \ldots, y_m \in U$ are pairwise row disjoint. However, to keep the equivalence of (z_i) to the l^2-basis, the elements admitted must have uniformly bounded ρ-norms. Fix $1 < p < 2$, and let

$$A = \{ y_1 + \ldots + y_m : m \in \mathbf{N}, y_1, \ldots, y_m \in U \text{ pairwise row disjoint}, \\ \|(\rho(y_1), \ldots, \rho(y_m))\|_{p,\infty} \le 1 \}.$$

For $y = y_1 + \cdots + y_m$, where $y_1, \ldots, y_m$ are as above, we say that the sum on the right is a *representative* of the element $y \in A$, and m is the *length* of the representative. Finally, let V be the convex hull of A. Being the convex hull of a solid set, V is solid as well [4, Proposition II.2.2].

Note that $\|x\|_\infty \le 1$ for all $x \in A$. Hence $\|x\|_\infty \le 1$ for all $x \in V$. Thus $\cap_{\lambda > 0} \lambda V = \{0\}$. It follows that the gauge functional τ of V is a lattice norm on G. Let E be the completion of G with respect to the norm τ. For every $(i, j) \in D$, let $e_{i,j} \in G$ be the characteristic function of $\{(i, j)\}$. Since $\|e_{i,j}\|_\infty = 1$, $\tau(e_{i,j}) \ge 1$. On the other hand, $e_{i,j} \in U$. Hence $\tau(e_{i,j}) = 1$. It is clear that $(e_{i,j})$ a normalized unconditional basis of E.

LEMMA 3. *Let $C = \sqrt{\sum_{n=1}^{\infty} n^{-2/p}}$. Then $\rho(x) \le C$ for all $x \in V$.*

Proof. Let $y = \sum_{i=1}^{m} y_i \in A$, where $y_1, \ldots, y_m \in U$ are pairwise row disjoint and

$$\|(\rho(y_1), \ldots, \rho(y_m))\|_{p,\infty} \le 1.$$

Then

$$\|(\rho(y_1), \ldots, \rho(y_m))\|_2 \le C.$$

Since $y_1, \ldots, y_m \in U$ are pairwise row disjoint,

$$\rho(y) = \|(\rho(y_1), \ldots, \rho(y_m))\|_2 \leq C.$$

The result now follows easily since $V = co(A)$. □

PROPOSITION 4. *E has a quotient space isomorphic to l^2.*

Proof. It suffices to show that the sequence $(z_i) \subseteq G' = E'$ defined above is equivalent to the l^2-basis. Let (b_i) be a finitely supported sequence on the unit sphere of l^2. For any $x \in V$,

$$\begin{aligned}\left\langle x, \sum b_i z_i \right\rangle &= \sum_i \frac{b_i}{i} \sum_{j=1}^{i} x(i,j) \\ &\leq \left\| \left(\frac{1}{i} \sum_{j=1}^{i} x(i,j) \right)_{i=1}^{\infty} \right\|_2 \\ &\leq \rho(x) \\ &\leq C.\end{aligned}$$

Hence $\tau'(\Sigma b_i z_i) \leq C$, where τ' denotes the norm dual to τ.

On the other hand, we claim that $\tau'(\Sigma b_i z_i) \geq 2/9$. Indeed, if $\|(b_1, \ldots, b_4)\|_2 \geq \sqrt{5}/3$, then since (z_i) is clearly pairwise disjoint (in the lattice sense) and $\tau'(z_i) \geq 1$ for all i,

$$\begin{aligned}\tau'\left(\sum b_i z_i\right) &\geq \sup |b_i| \\ &\geq \frac{1}{2} \|(b_1, \ldots, b_4)\|_2 \\ &\geq \sqrt{5}/6 \\ &\geq 2/9.\end{aligned}$$

Now if $\|(b_1, \ldots, b_4)\|_2 < \sqrt{5}/3$, then $\alpha \equiv \|(b_5, b_6, \ldots)\|_2 > 2/3$. For all $i > 4$, let m_i be the largest non-negative integer $\leq i|b_i|$. Then $m \equiv (0, 0, 0, 0, m_5/5, m_6/6, \ldots) \in B$. Let $y \in G$ be given by

$$y(i,j) = \begin{cases} \operatorname{sgn} b_i & \text{if } 1 \leq j \leq m_i, i > 4 \\ 0 & \text{otherwise.} \end{cases}$$

Then $|y| = x_m \in U$, and hence $y \in V$. Therefore,

$$\begin{aligned}\tau'\left(\sum b_i z_i\right) &\geq \left\langle y, \sum b_i z_i\right\rangle \\ &= \sum_i \frac{b_i}{i} \sum_{j=1}^{i} y(i,j) \\ &= \sum_{i>4} \frac{m_i |b_i|}{i}.\end{aligned}$$

Since $m_i/i \geq |b_i| - 1/i$ for $i > 4$, we have

$$\begin{aligned}\tau'\left(\sum b_i z_i\right) &\geq \sum_{i>4} |b_i|^2 - \sum_{i>4} \frac{|b_i|}{i} \\ &\geq \alpha^2 - \alpha \sum_{i>4} i^{-2} \\ &\geq \alpha^2 - \frac{\alpha}{3} \\ &> \frac{2}{9},\end{aligned}$$

since $\alpha > 2/3$. □

3. Proof that E is c_0-saturated

Let $k \in \mathbf{N}$, a collection of real sequences is *k-disjoint* if the pointwise product of any $k + 1$ members of the collection is the zero sequence; equivalently, if at most k of the sequences can be non-zero at any fixed coordinate. We begin with some elementary lemmas.

LEMMA 5. *Let $\{y_1, \dots, y_k\}$ be a finite subset of l^2, then*

$$\left\|\sum_{i=1}^{k} y_i\right\|_2 \leq \sqrt{k}\left(\sum_{i=1}^{k} \|y_i\|_2^2\right)^{1/2}.$$

LEMMA 6. *For any $k, n \in \mathbf{N}$, and any k-disjoint subset $\{x_1, \dots, x_n\}$ of the unit ball of l^2,*

$$\left\|\sum_{i=1}^{n} x_i\right\|_2 \leq \sqrt{kn}.$$

Proof. Write $x_i = (x_i(j))$ for $1 \le i \le n$. For each j, let $y_1(j), \ldots, y_n(j)$ be the decreasing rearrangement of $|x_1(j)|, \ldots, |x_n(j)|$. Then let $y_i = (y_i(j))$. By the k-disjointness, $y_i = 0$ for $i > k$. Hence $|\sum_{i=1}^n x_i| \le \sum_{i=1}^k y_i$ (pointwise order). Therefore,

$$\begin{aligned}
\left\| \sum_{i=1}^n x_i \right\|_2 &\le \left\| \sum_{i=1}^k y_i \right\|_2 \\
&\le \sqrt{k}\left(\sum_{i=1}^k \|y_i\|_2^2 \right)^{1/2} \qquad \text{by Lemma 5} \\
&= \sqrt{k}\left(\sum_{i=1}^k \sum_j y_i(j)^2 \right)^{1/2} \\
&= \sqrt{k}\left(\sum_{i=1}^n \sum_j x_i(j)^2 \right)^{1/2} \\
&= \sqrt{k}\left(\sum_{i=1}^n \|x_i\|_2^2 \right)^{1/2} \\
&\le \sqrt{kn}\,.
\end{aligned}$$

□

LEMMA 7. *Let $b_j \in B, 1 \le j \le n$, and write $b_j = (b_j(i))_i$. Then for all $(i_0, j_0) \in D$,*

$$\left(\sum_{j=1}^n x_{b_j} \right)(i_0, j_0) = |\{j : i_0 b_j(i_0) \ge j_0\}|\,.$$

Consequently, if $c_j = (c_j(i))_i$, $1 \le j \le l$, is another collection of elements in B such that, for every i, the nonzero numbers in the list $(c_j(i))_{j=1}^l$ is a rearrangement of the nonzero numbers in the list $(b_j(i))_{j=1}^k$, then $\sum_{j=1}^k x_{b_j} = \sum_{j=1}^l x_{c_j}$.

Proof. The second statement follows from the first. The first statement is verified by direct computation. □

In the sequel, we will have many occasions to compute with elements of the set U. The next lemma is a technical device which allows us to replace a general element of U by an average of elements from the set $\{x_b : b \in B\}$. The following easy remark is used in the proof. If, in a vector space, a vector y is expressed as an average $n^{-1}\sum_{i=1}^n y_i$, and $m \in \mathbb{N}$, then $y = (mn)^{-1}\sum_{i=1}^{mn} z_i$, where the sequence (z_i) is the m-fold repetition of (y_i).

LEMMA 8. *Let $x \in co\{x_b : b \in B\}$ with rational coefficients, and let r be a positive rational number. Then there exists $y \in co\{x_b : b \in B\}$ with rational coefficients so that $\|y\|_\infty \le r$ and $y(i,j) = \min\{x(i,j), r\}$ for all $(i,j) \in D$.*

Proof. Using the above remark, we can find $m, n \in \mathbf{N}$, and $b_1, \dots, b_n \in B$ such that

$$x = n^{-1} \sum_{j=1}^{n} x_{b_j} \quad \text{and} \quad r = m/n.$$

Let $I = \{i : x(i,1) > r\}$. By Lemma 7, $i \in I$ if and only if $|\{j : b_j(i) \neq 0\}| > m$. For each $i \in I$, choose a subset J_i of $L_i = \{j : b_j(i) \neq 0\}$ of cardinality m such that whenever $k \in J_i$ and $j \in L_i \setminus J_i$, $b_k(i) \ge b_j(i)$. For $1 \le j \le n$, define $c_j = (c_j(i))_i$ by

$$c_j(i) = \begin{cases} b_j(i) & \text{if } i \notin I \\ b_j(i) & \text{if } i \in I \text{ and } j \in J_i \\ 0 & \text{if } i \in I \text{ and } j \in \{1, \dots, n\} \setminus J_i. \end{cases}$$

Then $c_j \in B$ for $1 \le j \le n$. Therefore, $y = n^{-1}\sum_{j=1}^{n} x_{c_j} \in co\{x_b : b \in B\}$ with rational coefficients. For $(i,j) \in D$, apply Lemma 7 to compute $x(i,j)$ and $y(i,j)$. If $i \notin I$, then $c_j(i) = b_j(i)$, $1 \le j \le n$. Hence $r \ge x(i,1) \ge x(i,j) = y(i,j)$ for $1 \le j \le i$. Now consider the case when $i \in I$. Let $j_0 = \max\{ib_k(i) : k \in L_i \setminus J_i\}$. If $1 \le j \le j_0$, $ib_k(i) \ge j$ for at least one $k \in L_i \setminus J_i$. But by the choice of J_i, $ic_k(i) = ib_k(i) \ge j_0$ for all $k \in J_i$ as well. Hence

$$x(i,j) = \frac{1}{n}|\{k : ib_k(i) \ge j\}| \ge \frac{m+1}{n} > r,$$

and

$$y(i,j) = \frac{1}{n}|\{k : ic_k(i) \le j\}| = \frac{|J_i|}{n} = r.$$

Finally, if $i \in I$ and $j_0 < j$, then $ib_k(i) < j$ for all $k \in L_i \setminus J_i$. Certainly $ib_k(i) = 0 < j$ as well as for all $k \in \{1, \dots, n\} \setminus L_i$. Therefore $ib_k(i) \ge j$ only if $k \in J_i$, which implies $b_k(i) = c_k(i)$. Thus

$$\begin{aligned} x(i,j) &= \frac{1}{n}|\{k : ib_k(i) \ge j\}| \\ &= \frac{1}{n}|\{k \in J_i : ib_k(i) \ge j\}| \\ &= \frac{1}{n}|\{k \in J_i : ic_k(i) \ge j\}| \\ &= \frac{1}{n}|\{k : ic_k(i) \ge j\}| \\ &= y(i,j). \end{aligned}$$

Also, it follows from the second equality that

$$x(i,j) \leq \frac{|J_i|}{n} = \frac{m}{n} = r.$$

Hence in all cases, $y(i,j) = \min\{x(i,j), r\}$. □

LEMMA 9. *If $x \in U$ satisfies $\|x\|_\infty \leq \varepsilon$ for some $\varepsilon > 0$, then $\rho(x) \leq \sqrt{\varepsilon}$.*

Proof. Since $x \in U$, $|x| \leq z$ for some $z \in co\{x_b : b \in B\}$. Up to an arbitrarily small perturbation, we may even assume that z is a convex combination with rational coefficients. Given a rational number $r > \varepsilon$, apply Lemma 8 to obtain a $y \in co\{x_b : b \in B\}$ with rational coefficients so that $\|y\|_\infty \leq r$ and $y(i,j) = \min\{z(i,j), r\}$. Then $y \geq |x|$ and y can be expressed in the form $y = n^{-1}\sum_{i=1}^n x_{b_i}$, where $b_i \in B, 1 \leq i \leq n$. Let j be the greatest integer $\leq rn$. Then $\|y\|_\infty \leq r$ implies $\{b_1, \ldots, b_n\}$ is j-disjoint. Therefore

$$n^{-1}\left\| \sum_{i=1}^n b_i \right\|_2 \leq \sqrt{\frac{j}{n}} \leq \sqrt{r}$$

by Lemma 6. It remains to observe that the leftmost quantity in the above inequality is precisely $\rho(y)$, which is $\geq \rho(x)$, and $r > \varepsilon$ is arbitrary. □

The next lemma is a quantitative version of the fact that the unit vector basis of $l^{p,\infty}$ generates a c_0-saturated closed subspace.

LEMMA 10. *Let (a_i) be a real sequence, and $0 = n_0 < n_1 < \ldots$ a sequence of integers so that*
(1) $\|(a_{n_k+1}, \ldots, a_{n_{k+1}})\|_{p,\infty} \leq 1$, *and*
(2) $\|(a_{n_k+1}, \ldots, a_{n_{k-1}})\|_\infty \leq n_k^{-1/p}$
for all $k \geq 0$. Then $\|(a_1, a_2, \ldots)\|_{p,\infty} \leq 2$.

Proof. Assume the contrary. Then there exists n such that $a_n^* > 2n^{-1/p}$. Hence

$$J = \left\{ i : |a_i| > 2n^{-1/p} \right\}$$

has cardinality $\geq n$. Since $a_i \to 0$ as $i \to \infty$, J is finite. Let j be the largest element in J. Choose k so that $n_k < j \leq n_{k+1}$. Then

$$2n^{-1/p} < |a_j| \leq n_k^{-1/p}.$$

Hence $n_k < 2^{-p}n$. Therefore,

$$|J \cap \{n_k + 1, \dots, n_{k+1}\}| > n(1 - 2^{-p}).$$

Consequently,

$$\|(a_{n_k+1}, \dots, a_{n_{k+1}})\|_{p,\infty} \geq 2n^{-1/p}\|(\overbrace{1, \dots, 1}^{n(1-2^{-p})})\|_{p,\infty} > 1,$$

a contradiction. □

In preparation for the proof of the key Proposition 12 below, we introduce some more terminology. For $y \in A$, let

$$\phi(y) = \min\{\|(\rho(y_1), \dots, \rho(y_m))\|_\infty : y_1 + \dots + y_m \text{ is a representative of } y\}.$$

The minimum exists since y is finitely supported. A representative of y at which the above minimum is attained will be called a *good representative*. An element $y \in A$ with $\phi(y) \leq \varepsilon$ is said to be *ε-small*. A subset of A is *ε-small* if all of its members are ε-small. The *support* of an element $x \in G$ is written as supp x.

LEMMA 11. *Let $(y_i) \subseteq A$ be pairwise row disjoint, and let $(m_i) \subseteq \mathbf{N}$. Suppose each y_i has a good representative of length m_i, and $\phi(y_{i+1}) \leq (\sum_{k=1}^{i} m_k)^{-1/p}$ for all i. Then*

$$\sup_n \tau\left(\sum_{i=1}^{n} y_i\right) \leq 2.$$

Proof. For each i, choose a good representative $y_i(1) + \cdots + y_i(m_i)$ of length m_i. We may clearly assume that supp $y_i(k) \subseteq$ supp y_i for $1 \leq k \leq m_i$. Then, for any n, the elements $(y_i(k))_{k=1}^{m_i}{}_{i=1}^{n}$ are pairwise row disjoint. Using Lemma 10 and the assumptions, we see that $\|(\rho(y_i(k)))_{k=1}^{m_i}{}_{i=1}^{n}\|_{p,\infty} \leq 2$. Hence

$$2^{-1}\sum_{i=1}^{n} y_i = 2^{-1}\sum_{i=1}^{n}\sum_{k=1}^{m_i} y_i(k) \in A.$$

Therefore $\tau(\sum_{i=1}^{n} y_i) \leq 2$. □

DEFINITION. A sequence $(y_i) \subseteq V$ is called *strongly decreasing* if there exists a sequence (ε_i) decreasing to 0 such that for every i, there is a ε_i-small subset A_i of A with $y_i \in co(A_i)$.

PROPOSITION 12. *A τ-normalized, pairwise row disjoint, strongly decreasing sequence in V has a subsequence equivalent to the c_0-basis.*

Proof. Let $(y_i) \subseteq V$ be τ-normalized, pairwise row disjoint and strongly decreasing, and let (ε_i) be as in the above definition. Write $y_1 = n_1^{-1}\Sigma_{j=1}^{n_1} y_j^1$, where $y_j^1 \in A$, and $\operatorname{supp} y_j^1 \subseteq \operatorname{supp} y_1$ for $1 \le j \le n_1$. Choose m_1 so large that each y_j^1 has a good representative of length $\le m_1$. Going to a subsequence and relabeling, we may assume that $\varepsilon_2 \le m_1^{-1/p}$. Using the remark preceding Lemma 8, we can write

$$y_2 = (n_1 n_2)^{-1} \sum_{j=1}^{n_1 n_2} y_j^2,$$

with $y_j^2 \in A, \phi(y_j^2) \le \varepsilon_2$, and $\operatorname{supp} y_j^2 \subseteq \operatorname{supp} y_2$ for $1 \le j \le n_1 n_2$. Now choose m_2 so large that each y_j^2 has a good representative of length $\le m_2$. Relabel again to assume $\varepsilon_3 \le (m_1 + m_2)^{-1/p}$. Continuing inductively, we obtain a subsequence of (y_i), which we label as (y_i) again, so that for all i,

$$y_i = (n_1 \cdots n_i)^{-1} \sum_{j=1}^{n_1 \cdots n_i} y_j^i,$$

where $y_j^i \in A$, $\phi(y_j^i) \le \varepsilon_i$, and $\operatorname{supp} y_j^i \subseteq \operatorname{supp} y_i$ for $1 \le j \le n_1 \cdots n_i$. Furthermore, each y_j^i has a good representative of length $\le m_i$, and $\varepsilon_{i+1} \le (\Sigma_{k=1}^i m_k)^{-1/p}$. For all i, let

$$Y^i = \left\{y_j^i : 1 \le j \le n_1 \cdots n_i\right\}.$$

If a z_i is chosen from Y^i for each i, then (z_i) is a pairwise row disjoint sequence in A satisfying the assumptions of Lemma 11. Hence $\sup_l \tau(\Sigma_{i=1}^l z_i) \le 2$ by the same lemma. Fix l. Using again the remark preceding Lemma 8, we express each y_i, $1 \le i \le l$, as an average $y_i = r^{-1}\Sigma_{j=1}^r z_j^i$, where $r = n_1 \cdots n_l$, and $z_j^i \in Y^i$. Therefore,

$$\tau\left(\sum_{i=1}^l y_i\right) = \tau\left(r^{-1} \sum_{j=1}^r \sum_{i=1}^l z_j^i\right)$$
$$\le r^{-1} \sum_{j=1}^r \tau\left(\sum_{i=1}^l z_j^i\right).$$

Since it was observed that $\tau(\Sigma_{i=1}^l z_j^i) \le 2$ for every j, we see that $\tau(\Sigma_{i=1}^l y_i) \le 2$. Finally, since (y_i) is τ-normalized, a well known result of Bessaga and Pelczynski ([1], Proposition 2.e.4) asserts that it has a c_0-subsequence. □

What will be shown is that every sequence (y_i) in V with $\|y_i\|_\infty \to 0$ can be written as the sum of a strongly decreasing sequence and a τ-null sequence. The proof of this fact relies on the next result.

MAIN LEMMA. *If $x \in U$ satisfies $\|x\|_\infty \le \varepsilon$ for some $\varepsilon > 0$, then $\tau(x) \le 5\varepsilon^{1/4}$.*

The proof of the Main Lemma will be given in the next section. Assuming the result, we continue with:

LEMMA 13. *Let $y \in V$ satisfy $\|y\|_\infty \le \varepsilon$. Then there exist an $\varepsilon^{1/8p}$-small subset S of A, and $u \in co(S)$, such that $\tau(y - u) \le 5\varepsilon^{1/8}$.*

Proof. There is no loss of generality in assuming that $y \ge 0$. Express y as a convex combination $\sum_{i=1}^{j} \alpha_i z_i$ where $z_i \in A$, $1 \le i \le j$. Then $y \le \sum_{i=1}^{j} \alpha_i |z_i|$. By the Riesz Decomposition Property [4, Proposition II.1.6], $y = \sum_{i=1}^{j} \alpha_i y_i$ for some $0 \le y_i \le |z_i|$. Since A is solid, $y_i \in A$ for all i. Choose a representative $\sum_{l=1}^{m} y_i(l)$ for each y_i. (All the representatives can be made to have the same length m by adding on zeros if necessary.) Note that $y_i(l) \ge 0$ for all i and l. Let $\delta = \varepsilon^{1/8p}$ and let $A_i = \{1 : \rho(y_i(l)) > \delta\}$. Since $y \in V$,

$$\begin{aligned} 1 &\ge \|(\rho(y_i(1)), \ldots, \rho(y_i(m)))\|_{p,\infty} \\ &\ge \delta \|(\underbrace{1, \ldots, 1}_{|A_i|})\|_{p,\infty} \\ &= \delta |A_i|^{1/p}. \end{aligned}$$

Hence $|A_i| \le \delta^{-p}$ for all i. Let r be the greatest integer $\le \delta^{-p}$. Relabeling, we may assume that each A_i is an initial interval $\{1, \ldots, r_i\}$, where $r_i \le r$. Define

$$v_l = \sum_{i=1}^{j} \alpha_i y_i(l), \quad 1 \le l \le r,$$

and let

$$v = \sum_{l=1}^{r} v_l.$$

Then $v_1 \in U$, and $\|v_l\|_\infty \le \|y\|_\infty \le \varepsilon$ for all l. (The positivity of $y_i(l)$ is used here.) By the Main Lemma, we obtain

$$\begin{aligned}\tau(v) &\leq \sum_{l=1}^{r} \tau(v_l) \\ &\leq 5\varepsilon^{1/4} r \\ &\leq 5\varepsilon^{1/4}\delta^{-p} \\ &= 5\varepsilon^{1/8}.\end{aligned}$$

Now let $u_i = \sum_{l=r+1}^{m} y_i(l)$, $S = \{u_i : 1 \leq i \leq j\}$, and $u = \sum_{i=1}^{j} \alpha_i u_i$. It is clear that S is an $\varepsilon^{1/8p}$-small subset of A, $u \in co(S)$, and $\tau(y - u) = \tau(v) \leq 5\varepsilon^{1/8}$. □

The promised result is now immediate.

PROPOSITION 14. *Every sequence (y_i) in V satisfy $\|y_i\|_\infty \to 0$ can be written as the sum of a strongly decreasing sequence and a τ-null sequence.*

Using Propositions 1, 12, 14, and the standard perturbation result Proposition 1.a.9 in [1], we obtain:

THEOREM 15. *Every normalized, pairwise disjoint, finitely supported sequence (y_i) in E with $\|y_i\|_\infty \to 0$ has a subsequence equivalent to the c_0-basis. Consequently, E is c_0-saturated.*

4. Proof of the Main Lemma

In this section, we prove the Main Lemma. In fact, we will show that if $x \in U$ satisfies $\|x\|_\infty \leq \varepsilon$, then $x \in 5\varepsilon^{1/4}A$. The basic idea is as follows. Given such a x, Lemma 9 says that $\rho(x) \leq \sqrt{\varepsilon}$. Let n be any natural number $\leq 5^p\varepsilon^{-p/4}$. Then if x is written as a pairwise row disjoint sum $x = \sum_{i=1}^{n} x_i$,

$$\begin{aligned}\|(\rho(x_1), \ldots, \rho(x_n))\|_{p,\infty} &\leq \rho(x)\|(\overbrace{1, \ldots, 1}^{n})\|_{p,\infty} \\ &\leq \sqrt{\varepsilon}\, n^{1/p} \\ &\leq 5\varepsilon^{1/4}.\end{aligned}$$

What we need to show is that the x_i's can be chosen so that they come from a small multiple ($5\varepsilon^{1/4}$) of U. The key step in this regard is the decomposition result Lemma 17. The proof of the next lemma is left to the reader.

LEMMA 16. *Let $(a_i)_{i=1}^{l}$ be numbers in $[0, 1]$ such that $\sum_{i=1}^{l} a_i > 1$. Then there exists $S \subseteq \{1, \ldots, l\}$ such that $1/2 \leq \sum_{i \in S} a_i \leq 1$.*

For a real-valued matrix $a = (a_{i,j})_{i=1}^{k}{}_{j=1}^{l}$, we define $\Sigma(a) = \Sigma_{i,j} a_{i,j}$, and, for each j,

$$s_j(a) = \min\{i : a_{i,j} \neq 0\} \quad (\min \varnothing = 0).$$

LEMMA 17. *Let $a = (a_{i,j})_{i=1}^{k}{}_{j=1}^{l}$ be a $[0,1]$-valued matrix such that $\Sigma(a) \le M$. Then there is a partition $R_1, \dots, R_n$ of $R = \{(i,j) : 1 \le i \le k, 1 \le j \le l\}$ such that*
(1) $n \le 2M + k$,
(2) $\Sigma_{(i,j)\in R_m} a_{i,j} \le 1$, $1 \le m \le n$, *and*
(3) $|R_m \cap \{(i,j) : 1 \le i \le k\}| \le 1$ *for all* $1 \le m \le n$ *and* $1 \le j \le l$.

Proof. A $[0,1]$-valued matrix $b = (b_{i,j})_{i=1}^{k}{}_{j=1}^{l}$ will be called *reducible* if $\Sigma_{j=1}^{l} b_{s_j(b),j} > 1$, where $b_{s_j(b),j}$ is taken to be 0 if $s_j(b) = 0$. For a reducible matrix b, Lemma 16 provides a set $S(b) \subseteq \{(s_j(b), j) : s_j(b) > 0\}$ such that $1/2 \le \Sigma_{(i,j)\in S(b)} b_{i,j} \le 1$. We also let the *reduced* matrix b' be given by

$$b'_{i,j} = \begin{cases} b_{i,j} & \text{if } (i,j) \notin S(b) \\ 0 & \text{otherwise} \end{cases}$$

Now let the matrix a be as given. Since all the conditions are invariant with respect to permutations of the entities of a within columns, we may assume that $a_{i,j} \ge a_{i+1,j}$. Let $a^1 = a$. If a^1 is reducible, let $R_1 = S(a^1)$ and $a^2 = a^{1\prime}$. Inductively, if a^r is reducible, let $R_r = S(a^r)$ and $a^{r+1} = a^{r\prime}$. Note that $\Sigma(a^{r+1}) \le \Sigma(a^r) - 1/2$ if a^r is reducible. Therefore, there must be a $t < 2M$ such that a^{t+1} is not reducible. Now let

$$R_{t+i} = \{(i,j) : (i,j) \notin R_1 \cup \cdots \cup R_t\}$$

for $1 \le i \le k$. Let $n = t + k$. It is clear that the collection $R_1, \dots, R_n$ satisfies the requirements. □

Proof of the Main Lemma. Since $x \in U$ already implies $\tau(x) \le 1$, we may assume $\varepsilon < 1$. As in the proof of Lemma 9, it suffices to prove the Main Lemma for those x's which have the form $x = M^{-1}\Sigma_{i=1}^{M} x_{b_i}$. Write $b_i = (b_{i,j})_j$ for $1 \le i \le M$. Since $b_i \in c_{00}$, there exists l such that $b_{i,j} = 0$ for all $j > l$. Also, $\|x\|_\infty \le \varepsilon$ implies $\{b_1, \dots, b_M\}$ is k-disjoint, where k is the greatest integer $\le \varepsilon M$. For each j, let $a_{1,j}, \dots, a_{M,j}$ be the decreasing rearrangement of $b_{1,j}^2, \dots, b_{M,j}^2$. By the k-disjointness, $a_{i,j} = 0$ for all $i > k$. Now let $a = (a_{i,j})_{i=1}^{k}{}_{j=1}^{l}$. Then a is a $[0,1]$-valued matrix, and

$$\Sigma(a) = \sum_{i=1}^{M} \sum_{j=1}^{l} b_{i,j}^2 = \sum_{i=1}^{M} \|b_i\|_2^2 \le M.$$

Let $\eta = (2\varepsilon^{-p/4} - 1)^{-1}$. Since we are assuming that $\varepsilon < 1$, η is positive and $\le \varepsilon^{1/4}$. Choose the smallest integer j_1 such that $\sum_{i=1}^{k}\sum_{j=1}^{j_1} a_{i,j} > \eta M$. Since $\sum_{i=1}^{k} a_{i,j} \le k \le \varepsilon M$ for all j,

$$\sum_{i=1}^{k} \sum_{j=1}^{j_1} a_{i,j} \le (\eta + \varepsilon) M.$$

If $\sum_{i=1}^{k}\sum_{j=j_1+1}^{l} a_{i,j} > \eta M$, we choose the smallest integer j_2 such that

$$\sum_{i=1}^{k} \sum_{j=j_1+1}^{j_2} a_{i,j} > \eta M.$$

Continuing inductively, we obtain $0 = j_0 < j_1 < \dots < j_t = l$ such that

$$\eta M < \sum_{i=1}^{k} \sum_{j_m+1}^{j_m+1} a_{i,j} \le (\eta + \varepsilon) M$$

for $0 \le m \le t - 2$, and

$$\sum_{i=1}^{k} \sum_{j=j_{t-1}+1}^{j_t} a_{i,j} \le \eta M.$$

Note that

$$(t - 1)\eta M < \sum_{m=0}^{t-2} \sum_{i=1}^{k} \sum_{j=j_m+1}^{j_m+1} a_{i,j} \le \Sigma(a) \le M$$

implies $t \le \eta^{-1} + 1$. With M replaced by $(\eta + \varepsilon)M$, apply Lemma 17 to each submatrix $(a_{i,j})_{i=1}^{k}{}_{j=j_m+1}^{j_{m+1}}$ to obtain a partition $R_1^m, \dots, R_{r_m}^m$ of $\{(i,j) : 1 \le i \le k, j_m < j \le j_{m+1}\}$ $(0 \le m < t)$ such that

$$r_m \le 2(\eta + \varepsilon)M + k \le (2\eta + 3\varepsilon)M,$$
$$\sum_{(i,j) \in R_\nu^m} a_{i,j} \le 1, 1 \le \nu \le r_m,$$

and

$$|R_\nu^m \cap \{(i,j) : 1 \le i \le k\}| \le 1 \text{ for all } m, \nu, \text{ and } j.$$

Finally, if j is such that $|R_\nu^m \cap \{(i,j): 1 \le i \le k\}| = 1$, and (i,j) is the unique element of this set, let

$$d_\nu^m(j) = \sqrt{a_{i,j}};$$

otherwise, let $d_\nu^m(j) = 0$. Then for all m and ν, the sequence $d_\nu^m = (d_\nu^m(j))_{j=1}^\infty \in B$. Note that for every j, the nonzero numbers in the list $(d_\nu^m(j))_{\nu=1}^{r_m}{}_{m=0}^{t-1}$ is a rearrangement of the nonzero numbers in the list $(b_{1,j}, \dots, b_{M,j})$. Hence by Lemma 7,

$$\sum_{i=1}^{M} x_{b_i} = \sum_{m=0}^{t-1} \sum_{\nu=1}^{r_m} x_{d_\nu^m}.$$

Now let $y_m = M^{-1}\sum_{\nu=1}^{r_m} x_{d_\nu^m}, 0 \le m < t$. Then $y_0, \dots, y_{t-1}$ are pairwise row disjoint, and $x = y_0 + \cdots + y_{t-1}$. Also,

$$\begin{aligned} y_m &= \frac{r_m}{M}\left(r_m^{-1} \sum_{\nu=1}^{r_m} x_{d_\nu^m}\right) \\ &\in \frac{r_m}{M} U \\ &\subseteq (2\eta + 3\varepsilon) U \\ &\subseteq 5\varepsilon^{1/4} U. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|(\rho(y_0), \dots, \rho(y_{t-1}))\|_{p,\infty} &\le \rho(x) \|(\overbrace{1, \dots, 1}^{t})\|_{p,\infty} \\ &\le \varepsilon^{1/2} t^{1/p} \\ &\le 2\varepsilon^{1/4}. \end{aligned}$$

Therefore, $x \in 5\varepsilon^{1/4} A$, so $\tau(x) \le 5\varepsilon^{1/4}$, as required. □

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