ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

A GENERALIZATION OF MUMFORD'S THEOREM, II

JAMES D. LEWIS[1]

0. Introduction

Let X be a quasi-projective variety (over $\mathbf{C}$) of dimension n. In this paper, we want to study the Chow group $CH_k(X) = CH^{n-k}(X)$ of algebraic cycles of dimension k (respectively, codimension $n-k$) on X, modulo *rational* equivalence, and the corresponding subgroup $A_k(X) \subset CH_k(X)$ of cycles *algebraically* equivalent to zero (in the sense of [8]). This work can be seen as a continuation of [12, Ch. 15] and [13], where the smooth projective case was studied. According to Deligne [5], [6] the cohomology of X carries a canonical and functorial mixed Hodge structure (MHS). Using the isomorphism $H_i(X, \mathbf{Q}) \simeq H_c^i(X, \mathbf{Q})^{\vee}$, where $H_i(X)$ is *Borel-Moore* homology and $H_c^i(X)$ is cohomology with compact supports, it follows that $H_i(X)$ carries a *dual* MHS. The weights ω occurring in $H_i(X)$ satisfy $-i \leq \omega \leq 0$. There is a filtration by *niveau*, $N.H_i(X)$, which induces a corresponding filtration on $W_{-i}H_i(X)$. We will denote this by $N.W_{-i}H_i(X)$ (cf. Section 2). Let $H_{\mathbf{C}} = \bigoplus_{p,q} H^{p,q}$ be a Hodge structure. We define the level of H as follows:

$$\text{Level}(H) = \max\{|p-q| \; |H^{p,q} \neq 0\} \text{ if } H \neq 0$$

(otherwise Level$(H) = -\infty$). For $l \geq 0$, one checks that

$$\text{Level}(N_{k+l}W_{-2k-l}H_{2k+l}(X)) \leq l.$$

We prove:

(0.1) THEOREM. *Let X be quasi-projective, and assume the main standard Lefschetz conjecture. Suppose* Level$(N_{k+l}W_{-2k-l}H_{2k+l}(X)) = l$ *for some* $l \geq 1$. *Then*

$$A_k(X) \text{ is } \begin{cases} \text{non-zero} & \text{if } l = 1 \\ \text{infinite dimensional} & \text{if } l \geq 2 \end{cases}$$

Received January 20, 1993.
1991 Mathematics Subject Classification. Primary 14 C30.
[1]Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

Next, we define $Gr_j N. = N_j/N_{j-1}$. Now under the assumption of the general Hodge conjecture (GHC), or more precisely, a generalization of the GHC for singular varieties (Section 2), we can deduce the following:

(0.2) COROLLARY. *Let X be quasi-projective, and assume the GHC. If*

$$Gr_{k+l}N.W_{-2k-l}H_{2k+l}(X) \neq 0,$$

then

$$A_k(X) \text{ is } \begin{cases} \text{non-zero} & \text{if } l = 1 \\ \text{infinite dimensional} & \text{if } l \geq 2 \end{cases}$$

Note that the range of l that applies to (0.1) and (0.2) is given by $1 \leq l \leq n - k$. We now set $A_*(X) = \oplus_{k \geq 0} A_k(X)$. As a consequence of the main theorem, we deduce:

(0.3) COROLLARY. *Let X be quasi-projective, and assume the GHC. Then $A_*(X)$ finite dimensional implies* $\text{Level}(\oplus_{i \geq 0} W_{-i}H_i(X)) \leq 1$.

(0.4) *Remarks.* (1) More generally, one can introduce a notion of $\text{Level}(A_*(X))$ (cf. Section 4). One has $\text{Level}(A_k(X)) \leq 1$ if and only if $A_k(X)$ is finite dimensional ((1.7)). Then the conclusion of (0.3) generalizes, namely,

$$\text{Level}(A_*(X)) \leq l \Rightarrow \text{Level}\left(\bigoplus_{i \geq 0} W_{-i}H_i(X)\right) \leq l$$

(2) The following examples illustrate the importance of restricting to $\oplus_{i \geq 0} W_{-i}H_i(X)$ as opposed to say $\oplus_{i,j} Gr_{-i}W.H_j(X)$.

Example 1. *Quasi-projective smooth case.* Let $X \subset \mathbf{P}^n$ be the complement of a smooth hypersurface $Y \subset \mathbf{P}^n$. Let $j\colon X \hookrightarrow \mathbf{P}^n$ be the inclusion. There is an exact sequence

$$\cdots \to H_i(Y) \to H_i(\mathbf{P}^n) \xrightarrow{j^*} H_i(X) \to H_{i-1}(Y) \to H_{i-1}(\mathbf{P}^n) \to \cdots .$$

The weight filtration is given explicitly by

$$H_i(X) = W_{-(i-1)} \supset W_{-i} = j^* H_i(\mathbf{P}^n) \supset \{0\},$$

with

$$Gr_{-(i-1)}W.H_i(X) \simeq \ker\colon H_{i-1}(Y) \to H_{i-1}(\mathbf{P}^n).$$

Then $A_*(X) = 0$; moreover $\mathrm{Level}(\oplus_{i\geq 0} W_{-i}H_i(X)) = 0$. However $Gr_{-(n-1)}W.H_n(X)$ generally has Hodge level ≥ 2. This is, for example, the case when $\dim Y \geq 2$ and $\deg Y \geq 5$.

Example 2. *Singular projective case.* Let $X = X_1 \cup X_2 \subset \mathbf{P}^4$ be a union of smooth threefolds meeting transversally. Assume $\deg X_1 = 3$ and $\deg X_2 = 2$. Then $Y = X_1 \cap X_2 = \mathrm{Sing}(X)$ is a smooth surface of degree 6 with genus $Pg(Y) > 0$. One can readily check that $i_*(A_*(X_1)) = A_*(X)$ (where $i: X_1 \hookrightarrow X$ is the inclusion); hence $A_*(X)$ is finite dimensional. [Let us show, for example, $i_*(A_1(X_1)) = A_1(X)$ (the other cases are easier). Fix a line $l \subset X_2$ and let L be a hyperplane section of Y. Then it is easy to show that $CH_1(X_2) = \mathbf{Z}\{l\}$ and hence $L \overset{\mathrm{rat}}{\sim} 6l$ (in $CH_1(X_2)$). Let $\xi \in CH_1(X_2)$. Then $\xi \overset{\mathrm{rat}}{\sim} \deg(\xi)l$ and hence $6\xi \overset{\mathrm{rat}}{\sim} \deg(\xi)L$. We conclude, by divisibility, that $A_1(X) = 6A_1(X) \subset i_*(A_1(X_1))$, and hence $i_*(A_1(X_1)) = A_1(X)$.] If we consider the $M - V$ sequence

$$\cdots \to H_i(Y) \to H_i(X_1) \oplus H_i(X_2) \xrightarrow{\alpha} H_i(X) \to H_{i-1}(Y)$$
$$\xrightarrow{\beta} H_{i-1}(X_1) \oplus H_{i-1}(X_2) \to \cdots$$

then the weight filtration on $H_i(X)$ is given by

$$H_i(X) = W_{(i-1)}H_i(X) \supset W_{-i}H_i(X) = \mathrm{Im}(\alpha) \supset \{0\}$$

where

$$Gr_{-(i-1)}W.H_i(X) \simeq \ker\beta\colon H_{i-1}(Y) \to H_{i-1}(X_1) \oplus H_{i-1}(X_2).$$

For $i = 3$, $\mathrm{Level}(Gr_{-2}W.H_3(X)) = 2$, whereas $\mathrm{Level}(\oplus_{i\geq 0} W_{-i}H_i(X)) \leq 1$.

The results in (0.1), (0.2) and (0.3) remain valid if X is replaced by any separated, integral algebraic scheme over $\mathbf{C}$ (see Section 4). In this direction, we ask the following:

(0.5) *Question.* Let X be a separated, reduced algebraic scheme over $\mathbf{C}$. Is it the case that $\mathrm{Level}(A_*(X)) \leq l$ if and only if $\mathrm{Level}(\oplus_{i\geq 0} W_{-i}H_i(X)) \leq l$?

1. Some preliminary results

All varieties in this paper will be assumed quasi-projective and defined over the complex numbers. We will *not* assume our varieties are irreducible. Countable unions of projective subvarieties of some $\mathbf{P}^N$ are abbreviated *c-closed* (cf. [18], [19]). All homology will be Borel-Moore, and with $\mathbf{Q}$-coefficients.

(1.1) DEFINITION. *Let X be a variety, and G a subgroup of* $A_k(X)$. *We say that* $A_k(X)/G$ *is finite dimensional, if there exist a smooth (possibly reducible) projective curve* Γ *and cycle* $z \in CH_{k+1}(\Gamma \times X)$ *such that the homomorphism*

$$z_*: A_0(\Gamma) \to \frac{A_k(X)}{G},$$

induced by $t \in \Gamma \mapsto z_t \in CH_k(X)$ (*where* z_t *is defined in* [8, 10.3]), *is surjective.*

Let X be quasi-projective, with projective closure $\overline{X}$. Also let $Y = \overline{X} - X$ with inclusion $j: Y \hookrightarrow \overline{X}$. There is a s.e.s.

$$0 \to A_k(\overline{X}) \cap j_*(CH_k(Y)) \to A_k(\overline{X}) \to A_k(X) \to 0.$$

The following two lemmas are useful.

(1.2) LEMMA. *Let* V, W *be projective varieties and* $g: A_{l_1}(V) \to A_{l_2}(W)$ *a cycle induced homomorphism.* [*The examples we have in mind are those* g *arising in the case where* V *is smooth (as in* (1.1) *with* $V = \Gamma$) *or where* g *is induced from a morphism* $V \to W$.] *If* $A_{l_2}(W)/g(A_{l_1}(V))$ *is countable, then* $g(A_{l_1}(V)) = A_{l_2}(W)$.

Outline of proof. Let $\xi \in A_{l_2}(W)$. By a standard argument (cf. [19], and using the theory of Chow varieties) there exists an abelian variety B and homomorphism $\Psi: B \to A_{l_2}(W)$ such that $\Psi(B) \ni \xi, \ker(\Psi)$ is *countable*, and $\Psi^{-1}(g(A_{l_1}(V)))$ is a *countable* union of closed subvarieties of B. By construction, $B/\Psi^{-1}(g(A_{l_1}(V)))$ is countable, and by an argument using Baire's theorem, this implies $B = \Psi^{-1}(g(A_{l_1}(V)))$, *a fortiori* $\xi \in g(A_{l_1}(V))$.

(1.3) LEMMA. *Let* $f: V \to W$ *be a dominating morphism of projective varieties. There is an integer* $N \neq 0$ *such that* $N \cdot CH_*(W) \subset f_*(CH_*(V))$.

Proof. By taking general hyperplane sections of V we can assume $\dim V = \dim W$ and hence f generically finite to one of degree d say. Choose non-empty Zariski open sets $U_W \subset W$ and $U_V = f^{-1}(U_W) \subset V$ such that $\operatorname{res}(f): U_V \to U_W$ is (faithfully) flat. Also set $Y_V = V - U_V$, $Y_W = W - U_W$, and consider the following commutative diagram:

$$\begin{array}{ccccccc}
CH_*(Y_V) & \longrightarrow & CH_*(V) & \longrightarrow & CH_*(U_V) & \longrightarrow & 0 \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
CH_*(Y_W) & \longrightarrow & CH_*(W) & \longrightarrow & CH_*(U_W) & \longrightarrow & 0
\end{array}$$

Then $d \cdot CH_*(U_W) = f_* f^*(CH_*(U_W)) \subset f_*(CH_*(U_V))$ and by induction on $\dim W$, there is an integer $N_0 \neq 0$ such that $N_0 \cdot CH_*(Y_W) \subset f_*(CH_*(Y_V))$. Now set $N = d \cdot N_0$. A simple diagram chase shows that $N \cdot CH_*(W) \subset f_*(CH_*(V))$.

(1.4) COROLLARY. *$A_k(X)$ is finite dimensional if and only if $A_k(\overline{X})/j_*(A_k(Y))$ is finite dimensional.*

Proof. The implication $(\Leftarrow)$ is obvious. To show $(\Rightarrow)$, we use the fact that $A_k(X)$ is finite dimensional iff there is a smooth projective curve Γ and cycle $z \in CH_{k+1}(\Gamma \times \overline{X})$ such that the composite

$$z_*: A_0(\Gamma) \to \frac{A_k(\overline{X})}{A_k(\overline{X}) \cap j_*(CH_k(Y))}$$

is surjective. It then follows that the corresponding map

$$z_*: A_0(\Gamma) \to \frac{A_k(\overline{X})}{j_*(A_k(Y))}$$

has countable cokernel. Now apply (1.2).

(1.5) *Remark.* It is also the case that

$$\frac{A_k(\overline{X})}{A_k(\overline{X}) \cap j_*(CH_k(Y))} = 0 \quad \Leftrightarrow \quad \frac{A_k(\overline{X})}{j_*(A_k(Y))} = 0.$$

(1.6) COROLLARY. *Given the setting of* (1.3). *Then $f_*: A_*(V) \to A_*(W)$ is surjective.*

Proof. Use (1.2) and (1.3).

(1.7) COROLLARY. *$A_k(X)$ is finite dimensional if and only if there exists a closed algebraic subset $Z \subset X$ of dimension $k+1$ such that the map $A_k(X) \to CH_k(X - Z)$ is zero.*

Proof. The implication $(\Rightarrow)$ is clear. The implication $(\Leftarrow)$ is left to the reader. (Use (1.2), (1.6) and an argument involving a Poincaré divisor.)

For the remainder of this section, we will assume that X is a projective algebraic manifold of dimension n.

We recall [11] $J_a^k(X)$, the kth-Lieberman jacobian with (surjective) Abel-Jacobi map $\Phi_k\colon A^k(X) \to J_a^k(X)$. Now set $J_a^*(X) = \oplus_{k \geq 1} J_a^k(X)$. There is a corresponding Abel-Jacobi map $\Phi\colon A^*(X) \to J_a^*(X)$.

We set $CH^k(X)_{\mathbf{Q}} = CH^k(X) \otimes \mathbf{Q}$. We also recall ([10]):

Standard Lefschetz conjecture $B(*)$. Let L_X be the operation of cupping with the hyperplane class on X relative to a given (or any) embedding of X in some $\mathbf{P}^N$, and recall the isomorphism, for $i \leq n$, $L_X^{n-i}\colon H^i(X, \mathbf{Q}) \xrightarrow{\sim} H^{2n-i}(X, \mathbf{Q})$ (hard Lefschetz). Then the inverse to L_X^{n-i} is *algebraic*.

Coniveau ("arithmetic") filtration on $H^*(X, \mathbf{Q})$. $\{N^p H^*(X, \mathbf{Q})\}_{p \geq 0} \subset H^*(X, \mathbf{Q})$ is given by either of the two equivalent formulations below.

(1) $N^p H^l(X, \mathbf{Q}) = \bigcup \{\ker\colon H^l(X, \mathbf{Q}) \to H^l(X - Y, \mathbf{Q}) \mid Y \subset X$ closed, $\operatorname{codim}_X Y \geq p\}$.

(2) $N^p H^l(X, \mathbf{Q}) = \bigcup \{$Gysin images $\sigma_*\colon H^{l-2q}(\tilde{Y}, \mathbf{Q}) \to H^l(X, \mathbf{Q}) \mid Y \subset X$ closed, $\operatorname{codim}_X Y = q \geq p$ and $\tilde{Y} = \operatorname{desing}(Y)\}$.

The $\mathbf{Q}$ Hodge filtration on $H^*(X, \mathbf{Q})$. $F_{\mathbf{Q}}^p H^l(X, \mathbf{Q})$ is the maximal sub-Hodge structure in

$$F^p H^l(X, \mathbf{C}) \cap H^l(X, \mathbf{Q}) = \{H^{l-p,p}(X) \oplus \cdots \oplus H^{p,l-p}(X)\} \cap H^l(X, \mathbf{Q}).$$

The following inclusion is well known: $N^* H^*(X, \mathbf{Q}) \subset F_{\mathbf{Q}}^* H^*(X, \mathbf{Q})$, and the conjectured equality is the celebrated (Grothendieck amended) General Hodge Conjecture (GHC).

The main theorem (0.1) generalizes earlier results for the smooth projective case (over $\mathbf{C}$). In this case $W_{-i} H_i(X) = H_i(X)$. For example, there are these cases:

(i) $l = 1$, $A^k(X) = A_{n-k}(X)$. By Poincaré duality,

$$N_{n-k+1} H_{2(n-k)+1}(X) \simeq N^{k-1} H^{2k-1}(X),$$

and hence $N_{n-k+1} H_{2(n-k)+1}(X) \otimes_{\mathbf{Q}} \mathbf{R} \simeq$ Lie algebra to $J_a^k(X)$. Therefore

$$\begin{aligned} \operatorname{Level}\big(N_{n-k+1} H_{2(n-k)+1}(X)\big) = 1 &\Leftrightarrow J_a^k(X) \neq 0 \\ &\Rightarrow A^k(X) \neq 0. \end{aligned}$$

(ii) $A_0(X)$. We have

$$\operatorname{Level}\big(N_l H_l(X)\big) = l \overset{\text{weakLefschetz}}{\Longleftrightarrow} \operatorname{Level}\big(H_l(X)\big) = l \Leftrightarrow H^{0,l}(X) \neq 0;$$

moreover $A_0(X)$ is infinite dimensional if $l \geq 2$ and non-zero if $l = 1$. Thus we recover Roitman's generalization of Mumford's results (cf. [18] and [15]) for rational equivalence.

(iii) The results of [12, (15.34)] are recovered. Namely, under the assumption of $B(*)$, and if X is smooth and projective, then

$$\text{Level}\big(N^{k-l}H^{2k-l}(X)\big) = l \Rightarrow A^k(X) \text{ is} \begin{cases} \neq 0 & \text{if } l = 1 \\ \text{infinite dimensional} & \text{if } l \geq 2 \end{cases}.$$

Under the assumption of the GHC, $\text{Level}(N^{k-l}H^{2k-l}(X)) = l$ can be replaced by $Gr^{k-l}N \cdot H^{2k-l}(X) \neq 0$.

In the smooth projective case, (0.3) and (0.5) can be sharpened as follows:

(1.8) Corollary [12, (15.48)]. *Assume the GHC. Then*

$$A^*(X) \xrightarrow{\sim} J_a^*(X) \Rightarrow \text{Level}\big(H^*(X)\big) \leq 1$$

(1.9) *Conjecture* [12, (15.49)].

$$A^*(X) \xrightarrow{\sim} J_a^*(X) \Leftrightarrow \text{Level}\big(H^*(X)\big) \leq 1.$$

For some evidence in support of this conjecture, see [14]. From a philosophical perspective, we expect the following. Referring to (0.2) applied to $A^k(X) = A_{n-k}(X)$, the vector spaces $Gr_{n-k+l}N.W_{-2(n-k)-l}H_{2(n-k)+l}(X)$ are defined in terms of suitable graded pieces ([pure] niveau) of the niveau filtration; moreover the range of l is $\{0, \ldots, k\}$. In the smooth case, one expects a decreasing [functorial] filtration involving k steps:

$$\begin{aligned} CH^k(X) &= F^0CH^k(X) \supset A^k(X) = F^1CH^k(X) \\ &\supset \big\{\ker \Phi_k \colon A^k(X) \to J_a^k(X)\big\} \\ &= F^2CH^k(X) \supset \cdots \supset F^kCH^k(X) \supset \{0\}, \end{aligned}$$

such that $Gr^{k-l}N \cdot H^{2k-l}(X)$ influences the lth graded piece

$$Gr^lCH^k(X) = \frac{F^lCH^k(X)}{F^{l+1}CH^k(X)}$$

to some degree. A more thorough discussion along these lines appears in [12, Ch. 15]. We refer the reader to [21], [20], and [16], [17] for some recent developments in this direction.

We remark in passing that it is easy to see how one may define filtrations on Chow groups in the quasi-projective case, given a filtration in the projective smooth case. For example:

(a) X singular, projective with desingularization $\lambda\colon \tilde{X} \overset{\approx}{\to} X$. Then

$$F^l CH^k(X)_{\mathbf{Q}} \overset{\text{def}}{=} \lambda_* F^l CH^k\big(\tilde{X}\big)_{\mathbf{Q}}.$$

(b) X projective, $j\colon U \hookrightarrow X$ quasi-projective (where $\overline{U} = X$). Then

$$F^l CH^k(U)_{\mathbf{Q}} \overset{\text{def}}{=} j^* F^l CH^k(X)_{\mathbf{Q}}$$

(where $F^l CH^k(X)_{\mathbf{Q}}$ is defined using (a)).

One should check that these filtrations are independent of respective choices of $\tilde{X}$ and X (i.e., are well defined) and that this should follow from functoriality of the filtrations in the smooth case.

2. Filtration by niveau and the GHC for singular varieties

Let X be a quasi-projective variety and $\overline{X}$ a projective closure of X, with desingularization $\lambda\colon \tilde{\overline{X}} \overset{\approx}{\to} \overline{X}$. Also let $j\colon X \hookrightarrow \overline{X}$ be the inclusion.

We recall (cf. [3]) the filtration by *niveau*:

$$N_k H_i(X) = \{\textit{Images } H_i(W) \to H_i(X) \mid W \subset X \textit{ is closed algebraic, of dimension } \le k\}.$$

The mixed Hodge structure on $H_i(X)$ gives us a filtration $\{F^{-l}H_i(X)\}$ inducing a Hodge filtration

$$\Big\{F^{-l}W_{-i}H_i(X) \overset{\text{def}}{=} \{F^{-l}H_i(X)\} \cap \{W_{-i}H_i(X)\}\Big\}$$

of weight $-i$. We denote by $F_{\mathbf{Q}}^{-l}W_{-i}H_i(X)$ the *maximal* $\mathbf{Q}$ subHodge structure of $F^{-l}W_{-i}H_i(X)$. We also set

$$N_k W_{-i} H_i(X) = \{N_k H_i(X)\} \cap \{W_{-i}H_i(X)\},$$

and prove:

(2.1) PROPOSITION. (i) $N_k W_{-i} H_i(X) = j^* \circ \lambda_*(N_k H_i(\tilde{\overline{X}}))$.
(ii) $F_{\mathbf{Q}}^{-l} W_{-i} H_i(X) = j^* \circ \lambda_*(F_{\mathbf{Q}}^{-l} H_i(\tilde{\overline{X}}))$.
(iii) $\operatorname{Level}(N_k W_{-i} H_i(X)) \le 2k - i$.

Proof. By an argument involving weights, it follows that $j^* \circ \lambda_*: H_i(\tilde{\bar{X}}) \to W_{-i}H_i(X)$ is surjective [9, Lemmas 7.5, 7.6]. Let $Y \subset X$ be a closed algebraic subset of dimension k, and choose a closed subset $\tilde{Y} \subset \tilde{\bar{X}}$ such that $j^{-1}(\lambda(\tilde{Y})) = Y$. Let $\underline{Y} \xrightarrow{\approx} \tilde{Y}$ be a desingularization. Then the image of the composite $H_i(\underline{Y}) \to W_{-i}H_i(\tilde{Y}) \to H_i(\tilde{\bar{X}})$ is the same as the image $H_i(\tilde{Y}) \to H_i(\tilde{\bar{X}})$ [5, (8.2.7)], and hence agrees with the image $W_{-i}H_i(\tilde{Y}) \to H_i(\tilde{\bar{X}})$. Next, from the exact sequence

$$W_{-i}H_i(Y) \to W_{-i}H_i(X) \to W_{-i}H_i(X - Y) \to 0 = W_{-i}H_{i-1}(Y),$$

we deduce that

$$N_kW_{-i}H_i(X) = \{\text{images } W_{-i}H_i(Y) \to W_{-i}H_i(X) | Y \subset X \text{ and } \dim Y \le k\},$$

and therefore this agrees with $j^* \circ \lambda_*(N_kH_i(\tilde{\bar{X}}))$. This proves (i).

Part (ii) follows from the surjection $j^* \circ \lambda_*: H_i(\tilde{\bar{X}}) \to W_{-i}H_i(X)$ of *Hodge structures* together with semi-simplicity of Hodge structures over $\mathbf{Q}$.

Part (iii) follows immediately from (i) and (ii), the fact that

$$N^{n-k}H^{2n-i}(\tilde{\bar{X}}) \subset F_{\mathbf{Q}}^{n-k}H^{2n-i}(\tilde{\bar{X}}),$$

and Poincaré duality.

Now under the Poincaré duality (PD) isomorphism

$$H^{2n-i}(\tilde{\bar{X}}) \simeq H_i(\tilde{\bar{X}}),\ H^{p,q}(\tilde{\bar{X}}) \subset H^{2n-i}(\tilde{\bar{X}})$$

corresponds to

$$H_i^{p-n,q-n}(\tilde{\bar{X}}) \subset H_i(\tilde{\bar{X}}).$$

Corresponding to this is the isomorphism

$$F_{\mathbf{Q}}^{n-k}H^{2n-i}(\tilde{\bar{X}}) \overset{\text{PD}}{\simeq} F_{\mathbf{Q}}^{-k}H_i(\tilde{\bar{X}}).$$

(2.2) COROLLARY. *$N_kW_{-i}H_i(X) \subset F_{\mathbf{Q}}^{-k}W_{-i}H_i(X)$ with equality if the GHC holds.*

Proof. There is a commutative diagram

$$\begin{array}{ccccc}
N^{n-k}H^{2n-i}(\tilde{\bar{X}}) & \overset{\text{PD}}{\simeq} & N_kH_i(\tilde{\bar{X}}) & \xrightarrow{j^* \circ \lambda_*} & N_kW_{-i}H_i(X) \\
\downarrow & & \downarrow & & \downarrow \\
F_{\mathbf{Q}}^{n-k}H^{2n-i}(\tilde{\bar{X}}) & \overset{\text{PD}}{\simeq} & F_{\mathbf{Q}}^{-k}H_i(\tilde{\bar{X}}) & \xrightarrow{j^* \circ \lambda_*} & F_{\mathbf{Q}}^{-k}W_{-i}H_i(X).
\end{array}$$

By (2.1), the horizontal arrows are surjective, and the first two (from the left) of the vertical arows (inclusions) are surjective by the GHC. We deduce that the last vertical arrow is surjective as well.

(2.3) *Remarks.* (i) By an argument using Chow's lemma (cf. [9, 7.9]), Corollary (2.2) remains true for X a separated, reduced algebraic scheme over $\mathbf{C}$.

(ii) A generalization of the GHC for arbitrary varieties (separated, reduced algebraic schemes over $\mathbf{C}$) then takes the following form (compare with [9, 7.2]).

(2.4) *Conjecture (GHC for singular varieties).* The inclusion

$$N_k W_{-i} H_i(X) \subset F_{\mathbf{Q}}^{-k} W_{-i} H_i(X)$$

is an equality.

3. The main theorem

We assume $B(*)$ throughout the rest of this paper.

(3.1) THEOREM. *Suppose* $\mathrm{Level}(N_{k+l} W_{-2k-l} H_{2k+l}(X)) = l$. *Then*

$$A_k(X) \text{ is } \begin{cases} \text{non-zero} & \text{if } l = 1 \\ \text{infinite dimensional} & \text{if } l \geq 2 \end{cases}$$

Proof. The approach we will take is inspired by the ideas in [15] and [18], [19]. Another approach would be along the line of reasoning in [1], and we will have more to say about this in Section 4.

Let $\overline{X}$ be a projective closure of X, with desingularization $\lambda\colon \tilde{\overline{X}} \overset{\approx}{\to} \overline{X}$ and $Y = \overline{X} - X$. Choose $\tilde{Y} \subset \tilde{\overline{X}}$ dominating Y and corresponding commutative diagram

$$\begin{array}{ccccc} \tilde{Y} & \overset{\tilde{q}}{\hookrightarrow} & \tilde{\overline{X}} & & \\ \downarrow{\scriptstyle \lambda_Y} & & \downarrow{\scriptstyle \lambda} & & \\ Y & \overset{q}{\hookrightarrow} & \overline{X} & \overset{j}{\hookleftarrow} & X, \end{array}$$

where $q, \tilde{q}, j$ are the respective inclusions. Recall that if $A_k(X)$ is finite dimensional then $A_k(\overline{X})/A_k(Y)$ is finite dimensional ((1.4)). There is an algebraic subset $\Sigma_0 \overset{\mu}{\hookrightarrow} \tilde{\overline{X}}$ of pure dimension $k + l$ and a (possibly reducible) desingularization $\sigma\colon \tilde{\Sigma} \xrightarrow{\approx} \Sigma_0$ such that the composite

$$j^* \circ \lambda_* \circ \mu_* \circ \sigma_* : H_{2k+l}(\tilde{\Sigma}) \to N_{k+l}W_{-2k-l}H_{2k+l}(X)$$

is surjective. By taking k general hyperplane sections of $\tilde{\Sigma}$ and applying Bertini's theorem, we arrive at a smooth (and possibly reducible) projective algebraic submanifold $j_0\colon S \hookrightarrow \tilde{\Sigma}$ of pure dimension l such that $j_0^*\colon H^l(\tilde{\Sigma}) \hookrightarrow H^l(S)$ is injective. According to $B(*)$, the surjective left inverse $(j_0^*)^{-1}$: $H^l(S) \to H^l(\tilde{\Sigma})$ is algebraic (i.e., algebraic cycle induced), *a fortiori* via Poincaré duality, the composite $H_l(S) \to N_{k+l}H_{2k+l}(\bar{\tilde{X}})$ is algebraic and say induced by an algebraic cycle $w \in CH_{k+l}(S \times \bar{\tilde{X}}) \otimes \mathbf{Q}$. By taking a suitable integral multiple of w, we may assume $w \in CH_{k+l}(S \times \bar{X})$. We deduce that there exists S of dimension l and w as above such that $j^* \circ \lambda_* \circ w_*$: $H_l(S) \to N_{k+l}W_{-2k-l}H_{2k+l}(X)$ is surjective.

Now assume to the contrary that $A_k(X)$ is finite dimensional if $l \geq 2$ or zero if $l = 1$. Using (1.6), there exists a smooth curve Γ, a cycle $\xi \in CH_{k+1}(\Gamma \times \bar{X})$ such that

$$A_k(\bar{X}) = \lambda_* \circ \xi_* A_0(\Gamma) + \lambda_* \circ \tilde{q}_* A_k(\tilde{Y}).$$

By working with each irreducible component, it will easily follow from the proof that one can assume for simplicity that S is connected. Fix a point $s_0 \in S$ and consider the corresponding map $\lambda_w\colon S \to A_k(\bar{X})$ given by $s \mapsto \lambda_*\{w_*(s) - w_*(s_0)\}$. Based on some standard Chow variety and "c-closed" arguments in [18], [19] (also, cf. [22]), it is easy to show that there exists a smooth variety T_1 of dimension r say, and a cycle induced map ν_*: $T_1 \to A_k(\tilde{Y})$ (for some $\nu \in CH_{r+k}(T_1 \times \tilde{Y})$) for which $\lambda_w(S) \subset \lambda_* \circ \tilde{q}_* \circ \nu_*(T_1) + \lambda_* \circ \xi_* A_0(\Gamma)$. Again one can argue, as in [18], [19] (cf. [22]), that

$$V_0 \stackrel{\text{def}}{=} \left\{(s,t) \in S \times T_1 | \lambda_* \circ \tilde{q}_* \circ \nu_*(t) \overset{\text{rat}}{\sim} \lambda_w(s) \bmod \lambda_* \circ \xi_* A_0(\Gamma)\right\}$$

is c-closed.

By our construction, there exists a subvariety $\Sigma \subset V_0$ such that $Pr_1\colon \Sigma \to S$ is a surjective, generically finite to one map of degree d say. Note that Σ defines a corresponding cycle $\Sigma \in CH_l(S \times T_1)$ and that $\lambda_* \circ \tilde{q}_* \circ \nu_* \circ \Sigma_* = d \cdot \lambda_* \circ w_* \bmod \lambda_* \circ \xi_* A_0(\Gamma)$ on $A_0(S)$. Now set $\underline{w} = \tilde{q} \circ \nu \circ \Sigma - d \cdot w$. Then $\operatorname{Im}(\lambda_* \circ \underline{w}_*)$ is finite dimensional.

We first assume $l \geq 2$, and let $B \simeq A_0(\Gamma)$ be the corresponding abelian variety with homomorphism $\xi_*\colon B \to A_k(\bar{\tilde{X}})$ satisfying $\lambda_* \circ \underline{w}_*(A_0(S)) \subset \lambda_*(\xi_*(B))$. The subset

$$V = \{(s,q) \in S \times B | \lambda_* \circ \underline{w}_*(s - s_0) = \lambda_* \circ \xi_*(q)\}$$

is c-closed; moreover by definition of B, one can find a subvariety $\tilde{S} \subset V$ of dimension l (which we can presume smooth, after passing to a desingularization), dominating S, such that $\lambda_* \circ \tilde{\xi}_*(A_0(\tilde{S})) = 0$, where $\tilde{\xi} \in CH_{k+l}(\tilde{S} \times \bar{X})$ is the cycle given by the pullback of $Pr_{23}^*(\xi) - Pr_{13}^*(\underline{w})$ under the map $\tilde{S} \times \bar{X} \to S \times B \times \bar{X}$. Since $\xi_*(B)$ is supported on a subvariety in $\bar{X}$ of dimension $k + 1$, we conclude that the image $H_{2k+l}(|\xi_*(B)|) \to N_{k+1}H_{2k+l}(\bar{X})$ has level $\leq 2 - l$, which is less that l for $l \geq 2$. We conclude that the level of $H_l(\tilde{S})$ in $H_{2k+l}(\bar{X})$ is the same as that for $H_l(S)$.

We have shown that without modifying the Hodge level properties of $\underline{w}_*$: $H_l(S) \to H_{2k+l}(\bar{X})$, one can assume that $\lambda_* \circ \underline{w}_*$: $A_0(S) \to A_k(\bar{X})$ is zero. Now we assume $l \geq 1$ with $\lambda_* \circ \underline{w}_*$: $A_0(S) \to A_k(\bar{X})$ zero. Then by replacing $\underline{w}$ by $\underline{w} - \{S \times \underline{w}_*(s_0)\}$, we can further assume that $\lambda_* \circ \underline{w}_*$: $CH_0(S) \to CH_k(\bar{X})$ is zero. Let $C_k(-)$ represent the Chow variety of effective cycles of dimension k, and $C_k(-)_d \subset C_k(-)$ the subset of those cycles of degree d. By moving (via rational equivalence) the irreducible components of $\underline{w}$ in general position (Chow's moving lemma), we can assume $\underline{w}$ defines a rational map $\{\underline{w}\}$: $S \to C_k(\bar{X}) \times C_k(\bar{X})$, which restricts to a regular map (f, g): $S_0 \to C_k(\bar{X}) \times C_k(\bar{X})$ on some non-empty open subset $S_0 \subset S$. Likewise, we can assume (by possibly shrinking S_0 if necessary), that the corresponding map$(\lambda_* \circ f, \lambda_* \circ g)$: $S_0 \to C_k(\bar{X}) \times C_k(\bar{X})$ is also defined and regular. Using (only) the projectivity of $\bar{X}$, together with similar arguments to those in [18], [19] and [22], one can show that there exists a smooth quasi-projective variety T_0, a dominant morphism e: $T_0 \to S_0$ and a morphism H: $\mathbf{P}^1 \times T_0 \to C_k(\bar{X})_{d_1} \times C_k(\bar{X})_{d_2}$ such that

$$\lambda_* \circ f \circ e + \left(Pr_1 \circ H|_{\infty \times T_0}\right) + \left(Pr_2 \circ H|_{0 \times T_0}\right)$$
$$= \lambda_* \circ g \circ e + \left(Pr_1 \circ H|_{0 \times T_0}\right) + \left(Pr_2 \circ H|_{\infty \times T_0}\right)$$

where "+" denotes the sum on the Chow variety: $C_k(\bar{X})_{d_1} \times C_k(\bar{X})_{d_2} \to C_k(\bar{X})_{d_1+d_2}$.

Let $T = \bar{T}_0$ be a non-singular projective closure of T_0, with inclusion map ν_0: $T_0 \hookrightarrow T$. Note that H defines a cycle in $CH^{n-k}(\mathbf{P}^1 \times T_0 \times \bar{X})$ (cf. [22, (1.3)]), and by taking closures, a corresponding cycle $\Sigma_H \in CH^{n-k}(\mathbf{P}^1 \times T \times \bar{X})$. Likewise, there are cycles $\Sigma_f, \Sigma_g \in CH^{n-k}(S \times \bar{X})$ corresponding to f, g. Note that $\Sigma_H = \Sigma_{Pr_1 \circ H} - \Sigma_{Pr_2 \circ H}$, where $\Sigma_{Pr_j \circ H} \in CH^{n-k}(\mathbf{P}^1 \times T \times \bar{X})$ is the cycle associated to $Pr_j \circ H$. Now Σ_H defines a "cylinder map" on homology, $\Sigma_{H,*}$: $H_l(\mathbf{P}^1 \times T) \to H_{2k+l}(\bar{X})$. To see this, and quite generally, we consider the following setting. Let W be a smooth projective variety of dimension m, Z projective, and assume given a cycle $z \in CH_a(W \times Z)$. Then z determines a corresponding homology class $\mathrm{cl}(z) \in H_{2a}(W \times Z)$. By the Künneth formula applied to$H_{2a}(W \times Z)$, together with the intersection pairing on $H_*(W)$ (using W smooth, projective), the component of $\mathrm{cl}(z)$ in

$H_{2m-\cdot}(W) \otimes H_{2a-2m+\cdot}(Z)$ determines a cylinder homomorphism $z_*: H.(W) \to H_{2a-2m+\cdot}(Z)$. Applying these considerations to Σ_H, we obtained the aforementioned cylinder homomorphism $\Sigma_{H,*}$. Now let

$$i_0: T_0 \simeq 0 \times T_0 \hookrightarrow \mathbf{P}^1 \times T_0, \quad i_\infty: T_0 \simeq \infty \times T_0 \hookrightarrow \mathbf{P}^1 \times T_0$$

be respective inclusions. On *singular* homology, we clearly have

$$\begin{aligned} &\lambda_* \circ \Sigma_{f,*} \circ e_* + \Sigma_{Pr_1 \circ H,*} \circ (1 \times \nu_0)_* \circ i_{\infty,*} + \Sigma_{Pr_2 \circ H,*} \circ (1 \times \nu_0)_* \circ i_{0,*} \\ &\quad = \lambda_* \circ \Sigma_{g,*} \circ e_* + \Sigma_{Pr_1 \circ H,*} \circ (1 \times \nu_0)_* \circ i_{0,*} \\ &\qquad + \Sigma_{Pr_2 \circ H,*} \circ (1 \times \nu_0)_* \circ i_{\infty,*} \end{aligned}$$

Using the results in [5, 6], one can easily show that the dual map

$$[\Sigma_H]^*: H^{2k+l}(\overline{X}) \to H^l(\mathbf{P}^1 \times T)$$

is a morphism of mixed Hodge structures. Hence there is an induced map

$$[\Sigma_H]^*: Gr^{2k+l}W.H^{2k+l}(\overline{X}) \to H^l(\mathbf{P}^1 \times T).$$

Applying Hodge theory, we end up with the commutative diagram:

$$\begin{array}{ccccc} \{Gr^{2k+l}W.H^{2k+l}(\overline{X})\}^{k+l,k} & \xrightarrow{[\Sigma_H]^*} & H^{l,0}(\mathbf{P}^1 \times T) & & (i_0^* = i_\infty^*) \circ (1 \times \nu_0)^* \\ & & \| & & \searrow \\ & & Pr_2^* H^{l,0}(T) \simeq H^{l,0}(T) & & \overset{\nu_0^*}{\hookrightarrow} H^{l,0}(T_0). \end{array}$$

We conclude that on $\{Gr^{2k+l}W.H^{2k+l}(\overline{X})\}^{k+l,k}$,

$$\begin{aligned} e^* \circ [\Sigma_f]^* \circ \lambda^* &= e^* \circ [\Sigma_g]^* \circ \lambda^* \\ +(i_0^* - i_\infty^*) \circ (1 \times \nu_0)^* \circ [\Sigma_H]^* &= e^* \circ [\Sigma_g]^* \circ \lambda^* \end{aligned}$$

and since e is dominating, we arrive at

$$[\underline{w}]^* \circ \lambda^* = [\Sigma_f - \Sigma_g]^* \circ \lambda^* = 0$$

Translating this in terms of (Borel-Moore) homology, we deduce that level$(\lambda_* \circ \underline{w}_*(H_l(S))) \leq l - 2 < l$ (in $N_{k+l}W_{-2k-l}H_{2k+l}(\overline{X})$).

Now recall $d \cdot w_* = \tilde{q}_* \circ \nu_* \circ \Sigma_* - \underline{w}_*$. From the commutative diagram

$$
\begin{array}{ccccc}
 & H_l(S) & & & \\
\lambda_{Y,*} \circ \nu_* \circ \Sigma_* & \downarrow & \lambda_* \circ \tilde{q}_* \nu_* \circ \Sigma_* & & \\
 & & \searrow & & \\
 & H_{2k+l}(Y) & \xrightarrow{q_*} & H_{2k+l}(\overline{X}) \xrightarrow{j^*} & H_{2k+l}(X)
\end{array}
$$

we deduce that $\text{level}(j^* \circ w_*(H_l(S)) = \text{level}(N_{k+l}W_{-2k-l}H_{2k+l}(X)) < l$, contradiction.

4. Concluding remarks

(1) It is possible to give another proof of (3.1), along the line of reasoning in [1], based on an argument in [23]. To be specific, and referring to the notation in (3.1), we have that the image of $A_k(\overline{X})$ in $CH_k(\overline{X} - \{Y \cup |\lambda_* \circ \xi_*(B)|\})$ is zero. As in the proof of (3.1), one first reduces to the case where S is irreducible, with a choice of embedding $k(S) \subset \mathbf{C}$, where S can be viewed as defined over a subfield $k \subset \mathbf{C}$ of finite transcendence degree over $\mathbf{Q}$. Then one would show that $N \cdot (1 \times \lambda)_*(w) \overset{\text{rat}}{\sim} \Gamma_1 + \Gamma_2 + \Gamma_3$ in $CH_{k+l}(S \times \overline{X})$, where $N > 0$ is some integer and

(i) Γ_1 is supported on $D \times \overline{X}$ for some divisor D
(ii) Γ_2 is supported on $S \times Y$, and
(iii) Γ_3 is supported on $S \times |\lambda_* \circ \xi_*(B)|$.

Rather than give a precise proof, we comment that by a careful inspection of the proof of (3.1) in Section 3, and under the assumption of the GHC for S, one can arrive at the decomposition

$$N \cdot (1 \times \lambda)_*(w) \overset{\text{hom}}{\sim} \Gamma_1 + \Gamma_2 + \Gamma_3$$

in $H_{2(k+l)}(S \times \overline{X})$, where $N = (\text{deg: } \tilde{S} \to S) \cdot (d = \deg Pr_1\colon \Sigma \to S)$, (see proof of (3.1)).

Let us now assume the decomposition $N \cdot (1 \times \lambda)_*(w) = \Gamma_1 + \Gamma_2 + \Gamma_3$. We need to show that $\text{Level}(N_{k+l}W_{-2k-l}H_{2k+l}(X)) < l$ for $l = 1$ in the case $A_k(X) = 0$, and for $l \geq 2$ in the case $A_k(X)$ is finite dimensional. In this setting, it suffices to compute the level of the image of $j^* \circ \Gamma_{i,*}\colon H_l(S) \to H_{2k+l}(X)$.

(i) $\Gamma_{1,*}$. Choose $\overline{\Gamma}_1 \in CH_{k+l}(S \times \tilde{\overline{X}})_{\mathbf{Q}}$ such that $(1 \times \lambda)_*(\overline{\Gamma}_1) = \Gamma_1$ and $\text{supp}(\overline{\Gamma}_1) \subset D \times \tilde{\overline{X}}$. Then the Künneth component of $\{\overline{\Gamma}_1\}$ in $H_l(S) \otimes H_{2k+l}(\tilde{\overline{X}})$ maps (via $(1 \times \lambda)_*$) to the Künneth component of $\{\Gamma_1\}$ in $H_l(S) \otimes H_{2k+l}(\overline{X})$. Therefore, for $\gamma \in H_l(S)$, $\Gamma_{1,*}(\gamma) = \lambda_* \circ \overline{\Gamma}_{1,*}(\gamma)$. Now let $\tilde{D} \overset{\approx}{\to} D$ be a desingularization and let $\nu\colon \tilde{D} \to S$ be the composite, with corresponding $f = (\nu \times 1)\colon \tilde{D} \times \bar{\bar{X}} \to S \times \bar{\bar{X}}$. Also choose a cycle $\bar{\bar{\Gamma}}_1 \in CH_{k+l}(\tilde{D} \times \bar{\bar{X}})_{\mathbf{Q}}$ such that $f_*(\bar{\bar{\Gamma}}_1) = \bar{\Gamma}_1$. We now compute

$$\begin{aligned}\Gamma_{1,*}(\gamma) = \lambda_* \circ \bar{\Gamma}_{1,*}(\gamma) &= \lambda_* \circ Pr_{\bar{\bar{X}},*}\left\{\{Pr_S^*(\gamma) \cdot \bar{\Gamma}_1\}_{S\times\bar{\bar{X}}}\right\} \\ &= \lambda_* \circ Pr_{\bar{\bar{X}},*}\left\{\left\{Pr_S^*(\gamma) \cdot f_*\left(\bar{\bar{\Gamma}}_1\right)\right\}_{S\times\bar{\bar{X}}}\right\} \\ &= \lambda_* \circ Pr_{\bar{\bar{X}},*} \circ f_*\left\{\left\{f^* \circ Pr_S^*(\gamma) \cdot \bar{\bar{\Gamma}}_1\right\}_{\tilde{D}\times\bar{\bar{X}}}\right\} \\ &= \lambda_* \circ Pr_{\bar{\bar{X}},*} \circ f_* \circ \bar{\bar{\Gamma}}_{1,*}(\nu^*(\gamma)) \end{aligned}$$

In particular, $\Gamma_{1,*}\colon H_l(S) \to H_{2k+l}(\bar{X})$ factors through $\bar{\bar{\Gamma}}_{1,*}\colon H_{l-2}(\tilde{D}) \to H_{2k+l}(\bar{\bar{X}})$. We conclude that $\mathrm{Level}(j^* \circ \Gamma_{1,*}(H_l(S)) \le l - 2 < l$.

(ii) $\Gamma_{2,*}$. There is a factorization of $\Gamma_{2,*}$ in the diagram below (where the column is part of the exact sequence of Borel-Moore homology):

$$\begin{array}{ccc} & \nearrow & H_{2k+l}(Y) \\ & & \downarrow \\ H_l(S) & \xrightarrow{\Gamma_{2,*}} & H_{2k+l}(\bar{X}) \\ & & \downarrow \\ & & H_{2k+l}(X) \end{array}$$

It easily follows that $j^* \circ \Gamma_{2,*}\colon H_l(S) \to H_{2k+l}(X)$ is *zero*.

(iii) $\Gamma_{3,*}$. Let $\bar{Z} = |\lambda_* \circ \xi_*(B)|$, and recall $\dim \bar{Z} = k+1$. Our assumption that $A_k(X)$ is finite dimensional implies that $\bar{Z}$ is the projective closure of a closed algebraic subset $Z \subset X$ (cf. (1.7)). Using mixed Hodge structures, there is a commutative diagram below:

$$\begin{array}{ccc} H_l(S) = W_{-l}H_l(S) & \longrightarrow & W_{-2k-l}H_{2k+l}(\bar{Z}) \\ & \searrow^{\Gamma_{3,*}} & \downarrow \\ & & W_{-2k-l}H_{2k+l}(\bar{X}) \\ & & \downarrow \\ & & W_{-2k-l}H_{2k+l}(X). \end{array}$$

Let $\sigma\colon \bar{\bar{Z}} \xrightarrow{\approx} \bar{Z}$ be a desingularization, and recall that $\sigma_*\colon H_{2k+l}(\bar{\bar{Z}}) \to W_{-2k-l}H_{2k+l}(\bar{Z})$ is surjective. By Poincaré duality, $H_{2k+l}(\bar{\bar{Z}}) \simeq H^{2-l}(\bar{\bar{Z}})$, we deduce that Level($H_{2k+l}(\bar{\bar{Z}})$) $\leq 2 - l$. It then follows from the above diagram that Level($j^* \circ \Gamma_{3,*}(H_l(S))$) $\leq 2 - l$ in $W_{-2k-l}H_{2k+l}(X)$, *a fortiori* $< l$ if $l \geq 2$. In the case $l = 1$, our assumption is that $A_k(X) = 0$, hence $Z = \emptyset$ and $\Gamma_3 = 0$.

(2) The argument in (1) above generalizes as follows. We introduce (compare with [23, Def. 1.1], [20, Def. (0-6)]

$$\begin{aligned}\text{Level}(A_k(X)) = \min\{r \mid \exists \text{ a closed algebraic } Z \subset X,\\ \dim Z = k + r, \text{ such that } A_k(X) \to CH_k(X - Z) \text{ is zero}\}\end{aligned}$$

(Note the range, $0 \leq \text{Level}(A_k(X)) \leq \dim X - k$.) Then

$$\text{Level}(N_{k+l}W_{-2k-l}H_{2k+l}(X)) = l \quad \Rightarrow \quad \text{Level}(A_k(X)) \geq l.$$

(3) Using (1) above and Chow's lemma for complete varieties, one can show that X in (3.1) can be replaced by a separated, integral algebraic scheme over $\mathbf{C}$. To be specific, X can be embedded as an open subset of a complete variety $\bar{X}$; moreover by Chow's lemma, $\bar{X}$ can be birationally dominated by a projective variety $\bar{\bar{X}}$. The cycle $w \in CH_k(S \times \bar{\bar{X}})$ is constructed in the same way as before, using the hypothesis $B(*)$.

(4) *Example application of* (*3.1*). We will refer to the notation in the proof of (3.1). Let X be any quasi-projective variety. Instead of assuming the hypothesis $B(*)$, we will more specifically assume that $B(\bar{\bar{X}})$ holds (e.g., $\bar{\bar{X}}$ a complete intersection or an abelian variety). Also choose l and k such that $k + l = n$. According to the proof of (3.1), we deduce that if Level($W_{-n-k}H_{n+k}(X)$) $= n - k$, then

$$A_k(X) \text{ is } \begin{cases} \text{non-zero} & \text{if } n - k = 1 \\ \text{infinite dimensional} & \text{if } n - k \geq 2 \end{cases}.$$

(5) Let X be quasi-projective, with projective closure $\bar{X}$, and let $Y = \bar{X} - X$. Define

$$CH_*(-) = \bigoplus_{k \geq 0} CH_k(-) \text{ and } W_{-*}H_{*+m}(-) = \bigoplus_{i \geq 0} W_{-i}H_{i+m}(-).$$

There is the following schema below (with exact rows):

$$\begin{array}{ccccccccccc}
\to W_{-*}H_{*+m}(X) \to & \cdot \to & W_{-*}H_{*+1}(X) & \to & W_{-*}H_*(Y) & \to & W_{-*}H_*(\bar{X}) & \to & W_{-*}H_*(X) & \to & 0 \\
\updownarrow ? & & \updownarrow ? & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\to CH_{*+m}(X, m) \to & \cdot \to & CH_{*+1}(X, 1) & \to & CH_*(Y, 0) & \to & CH_*(\bar{X}, 0) & \to & CH_*(X, 0) & \to & 0
\end{array}$$

where $CH_{\dim Z+m-i}(Z, m) \stackrel{\text{def}}{=} CH^i(Z, m)$ are the higher Chow groups introduced by Bloch ([2]), and where $CH_*(-,0) = CH_*(-)$. A *näive question* would be to ask whether one can expect a relationship between $W_{-*}H_{*+m}(X)$ and $CH_*(X, m)$ involving an "influence" of graded pieces, as a generalization of the case $m = 0$.

References

1. S. Bloch, *On an argument of Mumford in the theory of algebraic cycles*, Algebraic Geometry Angers 1979, A. Beauville, editor, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, pp. 217–221.
2. ______, *Algebraic cycles and algebraic K-theory*, Adv. Math. **61** (1986), 267–304.
3. S. Bloch and A. Ogus, Gersten's conjecture and the homology of schemes, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–202.
4. A. Borel and J.C. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. **7** (1960), 137–159.
5. P. Deligne, *Théorie de Hodge III*, Publ. Math. I.H.E.S. **44** (1974), 5–77.
6. ______, *Poids dans la cohomologie des variétiés algébriques*, Actes du Congres International des Mathématiciens, Vancouver, 1974, 79–85.
7. C. Deninger and J.P. Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.
8. W. Fulton, *Intersection theory*, Erg. Mat. (3) **2** (1984).
9. U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Math. no. 1400, Springer-Verlag, New York, 1990.
10. S. Kleiman, "Algebraic cycles and the Weil conjectures" in *Dix exposés sur la cohomologie des schémas*, North Holland, Amsterdam, 1968, pp. 359–386.
11. D. Lieberman, *Higher Picard varieties*, Amer. J. Math. **90** (1968), 1165–1199.
12. J. Lewis, *A survey of the Hodge conjecture*, Les Publications Centre de Recherches Mathématiques, Montréal, 1990.
13. J.D. Lewis, *Towards as generalization of Mumford's theorem*, J. Math. Kyoto Univ. **29** (1989), 267–272.
14. ______, *Cylinder homomorphisms and Chow groups*, Math. Nachr. **160** (1993), 205–221.
15. D. Mumford, *Rational equivalence of* 0-*cycles on surfaces*, J. Math. Kyoto Univ. **9** (1969), 195–204.
16. J.P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.
17. ______, *On a conjectural filtration on the Chow groups of an algebraic variety*, preprint, 1991.
18. A. Roitman, *On Γ-equivalence of zero-dimensional cycles*, Math. USSR-Sb. **15** (1971), 555–567.
19. ______, *Rational equivalence of zero-cycles*, Math. USSR-Sb. **18** (1972), 571–588.
20. S. Saito, *Motives and filtrations on Chow groups*, preprint, 1992.
21. H. Saito, *On Bloch's metaconjecture*, preprint, 1990.
22. ______, *The Hodge cohomology and cubic equivalences*, Nagoya Math. J. **94** (1984), 1–41.
23. C. Schoen, *On Hodge structures and non-representability of Chow groups*, Compositio Math. **88** (1993), 285–316.