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APPROXIMATE VERSIONS OF CAUCHY'S FUNCTIONAL EQUATION

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1. Introduction

Ulam [U, page 63] raised the general problem of when a mathematical entity which nearly meets certain requirements must be close, in some sense, to one which does meet the requirements. A particular case is a result of Hyers [H]: if

$$|f(x+y) - f(x) - f(y)| < \varepsilon \quad \text{for all } x, y,$$

then there is a g satisfying Cauchy's equation with $|f(x) - g(x)| < \varepsilon$ for all x. A survey of related results appears in [HR].

In this note, we look at stronger assumptions ([H] did not even assume f was measurable) that imply $f(x) = \gamma x$ almost everywhere (we will use Lebesgue measure, denoted by μ, throughout). Our main results are:

THEOREM 1. *Let f, a, b be measurable functions and let*

$$\delta(x,y) \equiv f(x+y) - a(x) - b(y). \tag{1}$$

If there is a $J \in \mathbf{R}$ such that, for every $\varepsilon > 0$,

$$\mu(\{(x,y) | \, |\delta(x,y) - J| \geq \varepsilon\}) \tag{2}$$

is finite, then, for some γ and β, $f(x) = \gamma x + \beta$ almost everywhere.

Remarks

1. It is easy to see that, if $f = a = b$ and $J = 0$, then $\beta = 0$.

The referee points out that the case $f = a = b$ and $J \neq 0$ is related to Pexider's equation $f(x+y) = f(x) + f(y) + K$.

2. For any $p > 0$, $\delta \in L^p(R^2)$ implies that δ satisfies (2) with $J = 0$.

3. It can also be shown that, for some β', γ', $a(x) = \gamma' x + \beta'$ almost everywhere (the same argument applies to $b(x)$ by symmetry): replace f by

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$f'(x) = -a(-x)$ and let $a'(x) = -f(-x)$, and $b'(x) = b(x)$. Then

(3) $$\delta'(x,y) \equiv f'(x+y) - a'(x) - b'(y) \equiv \delta(-x-y,y)$$

satisfies the hypothesis of Theorem 1 if δ does, since the two are related by a measure-preserving transformation (look at the Jacobian), and the conclusion follows. Moreover, $\gamma = \gamma'$ (consider what happens with y fixed) and $\delta(x,y) = J$ almost everywhere.

THEOREM 2. *Let $f \in L^1[0,a]$ for all $a > 0$. For $x, y \geq 0$, define*

(4) $$\delta(x,y) \equiv f(x+y) - f(x) - f(y).$$

Suppose that for almost all x,

(5) $$\lim_{u\to\infty} \frac{1}{u}\int_0^u \delta(x,y)\,dy = 0$$

Then for some γ, $f(x) = \gamma x$ for almost all $x \geq 0$.

Notice the absence of absolute value signs in (5).

Elliott [E1] has shown that, for any $\alpha > 0$, $f(x) = \gamma x$ almost everywhere if $f \in L^\alpha(0,z)$ for all $z > 0$ and

(6) $$\lim_{z\to\infty} z^{-1}\int_0^z\int_0^z |f(x+y) - f(x) - f(y)|^\alpha\,dx\,dy = 0.$$

These results each cover certain cases not included in the others. Theorem 1 only assumes the measurability of f. Theorem 2 could be applied to cases in which $\int\delta$ is small but $\int|\delta|$ is large. For example, Theorem 2 implies that we could not have

(7) $$\delta(x,y) \equiv \sin\left(\left(x^2+y^2\right)^{1/2}\right).$$

We present proofs of these theorems in the next two sections. In our final section, we take a more elementary approach which, for the case of continuous functions, gives more information.

We thank Richard Rochberg for suggesting a related question to one of us (LAR).

2. Proof of Theorem 1

LEMMA 3. *If $D, E \subseteq \mathbf{R}$ and each set has finite measure, then for any $L \in \mathbf{R}$, there is $K \in \mathbf{R}$ with $K \notin D$ and $K + L \notin E$.*

Proof. Let $N = \mu(D) + \mu(E)$. Let K be any member of $[0, N+1]$ which is not a member of $D \cup (E - L)$, where the minus sign denotes translation. ∎

LEMMA 4. *Assume δ satisfies the assumptions of Theorem 1. For $\varepsilon, \theta > 0$ define*

$$A_{x,\varepsilon} = \{y \mid |\delta(x,y) - J| > \varepsilon\} \text{ and } B_{\varepsilon,\theta} = \{x \mid \mu(A_{x,\varepsilon}) > \theta\}. \tag{8}$$

Then $B_{\varepsilon,\theta}$ has finite measure for each ε, θ.

Proof. If the measure were not finite, Fubini's theorem would imply

$$|\delta(x,y) - J| \geq \varepsilon$$

on a set of infinite measure. ∎

LEMMA 5. *Define*

$$\begin{aligned} h(y,K,L) &\equiv \delta(K+L,y) - \delta(K,y) \\ &\equiv [f(y+K+L) - f(y+K)] - [a(K+L) - a(K)]. \end{aligned} \tag{9}$$

For any ε, $\theta > 0$ and $L \in \mathbf{R}$, there is a $K \in \mathbf{R}$ such that

$$\mu(\{y \mid |h(y,K,L)| \geq \varepsilon\}) \leq \theta. \tag{10}$$

Proof. Since $B_{\varepsilon/2,\theta/2}$ has finite measure, Lemma 3 implies that there is a K such that both K and $K+L$ are not members. Thus

$$\mu(A_{K,\varepsilon/2} \cup A_{K+L,\varepsilon/2}) \leq \theta \tag{11}$$

and, if y is not in the union, $|\delta(K+L,y) - \delta(K,y)| \leq \varepsilon$. ∎

LEMMA 6. *For any L, there is a number M_L such that*

$$f(y+L) - f(y) = M_L \tag{12}$$

for almost all y.

Proof. For $n = 1, 2, \ldots$, let K_n be given by Lemma 5 with $\varepsilon = \theta = 2^{-n}$, and let

$$s_n = a(K_n + L) - a(K_n), \tag{13}$$

$$C_n = \{y \mid |h(y, K_n, L)| < 2^{-n}\}. \tag{14}$$

C_n is the complement of the set in (10), so we may apply Lemma 3 with D and E the complements of C_n and C_{n+1} to conclude that there is $y \in C_n$ such that $y' = y + (K_n - K_{n+1}) \in C_{n+1}$, which implies that

$$|s_n - s_{n+1}| = |h(y, K_n, L) - h(y', K_{n+1}, L)| \leq 2^{-n+1}, \tag{15}$$

so s_n is a Cauchy sequence. We let M_L be its limit. The set of y for which (12) holds contains

$$\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} (C_n + K_n), \tag{16}$$

where the plus sign denotes translation. Since the complement of $C_n + K_n$ has measure $\leq 2^{-n}$, the complement of the set in (16) has measure 0. ∎

Finally, we show that, if f satisfies the conclusion of Lemma 6, then, for some β, $f(x) = M_1 x + \beta$ almost everywhere. We will assume $M_1 \geq 0$ in the proof. The case $M_1 < 0$ follows by considering $-f(x)$. Let

$$E_r = f^{-1}(-\infty, r) \cap [-1, 1], \tag{17}$$

$$\beta = \sup\{r | \mu(E_r) < 1\}, \tag{18}$$

$$g(x) = M_1 x + \beta. \tag{19}$$

Note that $\mu(E_\beta) \leq 1$.

If $f \neq g$ almost everywhere, then there is $\varepsilon > 0$ with $|f(x) - g(x)| > \varepsilon$ on a set of positive measure. We will show that both $f > g$ and $f < g$ lead to contradictions. The idea of the argument in both cases is that we begin by locating a small interval with f bounded away from g in most of the interval. Then we use (12) to conclude that f must be bounded away from g for most of $[-1, 1]$, and show that this leads to contradictions with the definition of β.

Case 1 (f too big). Define

$$T = \{x | f(x) > g(x) + \varepsilon\}. \tag{20}$$

If $\mu(T) > 0$, we can find, for any $\tau > 0$, a sequence I_t of intervals with rational endpoints with $T \subset \cup_t I_t$ and $\Sigma_t \mu(I_t) < (1 + \tau)\mu(T)$. For at least one t, $(1 + \tau)\mu(I_t \cap T) > \mu(I_t)$. Since I_t can be written as a union of subintervals (disjoint except for endpoints), we can find arbitrarily small intervals I with

$$\frac{\mu(T \cap I)}{\mu(I)} \tag{21}$$

arbitrarily close to 1. In particular, there is a natural number m and an integer k such that

$$\mu\left(T \cap \left[\frac{k}{m}, \frac{k+1}{m}\right]\right) > \left(\frac{M_1 + \varepsilon}{M_1 + 2\varepsilon}\right)\frac{1}{m}. \tag{22}$$

We require m to be so large that there is a natural number $j \leq m$ such that

$$M_1\left(\frac{j}{m}\right) < \varepsilon, \tag{23}$$

$$\left(1 - \frac{j}{m}\right) + \frac{\varepsilon}{M_1 + 2\varepsilon}\left(1 + \frac{j}{m}\right) \leq 1. \tag{24}$$

(The expression on the left in (24) is monotone decreasing in j/m, and ≤ 1 if $j/m = \varepsilon/M_1$. If $M_1 = 0$, then $j = m$.)

Let $\alpha = g(-j/m) + \varepsilon$. We will show that $\mu(E_\alpha) < 1$. Since (23) implies $\alpha > \beta$, this will contradict (18).

If $x \geq -j/m$, $f(x) > g(x) + \varepsilon$ implies $f(x) > \alpha$, so

$$\left\{x | f(x) > \alpha \text{ and } x \geq -\frac{j}{m}\right\} \supseteq T \cap \left[-\frac{j}{m}, \infty\right). \tag{25}$$

It is easy to show that, for any natural number m, $M_{(1/m)} = (1/m)M_1$. Hence, by Lemma 6 with $L = 1/m$,

$$x \in T \quad \text{if and only if } x + 1/m \in T \tag{26}$$

for almost every x. This implies that (22) holds for any integer k. If $k \geq -j$, (22) and (25) imply

$$\begin{aligned}\mu\left(\left\{x | f(x) \leq \alpha \text{ and } x \in \left[\frac{k}{m}, \frac{k+1}{m}\right]\right\}\right) \\ < \left(1 - \frac{M_1 + \varepsilon}{M_1 + 2\varepsilon}\right)\frac{1}{m} = \frac{\varepsilon}{(M_1 + 2\varepsilon)m}.\end{aligned} \tag{27}$$

We can write $[-j/m, 1]$ as a union of $j + m$ intervals of length $1/m$ and use (27) on each one to obtain

$$\mu\left(\left\{x | f(x) \leq \alpha \text{ and } x \in \left[-\frac{j}{m}, 1\right]\right\}\right) < \frac{\varepsilon(j+m)}{(M_1 + 2\varepsilon)m}. \tag{28}$$

Now, (28) and (24) together yield

$$\mu(f^{-1}(-\infty, \alpha) \cap [-1, 1]) < \mu\left(\left[-1, -\frac{j}{m}\right]\right) + \frac{\varepsilon(j+m)}{(M_1 + 2\varepsilon)m} \leq 1. \tag{29}$$

In other words, $\mu(E_\alpha) < 1$. As previously indicated, $\alpha > \beta$, so this contradicts (18).

Case 2 (f too small). The essential ideas are the same as in case 1. This time, we define

$$T = \{x \mid f(x) < g(x) - \varepsilon\}. \tag{30}$$

k, m, j are chosen so that they satisfy (22), (23), and

$$\frac{M_1 + \varepsilon}{M_1 + 2\varepsilon}\left(1 + \frac{j}{m}\right) \geq 1. \tag{31}$$

Define $\alpha = g(j/m) - \varepsilon$. By (23), $\alpha < \beta$. We will show that $\mu(E_\alpha) > 1$, which implies $\mu(E_\beta) > 1$, which is inconsistent with the construction of β.

For $x \leq j/m$, $f(x) < g(x) - \varepsilon$ implies $f(x) < \alpha$, so

$$\left\{x \mid f(x) < \alpha \text{ and } x \leq \frac{j}{m}\right\} \supseteq T \cap \left(-\infty, \frac{j}{m}\right]. \tag{32}$$

Just as in case 1, (26) implies (22) holds for any integer k. Hence, if $k + 1 \leq j$, (32) and (22) yield

$$\mu\left(\left\{x \mid f(x) < \alpha \text{ and } x \in \left[\frac{k}{m}, \frac{k+1}{m}\right]\right\}\right) > \left(\frac{M_1 + \varepsilon}{M_1 + 2\varepsilon}\right)\frac{1}{m}. \tag{33}$$

Write $[-1, j/m]$ as a union of $m + j$ intervals of length $1/m$, use (33) on each one, and apply (31) to obtain

$$\mu\left(\left\{x \mid f(x) < \alpha \text{ and } x \in \left[-1, \frac{j}{m}\right]\right\}\right) > \left(\frac{M_1 + \varepsilon}{M_1 + 2\varepsilon}\right)\frac{m+j}{m} \geq 1. \tag{34}$$

This establishes that $\mu(E_\alpha) > 1$, which leads to the desired contradiction.

3. Proof of Theorem 2

Iterating the equation (4) gives

$$f(y + nx) = nf(x) + f(y) + \sum_{k=0}^{n-1} \delta(x, y + kx). \tag{35}$$

Integrate equation (35) with respect to y to get

$$\frac{1}{nx}\int_0^x f(y + nx)\,dy = f(x) + \frac{1}{nx}\int_0^x f(y)\,dy + \frac{1}{nx}\int_0^{nx} \delta(x, y)\,dy. \tag{36}$$

If x satisfies (5), then

$$\lim_{n\to\infty} \frac{1}{nx}\int_0^x f(y+nx)\,dy = f(x). \tag{37}$$

To complete the proof, we first show that (37) implies, for any natural number r, that

$$f(rw) = rf(w) \quad \text{for almost all } w. \tag{38}$$

Next we show this implies $f(x) = \gamma x$, for some γ and almost all x.

Let S be the set of x for which (37) holds. We have seen that (5) implies almost every real number is in S. Hence, almost every x is in

$$\bigcap_{r=1}^{\infty} \frac{1}{r} S. \tag{39}$$

Hence, for almost every w, (37) holds for all $x \in \{w, 2w, 3w, \dots\}$. For such w,

$$\begin{aligned} f(rw) &= \lim_{n\to\infty} \frac{1}{nwr}\int_0^{rw} f(y+nrx)\,dy \\ &= \lim_{n\to\infty} \frac{1}{nwr}\sum_{k=0}^{r-1}\int_{kw}^{(k+1)w} f(y+nrw)\,dy = rf(w). \end{aligned} \tag{40}$$

This completes the proof of (38) for natural numbers r. It follows immediately that (38) holds for all rational $r > 0$.

The rest of the proof depends on theorems of Lebesgue about functions $f \in L^1$ and their "indefinite integrals" $F(x) \equiv \int_0^x f(w)\,dw$, which may be found, for example, in [KF, pp. 313–324]:

1. $F(rx) = r\int_0^x f(rw)\,dw$.
2. F is continuous.
3. $f(x) = F'(x)$ almost everywhere.

Let $\gamma/2 = F(1)$. For rational $r > 0$, we can use (38) to obtain

$$F(r) = r\int_0^1 f(rw)\,dw = r\int_0^1 rf(w)\,dw = r^2\gamma/2. \tag{41}$$

The continuity of F implies $F(x) = \gamma x^2/2$ for all x, so $f(x) = F'(x) = \gamma x$ almost everywhere. This completes the proof.

Theorem 2 can be extended to $f \in L^1[-a, a]$ for all $a > 0$. Theorem 2 implies that $f(x) = \gamma x$ for almost all $x \geq 0$. If $x < 0$ satisfies (5), then

$$(42)\quad 0 = \lim_{u\to\infty} \frac{1}{u}\int_0^u \delta(x,y)\,dy = \lim_{u\to\infty} \frac{1}{u+x}\int_{-x}^u \delta(x,y)\,dy$$
$$= \lim_{u\to\infty} \frac{1}{u+x}\int_{-x}^u \{[f(x+y) - f(y)] - f(x)\}\,dy = \gamma x - f(x)$$

and the conclusion follows.

4. A different analysis

The result we prove in this section is:

THEOREM 7. *Let f, a, b be continuous function and let*

$$(43)\qquad \delta(x,y) = f(x+y) - a(x) - b(y).$$

If $\delta \in L^p(\mathbf{R}^2)$ for some $p \geq 1$, then $f(x) \equiv \gamma x + \beta$ for some $\gamma, \beta \in \mathbf{R}$.

This follows from Theorem 1, but the method of proof here is more elementary. When f is not affine, we are able to identify regions in the plane (unions of infinite strips) on which $\int|\delta|$ is infinite.

Reasoning similar to that given in remark 3 following Theorem 1 can be used to conclude that $a(x) \equiv \gamma x + \beta'$ and $b(x) \equiv \gamma x + \beta''$, with $\delta(x,y) \equiv 0$.

LEMMA 8. *If we establish Theorem 7 for the case in which $a(x) \equiv b(x)$, this establishes the result in general.*

Proof. Make the replacements

$$(44)\qquad \delta'(x,y) \equiv \frac{\delta(x,y) + \delta(y,x)}{2},\ f'(x) \equiv f(x),$$
$$a'(x) \equiv b'(x) \equiv \frac{a(x) + b(x)}{2}.$$

δ', f', a', b' satisfy the assumptions of the theorem if δ, f, a, b do, so our hypothesis allows us to conclude that $f(x) = \gamma x + \beta$. ■

From now on, we will assume $a(x) \equiv b(x)$.

LEMMA 9. *If, for all $c, d, c', d' \in \mathbf{R}$, $c + d = c' + d'$ implies*

$$(45)\qquad a(c) + a(d) = a(c') + a(d'),$$

then for some $\gamma, \beta, a(x) \equiv \gamma x + \beta$ *and either* $f(x) \equiv \gamma x + 2\beta$ *(i.e.,* $\delta(x, y) \equiv 0$*) or there are* $\varepsilon > 0$ *and numbers* $K < L$ *with* $|\delta(x, y)| > \varepsilon$ *if* $K < x + y < L$.

Proof. For any numbers x, y, (45) implies $a(x) + a(y) = a(x + y) + a(0)$. If we define $a'(x) \equiv a(x) - a(0)$, then a' is a continuous solution to Cauchy's equation. This implies a' is linear and $a(x) \equiv \gamma x + a(0)$, for some γ. If

$$f(x) \not\equiv \gamma x + 2a(0),$$

continuity implies that there are ε, K, L with

$$|f(x) - \gamma x - 2a(0)| > \varepsilon$$

for $K < x < L$. ■

To complete the proof, the remaining case is treated using

LEMMA 10. *If there are* c, d, c', d' *with* $c + d = c' + d'$ *such that* (45) *does not hold, then there are* $\varepsilon, C > 0$ *such that if*

$$s(A) = \int_{R_1 \cup R_2 \cup R_3 \cup R_4} |\delta(x, y)|, \tag{46}$$

the integral over the union of four rectangles, where

$$R_1 = \{(x, y) |\ |x - c| < \varepsilon \text{ and } |y| < A\}$$
$$R_2 = \{(x, y) |\ |x - c'| < \varepsilon \text{ and } |y| < A\}$$
$$R_3 = \{(x, y) |\ |x - d'| < \varepsilon \text{ and } |y| < A\}$$
$$R_4 = \{(x, y) |\ |x - d| < \varepsilon \text{ and } |y| < A\},$$

then $s(A) > CA$ *for* A *sufficiently large.*

Proof. By continuity, we may assume c, c', d, d' are all different. Define

$$h(t) \equiv a(c + t) + a(d - t) - [a(c' + t) + a(d' - t)]. \tag{47}$$

Choose $\varepsilon > 0$ so that, for some $B > 0$, if $|t| \leq \varepsilon$, $|h(t)| > B$, and so that the R_i are disjoint.

Let $K = c' - c = d - d'$. For any $y \in \mathbf{R}$,

$$\begin{aligned} &-\delta(c + t, y) + \delta(c' + t, y - K) \\ &\quad + \delta(d' - t, y) - \delta(d - t, y - K) = h(t). \end{aligned} \tag{48}$$

If we take absolute values in (48), apply the triangle inequality, and integrate over $|y| < A$ and $|t| < \varepsilon$, we get

$$u(A) = \int_{S_1 \cup S_2 \cup S_3 \cup S_4} |\delta(x, y)| > 4AB\varepsilon, \tag{49}$$

where $S_1 = R_1$, $S_3 = R_3$, and S_2, S_4 are R_2, R_4 shifted downward by K. Since $s(A + K) \geq u(A) > 4AB\varepsilon$, this gives the desired result for any $C < 4B\varepsilon$. ■

This establishes Theorem 7 for the case $p = 1$. The case $p > 1$ may be obtained by Hölder's inequality.

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