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THE NUMBER OF ELEMENTS REQUIRED TO DETERMINE $(p, 1)$-SUMMING NORMS

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Introduction

A basic problem concerning the various summing norms of operators is to estimate the number of elements needed to determine the value of the norm (up to a constant multiple), either in terms of the rank of the operator or in terms of the dimension of its domain. As well as being of interest in their own right, such estimates play a vital part in evaluating ratios between different summing norms.

A good summary appears in [10], Chapter 4. The most satisfactory results concern the norm π_2: if the rank of T is n, then n elements are enough to give $(1/\sqrt{2})\pi_2(T)$ (this, the prototype of all theorems of this type, was originally proved in [9], with 2 instead of $\sqrt{2}$), and the exact value can be found using $\frac{1}{2}n(n+1)$ elements in the real case, or n^2 elements in the complex case. For other p, one finds that 4^n elements will give at least $\frac{1}{3}\pi_p(T)$, and when $p = 1$, the number of elements needed is of this order. More recently, Szarek [8] has shown that for operators on an n-dimensional space, π_1 can be estimated using $n \log n$ elements, and this result has been extended to other p by Johnson and Schechtman [6].

For the mixed summing norms $\pi_{p,2}$, König [7] showed that there is a constant C such that

$$\pi_{p,2}(T) \leq Cp/(p-2)\pi_{p,2}^{(n)}(T)$$

for operators of rank n. In [8] it was shown that $Cp/(p-2)$ can be replaced by a constant independent of p, and in [2] the constant was improved to $\sqrt{2}$. By applying some fairly deep theorems, these results can be applied to derive corresponding ones for $\pi_{p,1}$, where $p > 2$ [10, Proposition 24.9], but with intervening constants C_p that tend to infinity as $p \to 2$.

In this note we show by the easiest of arguments (simply discarding some of the elements) that the number of elements required for $\pi_{p,1}$ is not more than $c_p(\alpha/\beta)^{p^*}$, where $\alpha = \pi_1(T)$, $\beta = \pi_{p,1}(T)$ and c_p grows large when $p \to 1$. This converts the problem into estimating α/β. For the case $p = 2$, a result from [4] shows that α/β is of the order of $(n \log n)^{1/2}$ for operators of

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rank n, so that $n \log n$ elements are enough to determine $\pi_{2,1}(T)$ for such operators. It remains an open question (one which has been circulated informally for some years) whether the $\log n$ factor is really needed in these statements. In the case $p > 2$, we show that α/β is of the order n^{1/p^*}, thereby obtaining a new, simpler proof of the result mentioned above.

Preliminaries

For a finite sequence of elements of a normed linear space, write

$$\mu_1(x_1, \ldots, x_n) = \sup\left\{ \sum_{i=1}^{n} |f(x_i)| : f \in X^*, \|f\| \leq 1 \right\}.$$

Note that μ_1 is reduced when we pass to a subset. For any operator T, define

$$\pi_{p,1}^{(n)}(T) = \sup\left\{ \left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} : \mu_1(x_1, \ldots, x_n) \leq 1 \right\},$$

in which all sequences of length n are considered. This is the quantity obtained by restricting the number of elements to n in the definition of $\pi_{p,1}$. Obviously,

$$\pi_{p,1}^{(kn)}(T) \leq k^{1/p} \pi_{p,1}^{(n)}(T).$$

Before proceeding to our theorems on the relationship between $\pi_{p,1}$ and $\pi_{p,1}^{(k)}$, we mention some special cases where the answer is simple.

(1) If T maps on or into a Hilbert space, then $\pi_{p,1}(T) = \|T\|$ for $p \geq 2$, so one element is enough to determine $\pi_{p,1}(T)$. Similarly (up to a constant) for any space with the Orlicz property.

(2) For operators on l_∞^n, the exact value of $\pi_{p,1}(T)$ can be found using elements with disjoint support, hence not more than n elements (see e.g., [5], 14.4).

(3) For operators *into* l_∞^n, we have again $\pi_{p,1}(T) = \pi_{p,1}^{(n)}(T)$. To see this, take a finite sequence (x_i) at which $\pi_{p,1}(T)$ is nearly attained, and for each j combine together the elements x_i for which $\|Tx_i\|$ is attained at coordinate j.

The theorems

Our basic result applies to any 1-summing operator, whether or not of finite rank. It estimates the required number of elements in terms of the ratio between $\pi_1(T)(=\alpha)$ and $\pi_{p,1}(T)(=\beta)$. By way of motivation, consider the special situation where $\pi_{p,1}(T)$ is determined by elements $x_1, \dots, x_N$ with the norms $\|Tx_i\|$ all equal. This means that $\mu_1(x_1, \dots, x_N) = 1$ and $\|Tx_i\| = N^{-1/p}\beta$ for each i. Since $\sum_{i=1}^N \|Tx_i\| \le \alpha$, we have necessarily $N^{1-1/p}\beta \le \alpha$, or $N \le (\alpha/\beta)^{p/(p-1)}$.

THEOREM 1. *Let T be any 1-summing operator, $p > 1$. Let $\pi_1(T) = \alpha$, $\pi_{p,1}(T) = \beta$. Then there exists*

$$k \le c_p\left(\frac{\alpha}{\beta}\right)^{p^*} + 1$$

such that

$$\pi_{p,1}^{(k)}(T) \ge \left(1 - \frac{1}{2p}\right)^{1/p} \beta \quad \left(\ge \frac{1}{2}\beta\right),$$

where $p^* = p/(p-1)$ and

$$c_p = \left(1 - \frac{1}{p}\right) 2^{1/(p-1)} \quad (\textit{hence } c_p \le 1 \textit{ for } p \ge 2).$$

In particular, when $p = 2$ we have $k \le (\alpha/\beta)^2 + 1$ and $\pi_{2,1}(T) \ge (\sqrt{3}/2)\beta$.

Proof. Let $\varepsilon > 0$ and let $x_1, \dots, x_N$ be elements such that

$$\mu_1(x_1, \dots, x_N) = 1$$

and

$$\sum_{i=1}^N \|Tx_i\|^p \ge (1 - \varepsilon)\beta^p.$$

Note that $\sum_{i=1}^N \|Tx_i\| \le \alpha$. Let the elements be indexed so that $\|Tx_1\| \ge \cdots \ge \|Tx_N\|$. We simply restrict attention to the first k elements x_i for a suitably chosen k. We give the argument for $p = 2$ first, since this case is simpler as well as being the most important. Assume that $N > \alpha^2/\beta^2$, or there is nothing to prove. Let k be the least integer such that $k \ge \alpha^2/\beta^2$. Write $\sum_{i=1}^k \|Tx_i\| = s$. Then $\|Tx_k\| \le s/k$ and $\sum_{i=k+1}^N \|Tx_i\| \le \alpha - s$. Since $s(\alpha - s) \le \frac{1}{4}\alpha^2$ and $\alpha^2 \le k\beta^2$, we have

$$\sum_{i=k+1}^{n} \|Tx_i\|^2 \le \frac{s}{k}(\alpha - s) \le \frac{\alpha^2}{4k} \le \frac{1}{4}\beta^2.$$

Hence $\sum_{i=1}^{k} \|Tx_i\|^2 \geq (\frac{3}{4} - \varepsilon)\beta^2$, and the statement follows.

For other p, we use the inequality $s^{p-1}(\alpha - s) \leq (1/p)\alpha^p(1 - (1/p))^{p-1}$. From this we have

$$\sum_{i=k+1}^{N} \|Tx_i\|^p \leq \left(\frac{s}{k}\right)^{p-1}(\alpha - s) \leq k^{1-p}\frac{\alpha^p}{p}\left(1 - \frac{1}{p}\right)^{p-1}.$$

To ensure that this is not greater than $\beta^p/2p$, we take $k \geq c_p(\alpha/\beta)^{p^*}$, with c_p as stated. The statement that $c_p \leq 1$ for $p \geq 2$ is equivalent to $(p/(p-1))^{p-1} \geq 2$ for $p \geq 2$, which follows from the fact that $(1 + 1/x)^x$ is an increasing function for $x \geq 1$.

COROLLARY. *Suppose that $\pi_1(T) \leq C\pi_1^{(n)}(T)$ for some C, n. Then there exists $k \leq c_p C^{p^*} n + 1$ such that*

$$\pi_{p,1}^{(k)}(T) \geq \left(1 - \frac{1}{2p}\right)^{1/p} \pi_{p,1}(T).$$

Proof. From Hölder's inequality we have at once

$$\pi_1^{(n)}(T) \leq n^{1/p^*}\pi_{p,1}^{(n)}(T),$$

hence $(\alpha/\beta)^{p^*} \leq C^{p^*}n$.

It follows from Szarek's result on π_1 and Hölder's inequality again that for operators *on an n-dimensional space*, $\pi_1(T) \leq C(n \log n)^{1/2}\pi_{2,1}(T)$, and hence that $\pi_{2,1}(T)$ can be estimated by $n \log n$ elements for such operators. We now show that the same applies for all operators of *rank n*.

The following was proved in [4] (cf. [10], Corollary 21.8): *there is a constant $C(< 4)$ such that for every k and every operator T on l_∞^k of rank n (≥ 2), we have*

$$\pi_2(T) \leq C(\log n)^{1/2}\pi_{2,1}(T).$$

(The proof in [4] uses $\pi_2(T) \leq \sqrt{2}\,\pi_2^{(n)}(T)$, so does not apply to other p; however, the result has been generalized to other p in [1].)

LEMMA 1. *There is a constant C (< 4) such that for any operator T of rank n,*

$$\pi_1(T) \leq C(n \log n)^{1/2}\pi_{2,1}(T).$$

Proof. By the well-known premultiplication lemma [10, Proposition 9.7] or [5, 3.6], for any $\varepsilon > 0$, there exist k and an operator A: $l_\infty^k \to X$ such that

$\|A\| \leq 1$ and $\pi_1(TA) \geq (1-\varepsilon)\pi_1(T)$. With C as above, we have

$$\pi_2(TA) \leq C(\log n)^{1/2}\pi_{2,1}(TA) \leq C(\log n)^{1/2}\pi_{2,1}(T).$$

Since the rank of TA is not more than n, we have $\pi_1(TA) \leq n^{1/2}\pi_2(TA)$. The statement follows.

THEOREM 2. *There is a constant C (< 6) such that for all $n \geq 3$ and all operators T of rank n, there exists $k \leq n \log n$ such that*

$$\pi_{2,1}(T) \leq C\pi_{2,1}^{(k)}(T).$$

Proof. By Theorem 1 and Lemma 1, there exists $m \leq 17n \log n$ such that

$$\pi_{2,1}(T) \leq \left(2/\sqrt{3}\right)\pi_{2,1}^{(m)}(T).$$

Clearly, $m \leq 25k$ for some integer $k \leq n \log n$: then $\pi_{2,1}^{(m)}(T) \leq 5\pi_{2,1}^{(k)}(T)$.

LEMMA 2. *There is a constant C such that for all $p > 2$ and all operators T of rank n,*

$$\pi_1(T) \leq C\left(1 + (p-2)^{-1/2}\right)n^{1/p^*}\pi_{p,1}(T).$$

Proof. Let $\varepsilon > 0$ and $A\colon l_\infty^k \to X$ be as in Lemma 1. Write $TA = U$. Note that $\pi_{p,1}(U) \leq \pi_{p,1}(T)$ and

$$\pi_1(U) \leq n^{1/2}\pi_2(U) \leq (2n)^{1/2}\pi_2^{(n)}(U).$$

By Hölder's inequality,

$$\pi_2^{(n)}(U) \leq n^{1/2-1/p}\pi_{p,2}^{(n)}(U).$$

By a simple version of [10], Theorem 21.3, there is a constant C' such that for any operator U on l_∞^k,

$$\pi_{p,2}(U) \leq C'\left(1 + (p-2)^{-1/2}\right)\pi_{p,1}(U).$$

The statement follows, with $C = \sqrt{2}\,C'$.

THEOREM 3. *There is a constant C (independent of n and p) such that for all $p > 2$ and all operators T of rank n,*

$$\pi_{p,1}(T) \leq C\left(1 + \frac{1}{(p-2)^{1/2}}\right)\pi_{p,1}^{(n)}(T).$$

Proof. By Theorem 1 and Lemma 2, there exists $k \leq rn$ such that $\pi_{p,1}(T) \leq 2\pi_{p,1}^{(k)}(T)$, where for some C we have

$$r \leq C\left(1 + (p-2)^{-1/2}\right)^{p^*}.$$

Recall that $\pi_{p,1}^{(n)}(T) \leq r^{1/p}\pi_{p,1}^{(k)}(T)$. For $p \geq 3$, we see that $r \leq 3C$. For $2 < p < 3$, we have $r^{1/p} \leq C'(p-2)^{-1/2}$ for some suitable C', since $(p-2)^{p^*/2p} > (p-2)^{1/2}$. The given statement combines both cases.

Problems and remarks

(1) Perversely, Theorem 1 gives a larger estimate for the number of elements needed when $\pi_{p,1}(T)$ becomes smaller relative to $\pi_1(T)$. However, it is clear from the remark preceding the theorem that it represents the limit of what can be achieved by simply discarding some of the elements. Any further reduction must involve the more difficult processes of weighting or grouping them.

(2) Is the $\log n$ factor needed in Lemma 1 and Theorem 2? This question appears to be non-trivial even when restricted to identity operators. It was shown in [4] that $\log n$ cannot be removed from the theorem used in the proof of Lemma 1.

(3) Is the $(p-2)$ needed in Lemma 2 and Theorem 3? It seems highly probable that the answers to (2) and (3) are the same.

(4) The methods and results of this paper are further developed in the forthcoming paper [3]. In particular, an estimate of the form $c_p n^{p^*/2}$ (with $c_p \to \infty$ as $p \to 2$) is given for the number of elements needed when $p < 2$, and the results are extended to the norms $\pi_{p,q}$ where $1 < q < p$.

REFERENCES

1. B. CARL and A. DEFANT, *An inequality between the p- and $(p,1)$-summing norm of finite-rank operators from $C(K)$ spaces*, Israel J. Math. **74** (1991), 323–335.
2. M. DEFANT and M. JUNGE, *On absolutely summing operators with application to the (p,q)-summing norm with few vectors*, J. Funct. Anal. **103** (1992), 62–73.
3. ______, *How many vectors are needed to compute (p,q)-summing norms?*, Illinois J. Math., to appear.
4. G.J.O. JAMESON, *Relations between summing norms of mappings on l_∞^n*, Math. Z. **194** (1987), 89–94.

5. ______, *Summing and nuclear norms in Banach space theory*, Cambridge Univ. Press, Cambridge, England, 1987.
6. W.B. JOHNSON and G. SCHECHTMAN, *Computing p-summing norms with few vectors*, preprint.
7. H. KÖNIG, *Type constants and $(q, 2)$-summing norms defined by n vectors*, Israel J. Math. **37** (1980), 130–138.
8. S.J. SZAREK, *Computing summing norms and type constants on few vectors*, Studia Math. **98** (1990), 147–156.
9. N. TOMCZAK-JAEGERMANN, *Computing 2-summing norm with few vectors*, Ark. Mat. **17** (1979), 273–277.
10. ______, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics 38, Longman, Harlow, England, 1989.

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