ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

A GENERALIZED JENSEN'S INEQUALITY FOR BOUNDED ANALYTIC FUNCTIONS

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The classical Jensen's inequality, valid for any non-zero function in the Hardy space H^1 states that

$$\int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi} \geq \log|f(0)|. \tag{1}$$

This estimate plays an essential rôle, for instance in number theory (see [5]), in conjunction with Mahler's measure:

$$M(f) = \exp \int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}.$$

Estimate (1) has several drawbacks. First, it takes into account only the 0-th coefficient of f. It is also discontinuous, in the sense that applying it to the sequence of functions $f_n(z) = 1/n + z$ leads to the estimate $\log 1/n \to -\infty$, though the limit function satisfies $\int \log |f| = 0$.

In order to find better versions of Jensen's inequality, one is naturally led to the concept of *concentration at low degrees*, introduced by Beauzamy-Enflo in [1]. Say that a polynomial $P(z) = \sum_0^n a_j z^j$ has concentration d $(0 < d \leq 1)$ at degree k if

$$\sum_0^k |a_j| \geq d \sum_0^n |a_j|. \tag{2}$$

It is possible to improve on Jensen's inequality by finding a constant $\tilde{C}(d, k)$ depending only on the concentration of the polynomial (and not on the degree) such that

$$\int_0^{2\pi} \log|P(e^{i\theta})| \frac{d\theta}{2\pi} \geq \tilde{C}(d, k). \tag{3}$$

Received November 30, 1992.
1991 Mathematics Subject Classification. Primary 30A10; Secondary 30C10.
[1]Supported by the C.N.R.S. (France) and the N.S.F. (U.S.A.), by contracts E.T.C.A./C.R.E.A. nos 20367/91 and 20388/92 (Ministry of Defense, France) and by research contract EERP-FR 22, Digital Equipment Corporation.

Under condition (2), and assuming moreover the normalization $\Sigma_0^n |a_j| = 1$, it was proved by the author [2] that

$$\int_0^{2\pi} \log\left|P\left(e^{i\theta}\right)\right| \frac{d\theta}{2\pi} \geq C(d, k),$$

where $C(d, k)$ is the maximum value of the function

$$f_{d,k}(r) = \frac{1+r}{1-r} \log \frac{d\left(1 - \frac{1}{r}\right)}{1 - \frac{1}{r^{k+1}}} \quad \text{for } 0 < r < 1. \tag{4}$$

This implies the rough estimate

$$C(d, k) \geq 2 \log \frac{2d}{3^{k+1} - 1}.$$

The precise value of the best constant $\tilde{C}(d, k)$ in (3) is unknown, except in the following case. If we restrict ourselves to the class of normalized Hurwitz polynomials (that is, polynomials with real positive coefficients, with constant term 1, such that the roots have negative real parts) that satisfy concentration condition (2), then the best constant $C_{d,k}^H$ was determined by Rigler-Trimble-Varga [6]:

$$C_{d,k}^H = \log \frac{\rho}{(\rho+1)2^{\nu-1}},$$

where ν is the unique integer satisfying

$$\frac{1}{2^\nu} \sum_{j=0}^{k} \binom{\nu}{j} \leq d < \frac{1}{2^{\nu-1}} \sum_{j=0}^{k} \binom{\nu-1}{j},$$

and

$$\rho = \frac{\binom{\nu-1}{k}}{\sum_{j=0}^{k} \binom{\nu-1}{j} - d 2^{\nu-1}} - 1.$$

Recently, Richard Varga [7] proved that for any polynomial P satisfying (2) and the normalization $\Sigma_0^n |a_j| = 1$,

$$\int_0^{2\pi} \log\left|P(e^{i\theta})\right| \frac{d\theta}{2\pi} \geq \log \frac{d}{\sum_0^k \binom{n}{j}}. \tag{5}$$

This estimate improves upon Jensen's inequality, but depends on the degree of the polynomial.

Since the negative values of $\int \log|f|$ depend only on the set where $|f|$ is small, one may ignore the set where $|f|$ is large, that is assume for instance that f is bounded and that $\|f\|_\infty = 1$, where

$$\|f\|_\infty = \operatorname{supess}_\theta \left|f(e^{i\theta})\right|.$$

We can then define concentration by the formula

$$\sum_0^k |a_j| \geq d\|f\|_\infty \tag{6}$$

where $f(z) = \Sigma_0^\infty a_j z^j$ is the Taylor series of $f \in H^\infty$. Under the assumption (6) and the normalization $\|f\|_\infty = 1$, we can look for a lower bound

$$\int_0^{2\pi} \log\left|f(e^{i\theta})\right| \frac{d\theta}{2\pi} \geq C'(d, k).$$

Here again, the precise value of $C'(d, k)$ is unknown, except for $k = 1$, where it was shown by the author [3] that $C'(d, 1)$ is the unique number $c < 0$, solution of the equation

$$e^c(1 - 2c) = d.$$

Other definitions of concentration at low degrees can be taken, for instance

$$\left(\sum_0^k |a_j|^2\right)^{1/2} \geq d\|f\|_2, \tag{7}$$

for which a lower estimate of $\int \log|f|$ was given by the author in [4], Lemma 3.2.

The estimate we now give is valid with no assumption of concentration type. It simply uses the fact that the function is in H^∞.

THEOREM 1. *Let f be a function in H^∞, with $\|f\|_\infty = 1$. Then, for all r,* $0 \le r < 1$,

$$\int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \ge \frac{1+r}{1-r} \log M_f(r), \tag{8}$$

where $M_f(r) = \max|f(re^{i\theta})|$.

This estimate is best possible in the sense that for every ε, $0 < \varepsilon < 1$, *there exists a function f in H^∞, with* $\|f\|_\infty = 1$, *such that, for all r*

$$\int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \le (1-\varepsilon) \frac{1+r}{1-r} \log M_f(r). \tag{9}$$

Proof. Fix r, $0 \le r < 1$, and, for any t consider $z_0 = re^{it}$ and define

$$g_t(z) = f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right),$$

which is a function in H^∞, satisfying $\|g\|_\infty = \|f\|_\infty = 1$. We have

$$\begin{aligned} \int_0^{2\pi} \log|g_t(e^{i\theta})| \frac{d\theta}{2\pi} &= \int_0^{2\pi} \log\left| f\left(\frac{e^{i\theta}+z_0}{1+\bar{z}_0 e^{i\theta}}\right)\right| \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \frac{1-r^2}{|1+\bar{z}_0 e^{i\theta}|^2} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}. \end{aligned}$$

Since $|f| \le 1$, we have $\log|f| \le 0$, and since

$$\frac{1-r^2}{|1+\bar{z}_0 e^{i\theta}|^2} \ge \frac{1-r}{1+r},$$

we get

$$\int_0^{2\pi} \log|g_t(e^{i\theta})| \frac{d\theta}{2\pi} \le \frac{1-r}{1+r} \int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}.$$

Classical Jensen's inequality, applied to g_t, gives

$$\int_0^{2\pi} \log|g_t(e^{i\theta})| \frac{d\theta}{2\pi} \ge \log|g_t(0)| = \log|f(z_0)|$$

and therefore

$$\int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \geq \frac{1+r}{1-r} \log|f(re^{it})|.$$

Since t was arbitrary, our claim follows.

In order to prove that our formula is best possible, we take z_0 real, say $z_0 = r$. Elementary computations show that

$$\frac{1-r^2}{|1+\bar{z}_0 e^{i\theta}|^2} \leq \frac{1}{1-\varepsilon} \frac{1-r}{1+r}$$

if $|\theta| \leq \sqrt{2\varepsilon}$.

Now let's take f outer, with $\|f\|_\infty = 1$, and $|f(e^{i\theta})| = 1$ if $|\theta| > \sqrt{2\varepsilon}$. We have

$$\begin{aligned} \int_0^{2\pi} \log\left| f\left(\frac{e^{i\theta}+z_0}{1+\bar{z}_0 e^{i\theta}} \right) \right| \frac{d\theta}{2\pi} &= \int_0^{2\pi} \frac{1-r^2}{|1+\bar{z}_0 e^{i\theta}|^2} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \\ &= \int_{|\theta| \leq \sqrt{2\varepsilon}} \frac{1-r^2}{|1+\bar{z}_0 e^{i\theta}|^2} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \\ &\geq \frac{1}{1-\varepsilon} \frac{1-r}{1+r} \int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}. \end{aligned}$$

But the function $f(z+z_0/1+\bar{z}_0 z)$ is also outer, so

$$\begin{aligned} \int_0^{2\pi} \log\left| f\left(\frac{e^{i\theta}+z_0}{1+\bar{z}_0 e^{i\theta}} \right) \right| \frac{d\theta}{2\pi} &= \log|f(z_0)| \\ &\leq \log M_f(r), \end{aligned}$$

which shows that, for all r,

$$\int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \leq (1-\varepsilon) \frac{1+r}{1-r} \log M_f(r),$$

and proves our last claim.

For such a function f, of course, the quantity

$$\frac{1+r}{1-r} \log M_f(r) = c(r)$$

has only a small variation, since

$$\min_{0\le r<1} c(r) \ge \frac{1}{1-\varepsilon} \max_{0\le r<1} c(r).$$

If we take $r = 0$ in formula (8), we get

$$\int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \ge \log|f(0)|,$$

which is just the classical Jensen's inequality. Therefore, formula (8) can be viewed as a strengthening of this inequality.

COROLLARY 2. *If $f \in H^\infty$ has concentration d at degree k, measured as*

$$\left(\sum_0^k |a_j|^2\right)^{1/2} \ge d\|f\|_\infty, \tag{10}$$

and satisfies $\|f\|_\infty = 1$, then

$$\int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi} \ge \max_{0\le r<1} \frac{1+r}{1-r} \log(dr^k). \tag{11}$$

Proof. For every r,

$$M_f(r) \ge \left(\sum_0^\infty |a_j|^2 r^{2j}\right)^{1/2} \ge \left(\sum_0^k |a_j|^2 r^{2j}\right)^{1/2} \ge r^k \left(\sum_0^k |a_j|^2\right)^{1/2} \ge dr^k,$$

and the conclusion follows.

The function

$$\phi_{d,k}(r) = \frac{1+r}{1-r} \log(dr^k)$$

is larger than the function $f_{d,k}(r)$ introduced in (4), so formula (8) improves the estimate given by the author in [2], under assumption (10). However, the result in [2] held under the more general assumption (2).

Theorem 1 is especially suitable if the assumption on f is given by a formula of the sort

$$\left(\sum_0^\infty |a_j|^2 w^{2j}\right)^{1/2} \ge d\|f\|_\infty, \tag{12}$$

for a fixed w, $0 < w < 1$. Indeed, since $M_f(w) \geq d\|f\|_\infty$, we get:

COROLLARY 3. *Assume that $f \in H^\infty$ satisfies* (12) *and* $\|f\|_\infty = 1$. *Then*

$$\int_0^{2\pi} \log\left|f(e^{i\theta})\right| \frac{d\theta}{2\pi} \geq \frac{1+w}{1-w} \log d.$$

Condition (12) is indeed a way of measuring concentration at low degrees. If (10) holds, with $\|f\|_\infty = 1$, we have

$$\left(\sum_0^\infty |a_j|^2 w^{2j}\right)^{1/2} \geq \left(\sum_0^k |a_j|^2 w^{2j}\right)^{1/2} \geq w^k \left(\sum_0^k |a_j|^2\right)^{1/2} \geq dw^k,$$

which gives (12) with dw^k instead of d.

If (12) holds, with $\|f\|_\infty = 1$, we let $\sigma_k = \Sigma_0^k |a_j|^2$, $\sigma'_k = \Sigma_{k+1}^\infty |a_j|^2$, and we obtain

$$\sigma_k + w^{2(k+1)}\sigma'_k \geq d^2,$$

and since $\sigma_k + \sigma'_k \leq 1$, we find

$$\sigma_k \geq \frac{d^2 - w^{2(k+1)}}{1 - w^{2(k+1)}},$$

which is for instance $\geq d^2/4$ if w is correctly chosen.

We finally make two remarks on the function

$$\phi_f(r) = \frac{1+r}{1-r} \log M_f(r)$$

occurring in Theorem 1 for $f \in H^\infty$ of norm 1.

First, for $f \in H^\infty$ with $\|f\|_\infty = 1$, $M_f(r) \to 1$ when $r \to 1^-$. Therefore, when $r \to 1^-$,

$$\frac{\log M_f(r)}{1-r} \sim \frac{M_f(r) - 1}{1-r},$$

and the existence of the limit

$$\lim_{r \to 1^-} \frac{\log M_f(r)}{1-r}$$

is equivalent to that of the left derivative of $M_f(r)$, at $r = 1^-$. This derivative exists for any polynomial, and, more generally, for any function $f(z)$ such that the derivative $f'(z)$ has absolutely convergent Fourier series. However, it does not necessarily exist for any function with absolutely convergent Fourier series, as the example of

$$f(z) = \frac{6}{\pi^2} \sum_1^\infty \frac{1}{n^2} z^n$$

shows.

For some functions f, the r for which ϕ_f attains its maximum is unique (for instance, Theorem 1 shows that for an outer function, the maximum is attained at the only point $r = 0$), but this is not true in general: take

$$f(z) = \exp\left\{ -\frac{1+z}{1-z} \right\}$$

(which is a singular function) then

$$M_f(r) = \exp\left\{ -\frac{1-r}{1+r} \right\},$$

and $\phi_f(r) = -1$ for all r: the function ϕ_f is constant.

It might be interesting to have a classification of functions f in H^∞, in terms of properties or ϕ_f.

Theorem 1 has an obvious consequence in terms of Mahler's measure:

COROLLARY 4. *Let f be a function in H^∞ with $\|f\|_\infty = 1$. Then for every r, $0 \le r \le 1$,*

$$\max_{|z|=r} |f(z)| \le M(f)^{(1-r)/(1+r)},$$

where $M(f)$ is Mahler's measure of f. This estimate is best possible.

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