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LOWER BOUNDS FOR THE MEAN CURVATURE OF HOLLOW TUBES AROUND COMPLEX HYPERSURFACES AND TOTALLY REAL SUBMANIFOLDS

VICENTE MIQUEL[1] AND VICENTE PALMER[1]

1. Introduction

In this paper we get two comparison theorems for the mean curvature of a tubular hypersurface around a complex hypersurface and around a totally real submanifold in a Kähler manifold. The models for these comparisons are tubular hypersurfaces around the complex hyperquadric and the real projective space embedded in a complex projective space. Getting comparisons for the mean curvature of tubular hypersurfaces around submanifolds P of Riemannian manifolds M has become a useful tool to get bounds on several geometric invariants related to the volume and the Laplace operator of the ambient manifold (see for instance the books [Ch], [Gr3], the paper [MP2] and the references therein). In fact we shall apply our results to get comparison theorems for the relative volume, the first Dirichlet eigenvalue and the mean exit time.

This note is also a continuation of our paper [MP2], where we have got comparison theorems taking as a model tubular hypersurfaces around the complex projective space embedded as a totally geodesic submanifold in a complex projective space.

The plan of the paper is the following. In Section 2 we set up some notation and recall some definitions and known facts that we shall need later. Sections 3 and 4 are devoted to prove the comparison theorems. The main results are Theorems 3.1 and 4.1. In Section 5 we show the applications to the relative volume, the first Dirichlet eigenvalue and the mean exit time (Theorems 5.1 to 5.5). Finally in Section 6 we discuss what happens when equality is attained in the theorems of Sections 4 and 5. Here we get only partial results: we prove that equality implies many properties (Theorems 6.2 and 6.4) on the tubular hypersurfaces, but we have been unable to see if it characterizes the model spaces.

By doing the proofs of Theorems 3.1 and 4.1, we also answer partially a technical question stated in [MP2] (see also the introduction of the book

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[Gr3]). It is about the equivalence of Jacobi and Riccati equation methods to get comparison theorems. Although these methods are theoretically equivalent, we were compelled in [MP2] to use Jacobi equation method to get results with weaker hypothesis on the bounds of curvatures (instead of taking bounds on some curvatures, we consider bounds on the sum of them). In the proof of Theorems 3.1 and 4.1 we have developed a Riccati equation method that gives results as general as Jacobi equation method. The answer is partial because it does not give all possible results.

2. Notation and background

From now on, M will denote a connected, complete, Kähler manifold of real dimension $2n$, with riemannian metric $\langle\ ,\ \rangle$, and almost complex structure J. By P we shall denote a connected closed complex hypersurface or a totally real submanifold (of dimension n) of M. We shall denote by P_r the tube of radius r around P, and by ∂P_r its boundary, i.e., the tubular hypersurface of radius r around P. For the curvature and the Riemann Christoffel tensor R of M we shall adopt the following convention sign

$$R(X,Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z \text{ and } R_{XYZW} = \langle R(X,Y)Z, W\rangle.$$

Given a point $p \in M$, a vector $X \in T_pM$ and a totally real subspace Π of T_pM of real dimension $n-1$ and orthogonal to X and JX, the *totally real ricci curvatures* $K_r(X,\Pi)$ and $K_c(X,\Pi)$ of X at Π are defined by

$$K_r(X,\Pi) = \sum_{i=1}^{n-1} R_{Xe_iXe_i} \quad \text{and} \quad K_c(X,\Pi) = \sum_{i=1}^{n-1} R_{XJe_iXJe_i},$$

where $\{e_1, e_2, \ldots, e_{n-1}\}$ is an orthonormal basis of Π. These curvatures depend only on X and Π, and satisfy $K_r(X,\Pi) + K_c(X,\Pi) + K_H(X)|X|^2 = \rho(X,X)$, where $\rho(X,X)$ is the Ricci curvature of M and $K_H(X)$ is the holomorphic sectional curvature of the plane generated by X and JX.

We shall denote by $(\mathscr{SN}P)\mathscr{N}P$ the (unit) normal bundle of P in M, and by $\mathscr{N}_pP$ (resp. $\mathscr{SN}_pP$) the fibre of $\mathscr{N}P$ (resp. $\mathscr{SN}P$) over $p \in P$. Let $\mathscr{N}P(t)$ denote the set $\{\zeta \in \mathscr{N}P;\ |\zeta| = t\}$.

For every $N \in \mathscr{SN}P$, L_N will denote the Weingarten map of P associated to N.

If $\mathscr{B}$ is a subset of TM, $\exp_{\mathscr{B}}$ will denote the restriction of the exponential map to $\mathscr{B}$.

Given any fibre bundle B on P and $p \in P$, B_p will denote the fibre of B over $p \in P$.

We shall use $'$ to denote indistinctly the ordinary and the covariant derivative. Its exact meaning will be clear from the context.

Given $p \in P$ and $N \in \mathcal{S}\mathcal{N}_p P$, let $\gamma_N(t)$ be the geodesic such that $\gamma_N(0) = p$ and $\gamma_N'(0) = N$. Let

$$f(N) = \inf\{t > 0/\operatorname{rank} \exp_{\mathcal{N}P(t)*tN} < 2n - 1\}.$$

For every $t \in]0, f(N)[$, $S(t)$ will denote the Weingarten map of the tubular hypersurface of radius t about P, with respect to the unit normal vector $\gamma_N'(t)$. $S(t)$ satisfies the Riccati differential equation ([Gr1, Lemma 4.1] or [Gr3, page 37])

$$S'(t) = S^2(t) + R(t) \tag{2.1}$$

where $S'(t) = \nabla_{\gamma_N'(t)} S(t)$ and $R(t)\colon T_{\gamma_N(t)}\,\partial P_t \to T_{\gamma_N(t)}\,\partial P_t$ is defined by

$$R(t)U = R(\gamma_N'(t), U)\gamma_N'(t) \text{ for every } U \in T_{\gamma_N(t)}\,\partial P_t.$$

Moreover ([Gr1, page 210] or [Gr3, page 38]),

$$\lim_{t \to 0+} S(t) = L_N. \tag{2.2}$$

Given $p \in P$, $N \in \mathcal{S}\mathcal{N}_p P$ and a totally real subspace H of $T_p M$ contained in $T_p P$, orthogonal to JN and of dimension $n - 1$, let us denote by H_t the parallel transport of H along $\gamma_N(t)$. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of H and let $E_i(t)$ be the unit parallel vector fields along $\gamma_N(t)$ such that $E_i(0) = e_i$. Let us observe that

$$H_t = \langle \{E_1(t), \ldots, E_{n-1}(t)\} \rangle,$$

the vector space generated by $\{E_1(t), \ldots, E_{n-1}(t)\}$.

Let $\{Y_i(t)\}_{i=1}^{2n-1}$ be the unit vector fields defined by $Y_i(t) = E_i(t)$ and $Y_{n+i-1} = JE_i$ for $i = 1, \ldots, n-1$, and $Y_{2n-1} = J\gamma_N'(t)$. If we consider the functions

$$f_i(t) = \langle S(t)Y_i(t), Y_i(t) \rangle, \quad i = 1, \ldots, 2n-1,$$

using (2.1) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} f_i'(t) &= \langle S'(t)Y_i(t), Y_i(t) \rangle = \langle S^2(t)Y_i(t) + R(t)Y_i(t), Y_i(t) \rangle \\ &\geq \langle S(t)Y_i(t), Y_i(t) \rangle^2 + \langle R(t)Y_i(t), Y_i(t) \rangle \\ &= f_i^2(t) + \langle R(t)Y_i(t), Y_i(t) \rangle, \quad i = 1, \ldots, 2n-1, \end{aligned}$$

and, using the inequality between the square of the arithmetic mean and the mean of the squares, we have the differential inequalities

(2.3)

$$\left(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\right)' \geq \left(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\right)^2 + \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(t)Y_i(t), Y_i(t)\rangle$$

$$= \left(\frac{1}{n-1}\sum_{i=1}^{n-1} f_i\right)^2 + \frac{1}{n-1}K_r(\gamma_N'(t), H_t),$$

(2.4) $$\left(\frac{1}{n-1}\sum_{i=n}^{2n-2} f_i\right)' \geq \left(\frac{1}{n-1}\sum_{i=n}^{2n-2} f_i\right)^2 + \frac{1}{n-1}K_c(\gamma_N'(t), H_t)$$

and

(2.5) $$f_{2n-1}' \geq f_{2n-1}^2 + K_H(\gamma_N'(t))$$

We shall also need the following result.

2.1 LEMMA ([Gr2, Lemmas 5.1, 5.2] or [Gr3, pp. 174, 175]). *Let $f:]0, t_1[\to \mathbf{R}$ be a differentiable real valued function.*

(i) *Suppose $f' \geq f^2 + \lambda$ on $]0, t_1[$, and $\lim_{t\to 0} f(t) = -\infty$. Then, for $0 < t < t_1$,*

(2.6) $$f(t) \geq \frac{-\sqrt{\lambda}}{\tan(\sqrt{\lambda}\, t)}.$$

(ii) *Suppose $f' \geq f^2 + \lambda$ on $]0, t_1[$, and $\lim_{t\to 0} f(t) = f(0) \in \mathbf{R}$. Then, for $0 < t < t_1$,*

(2.7) $$f(t) \geq \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}\, t) + f(0)\cos(\sqrt{\lambda}\, t)}{\cos(\sqrt{\lambda}\, t) - \frac{f(0)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\, t)},$$

(2.8) $$\cos(\sqrt{\lambda}\, t) - \frac{f(0)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\, t) \geq 0.$$

2.2 LEMMA. *Let F be a C^∞ real function defined on $]a, b[\times\mathbf{R}$ and let $f, g: [a, b[\to \mathbf{R}$ be functions which are C^∞ on $]a, b[$ and continuous on $[a, b[$ satisfying:*

(i) *$f'(x) \geq F(x, f(x))$ and $g'(x) = F(x, g(x))$ for $x \in]a, b[$, and*

(ii) *either $f(a) > g(a)$, or $f(a) = g(a)$ and there exist ε, $\eta > 0$ such that for every $x \in]a, a+\varepsilon[$ the function $y \mapsto f(x, y)$ defined on $]g(a) - \eta, g(a) + \eta[$ is not increasing.*

Then

$$f(x) \geq g(x) \textit{ for all } x \in [a, b[. \tag{2.9}$$

Proof. The proof is like that of Theorem 7 in [BR, page 29].

We end this section by recalling that a P-Jacobi field along a geodesic $\gamma_N(t)$ is a Jacobi field $Z(t)$ such that $Z(0) = 0$ and $Z'(0) \in \mathscr{N}_pP$ or $Z(0) \in T_pP$ and $Z'(0) + L_N Z(0) = 0$. The operator $S(t)$ acting on these Jacobi fields satisfies the equation

$$S(t)Z(t) = -Z'(t), \tag{2.10}$$

where $'$ denotes the covariant derivative respect to $\gamma_N'(t)$ (see [Va, (120)] or [Ka, (1.2.6)], and [CV] for P a point).

3. The comparison theorem for complex hypersurfaces

Along all this section, P will be a complex hypersurface of M. When $M = \mathbf{C}P^n(\lambda)$, the complex projective space of constant holomorphic sectional curvature 4λ, and $P = \mathscr{Q}$, the complex hyperquadric in $\mathbf{C}P^n(\lambda)$, the operator $S(t)$ will be denoted by $\tilde{S}^{\mathscr{Q}}(t)$. Given $\tilde{p} \in \mathscr{Q}$ and $\tilde{N} \in \mathscr{S}\mathscr{N}_{\tilde{p}}\mathscr{Q}$, let $\{\tilde{e}_1, \ldots, \tilde{e}_{n-1}, J\tilde{e}_1, \ldots, J\tilde{e}_{n-1}\}$ be a J-orthonormal basis of $T_{\tilde{p}}\mathscr{Q}$ that diagonalizes the Weingarten map of $\mathscr{Q}$ ($L_{\tilde{N}}\tilde{e}_i = \sqrt{\lambda}\,\tilde{e}_i$ and $L_{\tilde{N}}J\tilde{e}_i = -\sqrt{\lambda}J\tilde{e}_i$). Let $\tilde{\gamma}_{\tilde{N}}(t)$ be the geodesic such that $\tilde{\gamma}_{\tilde{N}}(0) = \tilde{p}$ and $\tilde{\gamma}_{\tilde{N}}'(0) = \tilde{N}$. Then, if $\tilde{E}_i(t)$ are parallel unit vector fields along $\tilde{\gamma}_{\tilde{N}}(t)$ such that $\tilde{E}_i(0) = \tilde{e}_i$, $i = 0, \ldots, n-1$, we have ([Gr2, (4.3)] or [Gr3, page 138])

$$\begin{aligned}
\tilde{S}^{\mathscr{Q}}(t)\tilde{E}_i(t) &= \delta(t)\tilde{E}_i(t), \quad i = 1, \ldots, n-1 \\
\tilde{S}^{\mathscr{Q}}(t)J\tilde{E}_i(t) &= \nu(t)J\tilde{E}_i(t), \quad i = 1, \ldots, n-1 \\
\tilde{S}^{\mathscr{Q}}(t)J\tilde{\gamma}_{\tilde{N}}'(t) &= \eta(t)J\tilde{\gamma}_{\tilde{N}}'(t),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\delta(t) &= \sqrt{\lambda}\,\frac{\sin(\sqrt{\lambda}\,t) + \cos(\sqrt{\lambda}\,t)}{\cos(\sqrt{\lambda}\,t) - \sin(\sqrt{\lambda}\,t)} \\
\nu(t) &= \sqrt{\lambda}\,\frac{\sin(\sqrt{\lambda}\,t) - \cos(\sqrt{\lambda}\,t)}{\cos(\sqrt{\lambda}\,t) + \sin(\sqrt{\lambda}\,t)} = -\delta(-t) \\
\eta(t) &= -2\sqrt{\lambda}\cot(2\sqrt{\lambda}\,t)
\end{aligned}$$

Given $p \in P$ and $N \in \mathscr{S}\mathscr{N}_pP$, let $\{e_1, \ldots, e_{n-1}\}$ be a family of principal vectors with non-negative principal curvatures $\{\kappa_1, \ldots, \kappa_{n-1}\}$, and let $E_i(t)$,

$i = 1, \dots, n-1$ be parallel vectors along $\gamma_N(t)$ such that $E_i(0) = e_i$. We define the *transplanted operator* $S^{\mathscr{Q}} : T_{\gamma_N(t)}\, \partial P_t \mapsto T_{\gamma_N(t)}\, \partial P_t$, of $\tilde{S}^{\mathscr{Q}}(t)$ in (P, M) by

$$\begin{aligned} (3.2) \qquad S^{\mathscr{Q}}(t)E_i(t) &= \delta(t)E_i(t), \quad i = 1, \dots, n-1, \\ S^{\mathscr{Q}}(t)JE_i(t) &= \nu(t)JE_i(t), \quad i = 1, \dots, n-1, \\ S^{\mathscr{Q}}(t)J\gamma_N'(t) &= \eta(t)J\gamma_N'(t). \end{aligned}$$

We shall denote by $\tilde{R}^{\mathscr{Q}}(t)$ the operator $R(t)$ when $(P, M) = (\mathscr{Q}, \mathbf{C}P^n(\lambda))$. We define the transplanted operator $R^{\mathscr{Q}}(t)$ of $\tilde{F}^{\mathscr{Q}}(t)$ to (P, M) along $\gamma_N(t)$ as we did with $S^{\mathscr{Q}}$.

Now, let H be the totally real subspace of T_pM generated by $\{e_1, \dots, e_{n-1}\}$. Let H_t be defined from H as in Section 2.

3.1 THEOREM. *Let M be a Kähler manifold and P a complex hypersurface of M. Let us assume that, for every $p \in P$ and every $N \in \mathscr{SN}_pP$, one has*

$$\rho(\gamma_N'(t), \gamma_N'(t)) \geq (2n+2)\lambda, \; K_r(\gamma_N'(t), H_t) \geq (n-1)\lambda,$$
$$K_c(\gamma_N'(t), H_t) \geq (n-1)\lambda$$

for every $t \in [0, r(N)[$, $r(N) \leq f(N)$, and

$$\frac{1}{n-1}\sum_{i=1}^{n-1} \kappa_i \geq \sqrt{\lambda}.$$

Then

$$\operatorname{tr} S(t) \geq \operatorname{tr} S^{\mathscr{Q}}(t) \quad \textit{for every } t \in [0, r(N)].$$

Proof. First, let us suppose that

$$r(N) \leq \min\left\{\frac{\pi}{4\sqrt{\lambda}}, f(N)\right\}.$$

Let $Y_i(t)_{i=1}^{2n-1}$ be defined from the $\{e_i\}$ as in Section 2. It is obvious that $S^{\mathscr{Q}}$ and $R^{\mathscr{Q}}(t)$ also satisfy equation (2.1). Then, using (2.1), we have

$$\begin{aligned} (3.3) \qquad (S - S^{\mathscr{Q}})' &= S^2 + R - S^{\mathscr{Q}^2} - R^{\mathscr{Q}} \\ &= (S - S^{\mathscr{Q}})^2 + (R - R^{\mathscr{Q}}) + (S - S^{\mathscr{Q}})S^{\mathscr{Q}} + S^{\mathscr{Q}}(S - S^{\mathscr{Q}}). \end{aligned}$$

Thus

$$\begin{aligned}(3.4)\qquad \langle (S - S^{\mathscr{C}})Y_i, Y_i\rangle' &= \langle (S - S^{\mathscr{C}})'Y_i, Y_i\rangle \\ &= \langle (S - S^{\mathscr{C}})^2 Y_i, Y_i\rangle + \langle (R - R^{\mathscr{C}})Y_i, Y_i\rangle \\ &\quad + 2\langle (S - S^{\mathscr{C}})Y_i, S^{\mathscr{C}}Y_i\rangle\end{aligned}$$

for $i = 1, \dots, 2n - 1$. Therefore

$$\begin{aligned}&\left(\sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{C}})Y_i, Y_i\rangle\right)' \\ &\quad = \sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{C}})^2 Y_i, Y_i\rangle \\ &\qquad + \sum_{i=1}^{2n-1}\langle (R - R^{\mathscr{C}})Y_i, Y_i\rangle + 2\delta(t)\sum_{i=1}^{n-1}\langle (S - S^{\mathscr{C}})Y_i, Y_i\rangle \\ &\qquad + 2\nu(t)\sum_{i=n}^{2n-2}\langle (S - S^{\mathscr{C}})Y_i, Y_i\rangle + 2\eta(t)\langle (S - S^{\mathscr{C}})Y_{2n-1}, S^{\mathscr{C}}Y_{2n-1}\rangle \\ &\quad = \sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{C}})^2 Y_i, Y_i\rangle + \sum_{i=1}^{2n-1}\langle (R - R^{\mathscr{C}})Y_i, Y_i\rangle \\ &\qquad + 2\eta(t)\sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{C}})^2 Y_i, Y_i\rangle \\ &\qquad + 2(\delta(t) - \eta(t))\sum_{i=1}^{n-1}\langle (S - S^{\mathscr{C}})Y_i, Y_i\rangle \\ &\qquad + 2(\nu(t) - \eta(t))\sum_{i=n}^{2n-2}\langle (S - S^{\mathscr{C}})Y_i, Y_i\rangle.\end{aligned}$$

Notice that

$$\sum_{i=1}^{2n-1}\langle (R - R^{\mathscr{C}})Y_i, Y_i\rangle \geq 0,$$

because $\rho(\gamma_N'(t), \gamma_N'(t)) \geq (2n + 2)\lambda$.

On the other hand, from the hypotheses on $K_r(\gamma_N'(t), H_t)$, and $K_c(\gamma_N'(t), H_t)$, and the inequalities (2.3) and (2.4), we have

$$\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\langle SY_i, Y_i\rangle\right)' \geq \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\langle SY_i, Y_i\rangle\right)^2 + \lambda,$$

$$\left(\frac{1}{n-1}\sum_{i=n}^{2n-2}\langle SY_i, Y_i\rangle\right)' \geq \left(\frac{1}{n-1}\sum_{i=n}^{2n-2}\langle SY_i, Y_i\rangle\right)^2 + \lambda.$$

Moreover, since P is a complex submanifold, $\{Je_1, \ldots, Je_{n-1}\}$ are eigenvectors of L_N with eigenvalues $\{-\kappa_1, \ldots, -\kappa_{n-1}\}$. Then, using (2.2) and applying Lemma 2.1 (ii) to the above inequalities, we have

$$\frac{1}{n-1}\xi(t) \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\langle SY_i, Y_i\rangle$$

$$\geq \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}t) + \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i\right)\cos(\sqrt{\lambda}t)}{\cos(\sqrt{\lambda}t) - \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i\right)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}} \equiv \alpha(t) \geq 0$$

$$\frac{1}{n-1}\zeta(t) \equiv \frac{1}{n-1}\sum_{i=n}^{2n-2}\langle SY_i, Y_i\rangle$$

$$\geq \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}t) - \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i\right)\cos(\sqrt{\lambda}t)}{\cos(\sqrt{\lambda}t) + \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i\right)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}} \equiv \beta(t).$$

Now, let us observe that from the hypothesis

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i \geq \sqrt{\lambda}$$

it follows that

$$\alpha(t) \geq \delta(t) = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle S^{\mathscr{P}}Y_i, Y_i\rangle(t),$$

and

$$\alpha(t) + \beta(t) = \sqrt{\lambda}\,\frac{\left(\lambda + \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i\right)^2\right)\sin(2\sqrt{\lambda}\,t)}{\lambda\cos^2(\sqrt{\lambda}\,t) - \left(\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i\right)^2\sin^2(\sqrt{\lambda}\,t)}$$
$$\geq \delta(t) + \nu(t) = \sqrt{\lambda}\,\frac{\sin(4\sqrt{\lambda}\,t)}{\cos(2\sqrt{\lambda}\,t)} \geq 0$$

on the interval $[0, \pi/(4\sqrt{\lambda})]$.

Since $\delta(t) \geq \nu(t) \geq 0$ on $[0, \pi/(4\sqrt{\lambda})]$, we have

$$\begin{aligned}\delta(t) - \eta(t) &\geq \nu(t) - \eta(t)\\ &= \sqrt{\lambda}\,\frac{\cos^3(\sqrt{\lambda}\,t) - \sin^3(\sqrt{\lambda}\,t)}{\sin(\sqrt{\lambda}\,t)\cos^2(\sqrt{\lambda}\,t) + \sin^2(\sqrt{\lambda}\,t)\cos(\sqrt{\lambda}\,t)} \geq 0.\end{aligned}$$

Then

$$\begin{aligned}&(\delta(t) - \eta(t))(\xi(t) - (n-1)\delta(t))\\ &\quad + (\nu(t) - \eta(t))(\zeta(t) - (n-1)\nu(t))\\ &\geq (n-1)\{(\delta(t) - \eta(t))(\alpha(t) - \delta(t))\\ &\qquad + (\nu(t) - \eta(t))(\beta(t) - \nu(t))\}\\ &= (n-1)\{(\nu - \eta)(\alpha + \beta - (\delta + \nu))\\ &\qquad + (\delta - \eta - (\nu - \eta))(\alpha + \delta)\} \geq 0.\end{aligned}$$

Therefore we have

$$\begin{aligned}(3.5)\quad &\left(\sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{Q}})Y_i, Y_i\rangle\right)'\\ &\geq \sum_{i=1}^{2n-1}\left\langle (S - S^{\mathscr{Q}})^2 Y_i, Y_i\right\rangle + 2\eta(t)\sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{Q}})Y_i, Y_i\rangle\\ &\geq \sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{Q}})Y_i, Y_i\rangle^2 + 2\eta(t)\sum_{i=1}^{2n-1}\langle (S - S^{\mathscr{Q}})Y_i, Y_i\rangle.\end{aligned}$$

Then

$$\frac{1}{2n-1}\left(\sum_{i=1}^{2n-1}\langle (S-S^{\mathscr{Q}})Y_i, Y_i\rangle\right)'$$
$$\geq \left(\frac{1}{2n-1}\sum_{i=1}^{2n-1}\langle (S-S^{\mathscr{Q}})Y_i, Y_i\rangle\right)^2 + 2\eta(t)\frac{1}{2n-1}\sum_{i=1}^{2n-1}\langle (S-S^{\mathscr{Q}})Y_i, Y_i\rangle.$$

Let

$$b(t) := \frac{1}{2n-1}\sum_{i=1}^{2n-1}\langle (S-S^{\mathscr{Q}})Y_i, Y_i\rangle.$$

Then

$$b(t) = \frac{1}{2n-1}(\operatorname{tr} S(t) - \operatorname{tr} S^{\mathscr{Q}}(t)),$$

and we get the differential inequality

$$b'(t) \geq b^2(t) + 2\eta(t)b(t). \tag{3.6}$$

From (2.2) and the fact that both P and $\mathscr{Q}$ are complex submanifolds (then minimal), it follows that $b(0) = 0$. Then Lemma 2.2 applied to inequality (3.6) gives $b \geq 0$ in $]0, r(N)[$.

Now, let us show that

$$f(N) \leq \frac{\pi}{4\sqrt{\lambda}}.$$

Let $\{Z_i(t)\}_{i=1}^{2n-1}$ be a basis of P-Jacobi fields along $\gamma_N(t)$ such that $|Z_i(0)| = 1$ for $1 \leq i \leq 2n-2$ and $Z_{2n-1}(0) = 0$. From $b(t) \geq 0$ and (2.10), we have

$$\frac{d}{dt}\ln\prod_{i=1}^{2n-1}|Z_i| = \sum_{i=1}^{2n-1}\frac{\langle Z_i', Z_i\rangle}{|Z_i|^2} = -\sum_{i=1}^{2n-1}\left\langle S(t)\frac{Z_i}{|Z_i|}, \frac{Z_i}{|Z_i|}\right\rangle \tag{3.7}$$
$$= -\operatorname{tr} S(t) \leq -\operatorname{tr} S^{\mathscr{Q}}(t) = \frac{d}{dt}\ln a(t),$$

where

$$a(t) = \left(\cos(\sqrt{\lambda}t) - \sin(\sqrt{\lambda}t)\right)^{n-1}\left(\cos(\sqrt{\lambda}t) + \sin(\sqrt{\lambda}t)\right)^{n-1}\sin(2\sqrt{\lambda}t).$$

From the initial conditions on $|Z_i|$, we have $\lim_{t\to 0}((\prod_{i=1}^{2n-1}|Z_i(t)|)/a(t)) =$

1, which, together with (3.7), shows that

$$\prod_{i=1}^{2n-1} |Z_i(t)| \le a(t) \quad \text{for every } t \in \min\left\{f(N), \frac{\pi}{4\sqrt{\lambda}}\right\}.$$

Since $f(N)$ and $\pi/(4\sqrt{\lambda})$ are, respectively, the first zero of $\prod_{i=1}^{2n-1}|Z_i(t)|$ and of $a(t)$, we have $f(N) \le \pi/(4\sqrt{\lambda})$, and the proof is finished.

4. The comparison theorem for totally real submanifolds

In this section P will be a totally real submanifold of M of dimension n.

When $(P, M) = (\mathbf{R}P^n, \mathbf{C}P^n(\lambda))$, with $\mathbf{R}P^n$ embedded as a totally geodesic submanifold in $\mathbf{C}P^n(\lambda)$, the operators $S(t)$ and $R(t)$ will be denoted by $\tilde{S}^{\mathbf{R}P^n}(t)$ and $\tilde{R}^{\mathbf{R}P^n}(t)$ respectively.

Given $\tilde{p} \in \mathbf{R}P^n$ and $\tilde{N} \in \mathscr{S}\mathscr{N}_{\tilde{p}}\mathbf{R}P^n$, let $\{J\tilde{N}, \tilde{e}_1, \dots, \tilde{e}_{n-1}\}$ be an orthonormal basis of $T_{\tilde{p}}\mathbf{R}P^n$. Let $\tilde{\gamma}_{\tilde{N}}(t)$ be the geodesic such that $\tilde{\gamma}_{\tilde{N}}(0) = \tilde{p}$ and $\tilde{\gamma}'_{\tilde{N}}(0) = \tilde{N} = -J\tilde{e}_{2n-1}$. Then, if $\tilde{E}_i(t)$ are parallel unit vector fields along $\tilde{\gamma}_{\tilde{N}}(t)$ such that $\tilde{E}_i(0) = \tilde{e}_i$, $i = 1, \dots, n-1$, we have (see [CR])

$$\begin{aligned}
(4.1) \qquad \tilde{S}^{\mathbf{R}P^n}(t)\tilde{E}_i(t) &= \sqrt{\lambda}\tan(\sqrt{\lambda}t)\tilde{E}_i(t), \qquad i = 1, \dots, n-1,\\
\tilde{S}^{\mathbf{R}P^n}(t)J\tilde{E}_i(t) &= -\sqrt{\lambda}\cot(\sqrt{\lambda}t)J\tilde{E}_i(t), \quad i = 1, \dots, n-1,\\
\tilde{S}^{\mathbf{R}P^n}(t)J\tilde{\gamma}'_{\tilde{N}}(t) &= 2\sqrt{\lambda}\tan(2\sqrt{\lambda}t)J\tilde{\gamma}'_{\tilde{N}}(t).
\end{aligned}$$

Given $p \in P$ and $N \in \mathscr{S}\mathscr{N}_p P$, let $E_i(t)$, $i = 1, \dots, n-1$ be defined from an orthonormal basis $\{JN, e_1, \dots, e_{n-1}\}$ of T_pP as before. Then, we define the transplanted operators

$$S^{\mathbf{R}P^n}(t), R^{\mathbf{R}P^n}(t) : T_{\gamma_N(t)}\,\partial P_t \to T_{\gamma_N(t)}\,\partial P_t,$$

of $\tilde{S}^{\mathbf{R}P^n}(t)$, $\tilde{R}^{\mathbf{R}P^n}(t)$ to (P, M) as we did with $S^{\mathscr{C}}$.

We shall define the $Y_i(t)$ from the $E_i(t)$ as in section 2.

Let $h, k : \mathscr{S}\mathscr{N}P \to \mathbf{R}$ be the functions defined by

$$h(N) = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle L_N e_i, e_i\rangle \quad \text{and} \quad k(N) = \langle L_N JN, JN\rangle$$

We shall also define the operator

$$S_P^{\mathbf{R}}(t) : T_{\gamma_N(t)}\,\partial P_t \to T_{\gamma_N(t)}\,\partial P_t$$

by

$$
\begin{aligned}
(4.2)\qquad S_P^{\mathbf{R}}(t)E_i(t) &= \sqrt{\lambda}\,\omega(t)E_i(t), & i &= 1,\dots,n-1,\\
S_P^{\mathbf{R}}(t)JE_i(t) &= -\sqrt{\lambda}\cot(\sqrt{\lambda}t)JE_i(t), & i &= 1,\dots,n-1,\\
S_P^{\mathbf{R}}(t)J\gamma_N'(t) &= 2\sqrt{\lambda}\,\sigma(t)J\gamma_N'(t), & &
\end{aligned}
$$

where

$$\omega(t) = \frac{\sqrt{\lambda}\sin(\sqrt{\lambda}t) + h(N)\cos(\sqrt{\lambda}t)}{\sqrt{\lambda}\cos(\sqrt{\lambda}t) - h(N)\sin(\sqrt{\lambda}t)}$$

and

$$\sigma(t) = \frac{2\sqrt{\lambda}\sin(2\sqrt{\lambda}t) + k(N)\cos(2\sqrt{\lambda}t)}{2\sqrt{\lambda}\cos(2\sqrt{\lambda}t) - k(N)\sin(2\sqrt{\lambda}t)}.$$

Here H will denote the subspace of T_pP orthogonal to JN, and, then, H_t will the totally real subspace of $T_{\gamma_N(t)}M$ generated by $\{Y_1(t),\dots,Y_{n-1}(t)\}$ i.e. the parallel transport along $\gamma_N'(t)$ of the subspace of T_pP generated by $\{e_1,\dots,e_{n-1}\}$.

4.1 THEOREM. *Let M be a Kähler manifold and P a totally real submanifold of dimension n in M. Let us assume that, for every $p \in P$ and $N \in \mathcal{SN}_pP$,*

$$
\begin{gathered}
\rho(\gamma_N'(t),\gamma_N'(t)) \geq (2n+2)\lambda,\\
K_r(\gamma_N'(t),H_t) \geq (n-1)\lambda \quad \textit{and} \quad K_H(\gamma_N'(t)) \geq 4\lambda
\end{gathered}
$$

for every $t \in [0, r(N)]$, with $r(N) \leq f(N)$. Then

$$\operatorname{tr} S(t) \geq \operatorname{tr} S_P^{\mathbf{R}}(t) \quad \textit{for every } t \in]0, r(N)].$$

Proof. A straightforward computation shows that the operators $S_P^{\mathbf{R}}(t)$ and $R^{\mathbf{R}P^n}(t)$ also satisfy equation (2.1). From (2.1), it follows that formula (3.4) is also valid when we replace $S^{\mathscr{C}}$ and $F^{\mathscr{C}}$ by $S^{\mathbf{R}P^n}$ and $R^{\mathbf{R}P^n}$ respectively. Using this formula and the fact that

$$\sum_{i=1}^{2n-1} \langle (R - R^{\mathbf{R}P^n})Y_i, Y_i\rangle \geq 0$$

(which follows from the hypothesis $\rho(\gamma_N'(t), \gamma_N'(t)) \geq (2n+2)\lambda$) we have

$$
\begin{aligned}
(4.3)\quad &\left(\sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle'\right.\\
&= \sum_{i=1}^{2n-1} \left\langle (S - S_P^{\mathbf{R}})^2 Y_i, Y_i\right\rangle + 2\sqrt{\lambda}\,\omega(t) \sum_{i=1}^{n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle \\
&\quad - 2\sqrt{\lambda}\cot(\sqrt{\lambda}t) \sum_{i=n}^{2n-2} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle \\
&\quad + 4\sqrt{\lambda}\,\sigma(t)\langle (S - S_P^{\mathbf{R}})Y_{2n-1}, Y_{2n-1}\rangle \\
&\geq \sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle^2 - 2\sqrt{\lambda}\cot(\sqrt{\lambda}t) \sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle \\
&\quad + 2\sqrt{\lambda}\left(\omega(t) + \cot(\sqrt{\lambda}t)\right) \sum_{i=1}^{n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle \\
&\quad + 2\sqrt{\lambda}\left(2\sigma(t) + \cot(\sqrt{\lambda}t)\right)\langle (S - S_P^{\mathbf{R}})Y_{2n-1}, Y_{2n-1}\rangle
\end{aligned}
$$

Then, using inequalities (2.3) and (2.5), the hypotheses on the curvatures K_r and K_H, and Lemma 2.1, we have

$$
\begin{aligned}
(4.4)\quad &\frac{1}{2n-1}\left(\sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle\right)' \\
&\geq \left(\frac{1}{2n-1}\sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle\right)^2 \\
&\quad - 2\sqrt{\lambda}\cot(\sqrt{\lambda}t)\frac{1}{2n-1}\sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle
\end{aligned}
$$

If we put

$$
b = \frac{1}{2n-1}\sum_{i=1}^{2n-1} \langle (S - S_P^{\mathbf{R}})Y_i, Y_i\rangle,
$$

inequality (4.4) becomes

$$
b' \geq b^2 - 2\sqrt{\lambda}\cot(\sqrt{\lambda}t)b
$$

and the proof follows as for Theorem 3.1.

4.2 COROLLARY. *Let M and P be as in Theorem* 4.1, *but with P totally geodesic in M. Then*

$$\operatorname{tr} S(t) \geq \operatorname{tr} S^{\mathbf{R}P^n}(t) \quad \text{for every } t \in]0, r(N)].$$

5. Comparison theorems for the relative volume, the mean exit time and the first Dirichlet eigenvalue

In this section we give some applications of Theorems 3.1 and 4.1. We shall state them without giving the proof, and only indicate the references where the necessary arguments can be found.

For every $p \in P$, $N \in \mathscr{S}\mathscr{N}_p P$, let $c(N) = \sup\{t > 0; \text{distance}(P, \gamma_N(t)) = t\}$.

From Theorems 3.1 and 4.1, using arguments like those in [Gi, Theorem 3.3] or [Gr3, pp. 91–92], one gets:

5.1 THEOREM. (a) *Let M and P be as in Theorem* 3.1, *with the hypothesis on the bounds of the curvatures valid for $t \in [0, c(N)[$, then we have*

$$\frac{\text{volume}(P)}{\text{volume}(M)} \geq \frac{\text{volume}(\mathscr{C})}{\text{volume}(\mathbf{C}P^n(\lambda))}.$$

(b) *Let M and P be as in Corollary* 4.2, *with the hypothesis on the bounds of the curvatures value for $t \in [0, c(N)[$, then we have*

$$\frac{\text{volume}(P)}{\text{volume}(M)} \geq \frac{\text{volume}(\mathbf{R}P^n(\lambda))}{\text{volume}(\mathbf{C}P^n(\lambda))}.$$

In the next remark, $[(n-1)/2]$ will denote the integer part of the number $(n-1)/2$, ε will be a number defined by

$$\varepsilon = \begin{cases} 0, & \text{if } n-1 \text{ odd}, \\ -1, & \text{if } n-1 \text{ even}. \end{cases}$$

We shall also use the function

$$\varphi(t, \alpha, \beta) = \cos(2\sqrt{\lambda}\, t) \sum_{j=0}^{[(n-1)/2]} \binom{n-1}{2j} \left(\cos(\sqrt{\lambda}\, t)\right)^{n-1-2j} \alpha^{2j} \left(\frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}}\right)^{2j}$$
$$+ \frac{\sin(2\sqrt{\lambda}\, t)}{2\sqrt{\lambda}} \beta \sum_{j=0}^{[(n-1)/2]+\varepsilon} \binom{n-1}{2j+1} \left(\cos(\sqrt{\lambda}\, t)\right)^{n-1-2j-1} \alpha^{2j+1} \left(\frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}}\right)^{2j+1}$$

and by $z^+(\varphi)$ we shall mean the first positive zero of the function $\varphi(t, \alpha, \beta)$.

5.2 *Remark.* If we do not have the condition that P is totally geodesic in part (b) of Theorem 5.1 and assume that α is an upper bound of $|h|$ and β is an upper bound of $|k|$, we get the inequality

$$\text{volume}(M) \leq g(\alpha, \beta)\text{volume}(P),$$

where

$$g(\alpha, \beta) = \text{volume}(S^{n-1}) \int_0^{z^+(\varphi)} \varphi(t, \alpha, \beta) \left(\frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} \right)^{n-1} dt,$$

But now, unlike Theorem 5.1, there is no totally real submanifold $\mathscr{M}$ of $\mathbf{C}P^n(\lambda)$ such that $g(\alpha, \beta)$ is the quotient volume($\mathbf{C}P^n(\lambda)$)/volume($\mathscr{M}$), because this would imply that the tube around $\mathscr{M}$ has three different constant principal curvatures, and then $\mathscr{M}$ has to be $\mathbf{R}P^n(\lambda)$ (see [CR, Th.4] and [Ki, Prop. 3.4]).

Let $\mu_r^{\mathscr{Q}}$ (respectively $\mu_r^{\mathbf{R}P^n}$) be the first eigenvalue of the Dirichlet problem

$$\Delta f = \mu f, \; f|_{\partial \mathscr{Q}_r} = 0 \quad (\text{respectively } f|_{\partial \mathbf{R}P_r^n} = 0)$$

Let $\mu_1(\Omega)$ be the first eigenvalue of the first Dirichlet eigenvalue problem on any domain $\Omega \subset M$.

Let $c(P) = \inf\{c(N);\ N \in \mathscr{SNP}\}$. From now on $r \in [0, c(P)[$.

From Theorems 3.1 and 4.1, using arguments like those in [MP2, Theorems 4.4 and 4.8] and in [Le], having account that $\mathbf{R}P^n(\lambda)$ is the set of cut points of $\mathscr{Q}$ in $\mathbf{C}P^n(\lambda)$ at distance $\pi/(4\sqrt{\lambda})$, one gets:

5.3 THEOREM. (a) *Let P and M be as in* 5.1(a). *Then*

$$\mu_1(P_r) \leq \mu_r^{\mathscr{Q}} \quad \textit{and} \quad \mu_1(M - P_r) \geq \mu^{\mathbf{R}P^n}_{(\pi/4\sqrt{\lambda})-r}.$$

(b) *Let P and M be as in* 5.1(b). *Then*

$$\mu_1(P_r) \leq \mu_r^{\mathbf{R}P^n} \quad \textit{and} \quad \mu_1(M - P_r) \geq \mu^{\mathscr{Q}}_{(\pi/4\sqrt{\lambda})-r}.$$

Let E_r^P, F_r, $E_r^{\mathscr{Q}}$ and $E_r^{\mathbf{R}}$ denote the mean exit time functions from P_r, $M - P_r$, $\mathscr{Q}_r$ and $\mathbf{R}P_r^n$ respectively. From Theorems 3.1 and 4.1, using arguments like those in [MP2, Theorems 4.2 and 4.8] and having account that $\mathbf{R}P^n(\lambda)$ is the set of cut points of $\mathscr{Q}$ in $\mathbf{C}P^n(\lambda)$ at distance $\pi/(4\sqrt{\lambda})$, one gets:

5.4 THEOREM. (a) *Let P and M be as in* 5.1(a). *Let $\mathscr{E}_r^P : P_r \to \mathbf{R}$ be the function defined by*

$$\mathscr{E}_r^P(x) = E_r^{\mathscr{Q}}(d(P, x))$$

and let $\mathscr{F}_r : M - P_r \to \mathbf{R}$ *be the function defined by*

$$\mathscr{F}_r(x) = E^{\mathbf{R}}_{(\pi/4\sqrt{\lambda})-r}\left(\frac{\pi}{4\sqrt{\lambda}} - r - d(\partial P_r, x)\right).$$

Then

$$\mathscr{E}^P_r(x) \le E^P_r(x) \quad and \quad F_r(x) \le \mathscr{F}_r(x).$$

(b) *Let P and M be as in* 5.1(b). *Let* $\mathscr{E}^P_r : P_r \to \mathbf{R}$ *be the function defined by*

$$\mathscr{E}^P_r(x) = E^{\mathbf{R}}_r(d(P, x))$$

and let $\mathscr{F}_r : M - P_r \to \mathbf{R}$ *be the function defined by*

$$\mathscr{F}_r(x) = E^{\mathscr{C}}_{(\pi/4\sqrt{\lambda})-r}\left(\frac{\pi}{4\sqrt{\lambda}} - r - d(\partial P_r, x)\right).$$

Then

$$\mathscr{E}^P_r(x) \le E^P_r(x) \quad and \quad F_r(x) \le \mathscr{F}_r(x).$$

As in the comparison for the relative volume, it is still possible to get bounds for $\mu_1(P_r)$ and E^P_r in part (b) of Theorems 5.3 and 5.4 without the condition that P is totally geodesic, although in this case the bounds are not the eigenvalues nor the mean exit time function of any tube around a totally real submanifold of $\mathbf{C}P^n(\lambda)$, as we describe now.

Let α and β be, as above, the upper bounds of $|h|$ and $|k|$ respectively. Let $\omega(\alpha, t)$ (resp. $\sigma(\beta, t)$) be the function defined like $\omega(t)$ (resp. $\sigma(t)$) in Section 4, but changing $h(N)$ (resp. $k(N)$) by $-\alpha$ (resp. $-\beta$).

Let $\mu_{\alpha,\beta}$ be the first eigenvalue of the following Dirichlet eigenvalue problem on $[-r, r]$:

$$\begin{aligned} -f''(t) + \{(n-1)\sqrt{\lambda}\,\omega(\alpha, t) - (n-1)\sqrt{\lambda}\cot(\sqrt{\lambda}\,t) \\ +2\sqrt{\lambda}\,\sigma(\beta, t)\}f'(t) = \mu f(t), \\ f(-r) = f(r) = 0. \end{aligned}$$

Let $\mathscr{E}_{\alpha,\beta}$ be the solution of the following Poisson equation with Dirichlet condition:

$$\begin{aligned} -\mathscr{E}''_{\alpha,\beta}(t) + \{(n-1)\sqrt{\lambda}\,\omega(\alpha, t) - (n-1)\sqrt{\lambda}\cot(\sqrt{\lambda}\,t) \\ +2\sqrt{\lambda}\,\sigma(\beta, t)\}\mathscr{E}'_{\alpha,\beta}(t) = 1, \\ \mathscr{E}_{\alpha,\beta}(-r) = \mathscr{E}_{\alpha,\beta}(r) = 0. \end{aligned}$$

The same methods used to prove Theorems 5.3(b) and 5.4(b) allow us to get the following result from Theorem 4.1.

5.5 THEOREM. *Let P and M be as in* 5.1(b), *but with P not totally geodesic in M. Then we have*

$$\mu_1(P_r) \leq \mu_{\alpha,\beta} \quad \text{and} \quad \mathscr{E}_{\alpha,\beta}(x) \leq E_r^P(x).$$

6. Some remarks about the equality case

In this section we shall discuss what happens when equality is attained in Theorem 3.1 or Corollary 4.2. We think that it must characterize the pairs $(\mathscr{C}, \mathbf{C}P^n)$ and $(\mathbf{R}P^n, \mathbf{C}P^n)$, but we have only got partial results, which are contained in Theorems 6.2 and 6.3 below.

We shall use the following well known technical lemma. A proof of it can be found, for instance, in [MP2].

6.1 LEMMA. *For every $(p, N) \in \mathscr{S}\mathscr{N}P$ and every $r \in \mathbf{R}$, the kernel of $\exp_{\mathscr{N}P*(p,rN)}$ is the set of vectors $(c'(0), r\xi'(0)) \in T_{(p,rN)}\mathscr{N}P$ tangent to curves $(c(s), r\xi(s))$ in $\mathscr{N}P$ with $(c(0), \xi(0)) = (p, N)$ and such that the P-Jacobi field $Y(t)$ along $\gamma_N(t)$ satisfying $Y(0) = c'(0)$ and $Y'(0) = (\nabla/dt)\xi(0)$ also satisfies $Y(r) = 0$.*

6.2 THEOREM. *Let P and M be as in* 3.1. *If $\operatorname{tr} S(t) = \operatorname{tr} S^{\mathscr{C}}(t)$ for every $p \in P$, $N \in \mathscr{S}\mathscr{N}_pP$ and $t \in [0, f(N)[$.*

(a) *The principal curvatures of P have only the values $\sqrt{\lambda}$ and $-\sqrt{\lambda}$.*

(b) *On the open sets of ∂P_t where ∂P_t is a regular submanifold of M, the distributions H_t and JH_t are integrable, and satisfy*:

(b.i) *The distribution H_t is the one defined by the eigenspace of $S(t)$ of eigenvalue $\delta(t)$. Its leaves have constant sectional curvature*

$$\frac{2\lambda}{1-\sin(2\sqrt{\lambda}\,t)}.$$

(b.ii) *The distribution JH_t is the one defined by the eigenspace of $S(t)$ of eigenvalue $\nu(t)$. Its leaves have constant sectional curvature*

$$\frac{2\lambda}{1+\sin(2\sqrt{\lambda}\,t)}.$$

(b.iii) *The leaves of the distribution $\mathscr{N}$ defined as the orthogonal complement of $H_t \oplus JH_t$ in $T\partial P_t$ are geodesics in ∂P_t.*

(c) *For $r < c(P)$, the leaves of H_t and JH_t are compact and the leaves of $\mathscr{N}$ are closed geodesics.*

Proof. From the proof of 3.1 it follows that the equality $\operatorname{tr} S(t) = \operatorname{tr} S^{\mathscr{Q}}(t)$ implies that $S(t) = S^{\mathscr{Q}}(t)$ and $R(t) = R^{\mathscr{Q}}(t)$. Then (2.2) implies that $L_N = L_N^{\mathscr{Q}}$, where $L_N^{\mathscr{Q}}$ denotes the Weingarten map of $\mathscr{Q}$ in $\mathbf{C}P^n$ (observe that this means that H is the eigenspace corresponding to the eigenvalue $+\sqrt{\lambda}$ of L_N). This proves (a). From these facts and (2.10), we have that the P-transverse Jacobi fields along the normal geodesic $\gamma_N(t)$ satisfy the same equation and the same initial conditions in (P, M) that in $(\mathscr{Q}, \mathbf{C}P^n(\lambda))$, then a basis of them has the form

$$\begin{aligned} (6.1) \qquad Z_i(t) &= \left(-\sin(\sqrt{\lambda}\,t) + \cos(\sqrt{\lambda}\,t)\right)E_i(t), \quad i = 1, \dots, n-1, \\ Z_{n-1+i}(t) &= \left(\sin(\sqrt{\lambda}\,t) + \cos(\sqrt{\lambda}\,t)\right)JE_i(t), \quad i = 1, \dots, n-1, \\ Z_{2n-1}(t) &= \frac{\sin(2\sqrt{\lambda}\,t)}{2\sqrt{\lambda}} J\gamma_N'(t), \end{aligned}$$

where $\{E_i(t)\}_{i=1}^{n-1}$ are defined as in Section 3. This expression of the P-Jacobi fields implies that $f(N) = \pi/(4\sqrt{\lambda})$ for every $N \in \mathscr{S}\mathscr{N}P$. Let $\phi : \partial P_r \to M$ be defined by

$$\phi(\gamma_N(r)) = \gamma_N\left(\frac{\pi}{4\sqrt{\lambda}}\right).$$

Then, defining

$$\mu : \mathscr{N}P(r) \to \mathscr{N}P\left(\frac{\pi}{4\sqrt{\lambda}}\right)$$

as the canonical isomorphism

$$(p, rN) \mapsto \left(p, \frac{\pi}{4\sqrt{\lambda}} N\right),$$

we have

$$\phi \circ \exp_{\mathscr{N}P(r)} = \exp_{\mathscr{N}P(\frac{\pi}{4\sqrt{\lambda}})} \circ \mu.$$

Then

$$\phi_* \circ \exp_{\mathscr{N}P(r)*} = \exp_{\mathscr{N}P(\frac{\pi}{4\sqrt{\lambda}})*} \circ \mu_*.$$

Using this equality and the fact that exp is a local diffeomorphism, we have

$$(6.2) \qquad \operatorname{Ker} \phi_* = \exp_*\left(\mu_*^{-1}(\operatorname{Ker} \exp_*)\right).$$

To determine $\operatorname{Ker}\exp_{*(p,\pi/4\sqrt{\lambda})}$ we shall use Lemma 6.1. The condition $Z(\pi/4\sqrt{\lambda}) = 0$ for a P-Jacobi field Z, and the formulae (6.1) imply that

$$Z(0) = \sum_{i=1}^{n-1} \alpha_i e_i \in H$$

and

$$Z'(0) = \sum_{i=1}^{i=n-1} \alpha_i Z_i'(0) = -\sqrt{\lambda} \sum_{i=1}^{n-1} \alpha_i e_i = -L_N Z(0).$$

Then the condition $Z'(0) = (\nabla/dt)\xi(0)$ is equivalent to $\nabla^{\perp}_{c'(0)}\xi = 0$, and Lemma 6.1 gives

(6.3)
$$\operatorname{Ker}\exp_{*(p,\pi/4\sqrt{\lambda})} = \left\{\left(c'(0), \frac{\pi}{4\sqrt{\lambda}}\xi'(0)\right); c'(0) \in V_+ \text{ and } \nabla^{\perp}_{c'(0)}\xi = 0\right\}$$

From (6.2) and (6.3) it is easy to see that

$$\begin{aligned}\operatorname{Ker}\phi_{*\gamma_N(r)} &= H_r = \langle\{E_1(r), \dots, E_{n-1}(r)\}\rangle \\ &= \left\{X \in T_{\gamma_N(r)}\,\partial P_r; S(r)X = \delta(r)X\right\}.\end{aligned}$$

Obviously $\operatorname{Ker}\phi_*$ defines an integrable foliation on ∂P_r. The sectional curvature of the leaves of this foliation is computed as follows: Let $\mathscr{V}_+^r$ be a leaf of $\operatorname{Ker}\phi_*$ passing through $q \in \partial P_r$. Then $\phi(\mathscr{V}_+^r) = y \in \partial P_{\pi/4\sqrt{\lambda}}$, because $T_q\mathscr{V}_+ = \operatorname{Ker}\phi_{*q}$. Let

$$\Psi : \mathscr{V}_+^r \to S^{n-1} \subset T_y M$$

be defined by

$$\Psi(m) = \gamma'_{N^r(m)}\left(\frac{\pi}{4\sqrt{\lambda}} - r\right),$$

where N^r is the unit vector field normal to the tubular hypersurface ∂P_r pointing outward and $\gamma_{N^r(m)}$ is the geodesic starting from m with tangent vector $N^r(m)$. Then if $f(r) = -\sin(\sqrt{\lambda}r) + \cos(\sqrt{\lambda}r)$, for $i = 1, \dots, n-1$, we have

$$\Psi_{*m}(E_i(r)) = \frac{1}{f(r)}\Psi_{*m}(f(r)E_i(r)) = \frac{1}{f(r)}\Psi_{*m}(Z_i(r)),$$

where the $Z_i(t)$ are the P-Jacobi fields along $\gamma_N(t)$ (with $N = \gamma'_{N^r(m)}(-r)$) defined by (6.1). Then

$$Z_i(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{c_i(s)} tN(c_i(s)),$$

where $c_i(s)$ is a curve in P such that $c_i(0) = \gamma_{N^r(m)}(-r)$ and $c_i'(0) = e_i$. Therefore

$$\begin{aligned} \Psi_{*m}(E_i(r)) &= \frac{1}{f(r)} \Psi_{*m}\left(\frac{\partial}{\partial s}\Big|_{s=0} \exp_{c(s)} tN(c(s))\right) \\ &= \frac{1}{f(r)} \frac{\partial}{\partial s}\Big|_{s=0} \left(\Psi\left(\exp_{c(s)} rN(c(s))\right)\right) \\ &= \frac{-\sqrt{2\lambda}}{f(r)} E_i\left(\frac{\pi}{4\sqrt{\lambda}}\right). \end{aligned}$$

Let g be the standard metric on $S^{n-1} \in T_y M$ and let $\tilde{g}$ be the metric on S^{n-1} for which Ψ_{*m} is an isometry. Then

$$g = \frac{2\lambda}{f^2(r)} \tilde{g}$$

and, therefore the sectional curvature of $\mathscr{V}_+^r$ is

$$\frac{2\lambda}{1 - \sin(2\sqrt{\lambda}\, r)}.$$

This proves (b.i).

In order to prove (b.ii), let us define $\phi^- : \partial P_r \to M$ by

$$\phi^-(\gamma_N(r)) = \gamma_N\left(-\frac{\pi}{4\sqrt{\lambda}}\right) = \gamma_{-N}\left(\frac{\pi}{4\sqrt{\lambda}}\right).$$

Then, by arguments similar to those used before, it is possible to see that

$$\begin{aligned} \operatorname{Ker} \phi^-{}_{*\gamma_N(r)} &= JH_r = \langle \{JE_1(r), \dots, JE_{n-1}(r)\}\rangle \\ &= \{X \in T_{\gamma_N(r)}\, \partial P_r;\, S(r)X = \nu(r)X\} \end{aligned}$$

and that the leaves of JH_r have sectional curvature

$$\frac{2\lambda}{1 + \sin(2\sqrt{\lambda}\, r)}.$$

Since $T_{\gamma_N(r)}\partial P_r = H_r \oplus JH_r \oplus \langle\{JN^r\}\rangle$, the foliation $\mathcal{N}$ on ∂P_r is given by $m \mapsto JN^r(m)$. Since $L_N = L_N^{\mathcal{Q}}$, if D is the riemannian connection on ∂P_r, the Gauss equation gives $D_{JN^r}JN^r = 0$, and the integral curves of $\mathcal{N}$ are geodesics in ∂P_r. This proves (b.iii).

Now, if $r < c(P)$, ∂P_r is a complete manifold then the leaves of H_r and JH_r are complete (see [Rh, page 143]), then compact, because they have positive sectional curvature.

On the other hand, it is easy to see that $c(s) = \exp_p r(\cos sN + \sin sJN)$, $p \in P$ and $N \in \mathcal{SN}_pP$ are the integral curves of $\mathcal{N}$ if $r < c(P)$, which shows that they are closed.

From this theorem, using known arguments (see [Gi], [Gr3], [MP1, 2] and [GM2]) and the results of Section 5, the following corollary is obvious.

6.3 COROLLARY. *Let us suppose that we have equality in* 5.1(a), *or that* $\mu_1(M - P_r) = \mu^{\mathbf{R}P^n}_{(\pi/4\sqrt{\lambda})-r}$ *in* 5.3(α) *or that* $F_r(x) = \mathcal{F}_r(x)$ *in* 5.4.(a). *Then for every* $p \in P$ *and* $N \in \mathcal{SN}_pP$, *we have* $c(N) = f(N) = \pi/(4\sqrt{\lambda})$ *and the statements* (a), (b) *and* (c) *in Theorem* 6.2 *hold.*

6.4 THEOREM. *Let P and M be as in* 4.2. *If* $\operatorname{tr} S(t) = \operatorname{tr} S^{\mathbf{R}P^n}(t)$ *for every* $p \in P$, $N \in \mathcal{SN}_pP$ *and* $t \in [0, f(N)[$.

(a) *On the open sets of* ∂P_t *where* ∂P_t *is a regular submanifold of* M, *the distributions* H_t *and* JH_t *are integrable, and satisfy*:

(a.i) *The distribution* H_t *is the one defined by the eigenspace of* $S(t)$ *of eigenvalue* $\sqrt{\lambda}\tan(\sqrt{\lambda}\, t)$. *Its leaves have constant sectional curvature*

$$\frac{2\lambda}{1 - \cos(2\sqrt{\lambda}\, t)}.$$

(a.ii) *The distribution* JH_t *is the one defined by the eigenspace of* $S(t)$ *of eigenvalue* $-\sqrt{\lambda}\cot(\sqrt{\lambda}\, t)$. *Its leaves have constant sectional curvature*

$$\frac{2\lambda}{1 + \cos(2\sqrt{\lambda}\, t)}.$$

(a.iii) *The leaves of the distribution* $\mathcal{N}$ *defined as the orthogonal complement of* $H_t \oplus JH_t$ *in* $T\partial P_t$ *are geodesics in* ∂P_t.

(b) *For* $r < c(P)$, *the leaves of* H_t *and* JH_t *are compact and the leaves of* $\mathcal{N}$ *are closed geodesics.*

Proof. From the proof of 4.1 it follows that the equality $\operatorname{tr} S(t) = \operatorname{tr} S^{\mathbf{R}P^n}(t)$ implies that $S(t) = S^{\mathbf{R}P^n}(t)$ and $R(t) = R^{\mathbf{R}P^n}(t)$. From these facts and (2.10), we have that the P-transverse Jacobi fields along the normal geodesic $\gamma_N(t)$ satisfy the same equation and the same initial conditions in (P, M) that in

$(\mathbf{R}P^n, \mathbf{C}P^n(\lambda))$, then a basis of them has the form

$$
\begin{aligned}
(6.4) \qquad Z_i(t) &= \cos(\sqrt{\lambda}\, t) E_i(t), \qquad i = 1, \dots, n-1, \\
Z_{n-1+i}(t) &= \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}} J E_i(t), \quad i = 1, \dots, n-1, \\
Z_{2n-1}(t) &= \cos(2\sqrt{\lambda}\, t) J \gamma_N'(t),
\end{aligned}
$$

where $\{E_i(t)\}_{i=1}^{n-1}$ are defined as in Section 4. This expression of the P-Jacobi fields implies that $f(N) = \pi/(4\sqrt{\lambda})$ for every $N \in \mathscr{S}\mathscr{N}P$. Let $\phi : \partial P_r \to M$ be defined by

$$\phi(\gamma_N(r)) = \gamma_N\left(\frac{\pi}{4\sqrt{\lambda}}\right) = \exp_{\mathscr{N}P}\big((\pi/4\sqrt{\lambda})N\big).$$

The formulae (6.4) imply that $\operatorname{rank} \phi_{*\gamma_N(r)} = \operatorname{rank} \exp_{\mathscr{N}P(\pi/4\sqrt{\lambda})*N} = 2n-2$ for every $N \in \mathscr{S}\mathscr{N}P$. Then, from the constant rank theorem, there is an open set W of ∂P_r containing $\gamma_N(r)$ such that $\phi(W)$ is a complex hypersurface of M. Obviously, W is an open set of $\partial\phi(W)_{(\pi/4\sqrt{\lambda})-r}$ and, in the common points, $H_r = H^{\mathscr{Q}}_{(\pi/4\sqrt{\lambda})-r}$ (where $H^{\mathscr{Q}}_t$ denotes the distribution H_t of Theorem 6.2). Then the Weingarten map of $\phi(W)$ in the direction of $\xi = -\gamma_N'(r)$ is

$$\lim_{t \to (\pi/4\sqrt{\lambda})} -S(t)|_{\langle\{Y_1, \dots, Y_{2n-2}\}\rangle} = L^{\mathscr{Q}}_{\xi}.$$

Then for the points in W, ∂P_r is like $\partial P_{(\pi/4\sqrt{\lambda})-r}$ in the proof of Theorem 6.2, and Theorem 6.e follows from Theorem 6.2 and the fact that $S^{\mathbf{R}P^n}(t) = -S^{\mathscr{Q}}((\pi/4\sqrt{\lambda}) - t)$.

As before, the next result follows from this theorem.

6.5 COROLLARY. *Let us suppose that we have equality in* 5.1(b), *or that* $\mu_1(M - P_r) = \mu^{\mathscr{Q}}_{(\pi/4\sqrt{\lambda})-r}$ *in* 5.3(b) *or that* $F_r(x) = \mathscr{F}_r(x)$ *in* 5.4(b). *Then for every* $p \in P$ *and* $N \in \mathscr{S}\mathscr{N}_pP$, *we have* $c(N) = f(N) = \pi/(4\sqrt{\lambda})$ *and the statements* (a) *and* (b) *in Theorem* 6.4 *hold.*

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