ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 3, Fall 1995

MINIMIZING NORMS OF POLYNOMIALS UNDER CONSTRAINTS ON THE DISTRIBUTION OF THE ROOTS

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Let $P(z) = \Sigma_{j=0}^{n} a_j z^j$ be an univariate polynomial with complex coefficients; we note $\|P\|_\infty = \max_{|z|=1} |P(z)|$. A well-known result of Erdös-Turan (see [6]) asserts that if the ratio

$$\frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}}$$

is not too large, then the roots are uniformly distributed in different angles with vertex at the origin. To be precise, for every α, β with $0 \leq \alpha \leq \beta \leq 2\pi$, if $N_{\alpha,\beta}$ is the number of roots z_j with $\arg z_j \in [\alpha, \beta]$, then

$$\left| N_{\alpha,\beta} - \frac{\beta - \alpha}{2\pi} n \right| \leq c \sqrt{n \log \frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}}}$$

where n is the degree of P.

Erdös-Turan obtained $c = 16$, a value which was improved later by Ganelius (see [7]): $c \sim 2.619$. Taking the polynomial $(z-1)^n$, we see that $c \geq 1/\sqrt{\log 2} \sim 1.201$.

The result of Erdös-Turan concerns the distribution of roots in the whole plane, but it does not yield optimal estimates if specific constraints are laid upon the roots, especially if they are required to lie in a half-plane, or a sector. However, the polynomials whose roots lie in the open half-plane $\{\text{Re } z < 0\}$ play a particularly prominent role in physics; they are called stable polynomials (see [8]). By extension, we call "stable" any polynomial all of whose roots lie in the closed half-plane $\{\text{Re } z \leq 0\}$.

In the present paper, we consider the following problem: let P be a polynomial whose roots lie in a sector $\{|\text{Arg } z| \geq \theta \geq \pi/2\}$; what distribution of the roots minimizes the quantity $\|P\|_\infty / \sqrt{|a_0 a_n|}$?

Beside the norm $\|\cdot\|_\infty$, we consider $\|P\|_2 = (\Sigma_{j=0}^{n} |a_j|^2)^{1/2}$. These two norms have very different implications: the first one, for a signal, controls

Received June 24, 1993.
1991 Mathematics Subject Classification. Primary 30C15.

[1]Supported by the C.N.R.S. (France) and the N.S.F. (U.S.A.) and by the Digital Eq. Corp.

maximal values whereas the second one controls the quadratic mean, that is the energy.

We also consider the norm $[P]$, introduced by Bombieri in [2], and defined by

$$[P] = \left(\sum_{j=0}^{n} \frac{|a_j|^2}{\binom{n}{j}} \right)^{1/2},$$

where $\binom{n}{j}$ is the binomial coefficient $n!/j!(n-j)!$. This norm is closely linked with the representation of the polynomial on a hypercube (see [4]). Together with the associated scalar product, it is very useful in the area of multivariate polynomials (see [2], [3], [4], [9], [10]).

For this last norm, we get satisfactory results only in the case $\theta = \pi/2$ (stable polynomials).

On account of the result of Erdös-Turan, we might expect the minimum of the norms to be reached for the uniform distribution of the roots inside the considered sector. But this is completely wrong: on the contrary, the distribution at the extremities of the sector gives the minimum.

To be precise, we obtain the following result.

THEOREM. *Let θ be real, $\pi/2 \leq \theta \leq \pi$. In the set of polynomials P with complex coefficients, of degree n ($n > 1$), having all roots in the sector* $\{|\operatorname{Arg} z| \geq \theta\}$, *the polynomial*

$$P_n(z) = (z - e^{i\theta})^{\lfloor n/2 \rfloor}(z - e^{-i\theta})^{n - \lfloor n/2 \rfloor}$$

(where $\lfloor n/2 \rfloor$ is the integral part of $n/2$) makes the quantity $f(P) = N(P)/\sqrt{|a_0 a_n|}$ minimal, where $N(P)$ is either $\|P\|_\infty$ or $\|P\|_2$.

If $\theta = \pi/2$, this result is also valid for $N(P) = [P]$.

Moreover P_n is, up to complex conjugation of the roots, the only monic ($a_n = 1$) polynomial minimal for f, except if $\theta = \pi/2$, n is odd and $N(P)$ is either $\|P\|_2$ or $[P]$. In this case, the minimal monic polynomials have one free root; that is they are of the form

$$P_\varphi(z) = (z - e^{i\varphi})(z^2 + 1)^{(n-1)/2}, \quad \text{with } \pi/2 \leq |\varphi| \leq \pi.$$

Obviously, rotating the roots, one can get a similar statement for other convex sectors.

We now prove the theorem. We must distinguish between the different norms.

1. Proof of the theorem for the norm $\|\cdot\|_\infty$

Let $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$, with $z_\nu = \rho_\nu e^{i\varphi_\nu}$, $1 \le \nu \le n$. We introduce $Q(z) = \prod_{\nu=1}^{n}(z - e^{i\varphi_\nu})$. A well-known lemma due to Schur (see [6]) asserts that for every z of modulus 1,

$$\frac{|P(z)|^2}{|a_0 a_n|} \ge |Q(z)|^2,$$

with equality only if all roots z_j have modulus 1.

Thus $f(P) \ge f(Q)$, with equality only if P has all its roots of modulus 1, and it suffices to minimize $\|P\|_\infty$ over the set of monic polynomials (the quantity $f(P)$ is homogeneous) with degree n and such that all roots lie on the arc $\{|\operatorname{Arg} z| \ge \theta\}$ of the unit circle.

The following proposition shows that any minimal polynomial has no root inside the arc.

PROPOSITION 1.1. *If P is minimal for $\|P\|_\infty$ in the set of monic polynomials of degree n having all roots on the arc $[|\operatorname{Arg} z| \ge \theta\}$ of the unit circle, then P has all its roots at the boundary of this arc; that is P is of the form*

$$P(z) = (z - e^{i\theta})^p (z - e^{-i\theta})^q \quad \textit{with } p + q = n.$$

Proof. We say that a point z_0, $|z_0| = 1$ is a *maximal* point of the polynomial P if $|P(z_0)| = \|P\|_\infty$. We need two lemmas.

LEMMA 1.2. *Let $P(z) = a_n \prod_{\nu=1}^{n}(z - e^{i\varphi_\nu})$, with $0 \le \varphi_1 \le \cdots \le \varphi_n \le 2\pi$, be a polynomial such that all roots have modulus 1. On each arc $(\varphi_\nu, \varphi_{\nu+1})$ of the unit circle between two consecutive roots, the polynomial P has at most one maximal point.*

Proof. We consider the real function g, defined on $\mathbf{R}$ by

$$g(t) = |P(e^{it})|^2 = |a_n|^2 \prod_{\nu=1}^{n} (2 - 2\cos(t - \varphi_\nu)),$$

and its logarithmic derivative $h(t) = g'(t)/g(t)$, defined on $(\varphi_\nu, \varphi_{\nu+1})$.

Computing the derivative of h, we get

$$h'(t) = -\frac{1}{2} \sum_{\nu=1}^{n} \frac{1}{\sin^2\left(\frac{t - \varphi_\nu}{2}\right)} < 0,$$

which ensures that g has at most once local maximum in $(\varphi_\nu, \varphi_{\nu+1})$.

The next lemma is an immediate consequence of the previous one.

LEMMA 1.3. *Let P be a stable polynomial such that all roots have modulus* 1 *and with at least one root inside the half-plane* $\{\operatorname{Re} z < 0\}$. *Then P has only one maximal point z_0, and this maximal point satisfies* $\operatorname{Re} z_0 > 0$.

Proof. We consider, as in Lemma 1.2, the real functions

$$g(t) = |P(e^{it})|^2 \quad \text{and} \quad h(t) = \frac{g'(t)}{g(t)}.$$

The stability of P implies that $|P(e^{it})| \geq |P(e^{i(\pi-t)})|$ for every t, $|t| \leq \pi/2$; and since P has at least one root inside the half-plane $\{\operatorname{Re} z < 0\}$, the previous inequality is strict for $|t| < \pi/2$.

This ensures that all maximal points of P have non-negative real parts.

Then, in view of Lemma 1.2, it suffices to show that i and $-i$ are not maximal for P: if $P(i) = 0$, then i is not maximal for P, if $P(i) \neq 0$, then computing $h(\pi/2)$, we get

$$h\left(\frac{\pi}{2}\right) = \sum_{\nu=1}^{n} \cot\left(\frac{\pi/2 - \varphi_\nu}{2}\right) < 0.$$

Thus $g'(\pi/2) < 0$ and $\pi/2$ is not extremal for g.

In the same way we can show that $-i$ is not maximal for P, which completes the proof of Lemma 1.3.

Remark. Lemma 1.3 is also valid for stable polynomials with roots of any modulus (and at least one root in the open half-plane $\{\operatorname{Re} z < 0\}$). The proof will appear in a forthcoming paper [1].

We can now prove Proposition 1.1. Fixing $\varphi_2, \ldots, \varphi_n \in \mathbf{R}$ with $\theta \leq |\varphi_j| \leq \pi$, $2 \leq j \leq n$, we write, for $\varphi \in \mathbf{R}$, $\theta \leq |\varphi| \leq \pi$,

$$P_\varphi(z) = (z - e^{i\varphi}) \prod_{\nu=2}^{n} (z - e^{i\varphi_\nu}).$$

Let $\varphi \in \mathbf{R}$, $\theta < |\varphi| \leq \pi$. Our claim will be proved if we find $\varphi' \in \mathbf{R}$, $\theta \leq |\varphi'| \leq \pi$, such that $\|P_{\varphi'}\|_\infty < \|P_\varphi\|_\infty$.

The polynomial P_φ has, on account of Lemma 1.3, only one maximal point z_0, and it satisfies $\operatorname{Re} z_0 > 0$.

We must distinguish between two cases.

First case. $z_0 \neq -e^{i\varphi}$. Writing $t_0 = \operatorname{Arg} z_0$, this means

$$|t_0 - \varphi| < \pi.$$

Let $\varepsilon > 0$ be small enough so that

$$\{|t - t_0| < \varepsilon\} \subset \{|t - \varphi| < \pi\}.$$

Writing $M = \max\{|P_\varphi(e^{it})|,\ |t - t_0| \geq \varepsilon\}$, we have $M < \|P_\varphi\|_\infty$ and for φ' close enough to φ, we again have

$$M' = \max\left\{\left|P_{\varphi'}(e^{it})\right|, |t - t_0| \geq \varepsilon\right\} < \|P_\varphi\|_\infty.$$

But if φ' satisfies $|\varphi' - t_0| < |\varphi - t_0|$, then for every t with $|t - t_0| < \varepsilon$ we have

$$|\varphi' - t| < |\varphi - t| < \pi;$$

thus

$$|e^{it} - e^{i\varphi'}| < |e^{it} - e^{i\varphi}|,$$

and finally $|P_{\varphi'}(e^{it})| < |P_\varphi(e^{it})| \leq \|P_\varphi\|_\infty$.

Hence we see that an appropriate choice of φ' gives

$$\|P_{\varphi'}\|_\infty < \|P_\varphi\|_\infty.$$

Second case. $z_0 = -e^{i\varphi}$. Writing $Q(z) = \prod_{\nu=2}^{n}(z - e^{i\varphi_\nu})$, we have $P_\varphi(z) = (z - e^{i\varphi})Q(z)$.

We will prove that z_0 is a maximal point for Q.

Let us consider the functions

$$f(t) = \left|P_\varphi(e^{it})\right|, \quad f_1(t) = |e^{it} - e^{i\varphi}|, \quad f_2(t) = \left|Q(e^{it})\right|,$$

$$g(t) = \frac{f'(t)}{f(t)}, \quad g_1(t) = \frac{f_1'(t)}{f_1(t)}, \quad g_2(t) = \frac{f_2'(t)}{f_2(t)}.$$

We have $f(t) = f_1(t)f_2(t)$ and $g(t) = g_1(t) + g_2(t)$.

Since z_0 is maximal for P_φ, we have $g(t_0) = 0$. Moreover, since $e^{i\varphi} = -z_0$, we also have $g_1(t_0) = 0$. This gives $g_2(t_0) = 0$ and $f_2'(t_0) = 0$.

Since the polynomial Q is stable with all roots of modulus 1, the proof of Lemma 1.3 implies that f_2' vanishes only once on $]-\pi/2, \pi/2[$. So z_0 is the only maximal point for Q in $\{\operatorname{Re} z > 0\}$.

Hence we have

$$\left|P_\varphi(z_0)\right| = \|P_\varphi\|_\infty, \quad |z_0 - e^{i\varphi}| = \|z - e^{i\varphi}\|_\infty = 2, \quad \left|Q(z_0)\right| = \|Q\|_\infty.$$

So let $\varphi' \neq \varphi$, $\pi/2 \leq |\varphi'| \leq \pi$. The point z_0 is not maximal for $z - e^{i\varphi'}$; hence

$$|z_0 - e^{i\varphi'}| < \|z - e^{i\varphi}\|_\infty = 2$$

and

$$\left|P_{\varphi'}(z_0)\right| < 2\|Q\|_\infty = \|P_\varphi\|_\infty.$$

Now, if $z \neq z_0$ satisfies $\operatorname{Re} z > 0$, then z is not maximal for Q; hence

$$|Q(z)| < \|Q\|_\infty$$

and

$$\left|P_{\varphi'}(z)\right| < 2\|Q\|_\infty = \|P_\varphi\|_\infty.$$

Therefore $\|P_{\varphi'}\|_\infty < \|P_\varphi\|_\infty$, and Proposition 1.1 is proved.

PROPOSITION 1.4. *In the set of the polynomials $P_{p,q}(z) = (z - e^{i\theta})^p(z - e^{-i\theta})^q$ with $p + q = n$, the polynomial $P_n = P_{\lfloor n/2\rfloor, n-\lfloor n/2\rfloor}$ is, up to the complex conjugation of the roots, the only polynomial minimal for $\|\cdot\|_\infty$.*

Proof. If we take $p, q \in \mathbf{N}$, with $p + q = n - 1$, it suffices to prove that if $p > q$, then $\|P_{p+1,q}\|_\infty > \|P_{p,q+1}\|_\infty$. This means that if one wants to add another root, in order to decrease the norm of the polynomial, one has to put it on the side where there are fewer roots.

So let $p, q \in \mathbf{N}$, $p + q = n - 1$, $p > q$. Reasoning as in the proof of Lemma 1.2, one shows that any maximal point of $P_{p+1,q}$ (resp. of $P_{p,q+1}$) lies in the open half-plane $\{\operatorname{Im} z < 0\}$ (resp. in the closed half-plane $\{\operatorname{Im} z \leq 0\}$), which proves our claim, on account of the fact that for every t, $\pi < t < 2\pi$, one has $|P_{p+1,q}(e^{it})| > |P_{p,q+1}(e^{it})|$.

2. Proof of the theorem for the norm $\|\cdot\|_2$

For $P(z) = a_n\prod_{\nu=1}^{n}(z - \rho_\nu e^{i\varphi_\nu})$ and $Q(z) = \prod_{\nu=1}^{n}(z - e^{i\varphi_\nu})$, we use again Schur's argument:

$$\frac{|P(z)|^2}{|a_0 a_n|} \geq |Q(z)|^2$$

for every z of modulus 1. Integrating over the unit circle, we get $f(P) \geq f(Q)$, with equality only if P has all its roots of modulus 1. Hence, once again, it suffices to minimize $\|P\|_2$ over the set of monic polynomials with degree n such that all roots lie on the arc $\{|\operatorname{Arg} z| \geq \theta\}$ of the unit circle.

In order to state an analogue of Proposition 1.1, we must distinguish between the case $\theta > \pi/2$ and the case $\theta = \pi/2$.

PROPOSITION 2.1. *Let P be a polynomial minimal for $\|\cdot\|_2$ in the set of monic polynomials with degree n having all roots on the arc $\{|\operatorname{Arg} z| \geq \theta\}$ of the unit circle.*

If $\theta > \pi/2$, then P has no root inside the arc; that is, P has the form

$$P(z) = P_{p,q}(z) = (z - e^{i\theta})^p (z - e^{-i\theta})^q$$

with $p + q = n$.

If $\theta = \pi/2$, then P has either the form $P_{p,q}$ with $p + q = n$, or (and only if n is odd) the form $P(z) = (z - e^{i\varphi})(z^2 + 1)^{(n-1)/2}$ and φ, $\pi/2 \leq |\varphi| \leq \pi$, does not affect $\|P\|_2$.

Proof. Fixing $z_2 = e^{i\varphi_2}, \ldots, z_n = e^{i\varphi_n}$ with $\theta \leq |\varphi_\nu| \leq \Pi$, $2 \leq \nu \leq n$ we set $Q(z) = \prod_{\nu=2}^n (z - z_\nu)$.

Then, for $\varphi \in \mathbf{R}$, $\theta \leq |\varphi| \leq \pi$, we write $P_\varphi(z) = (z - e^{i\varphi})Q(z)$, and find

$$\begin{aligned}
\|P_\varphi\|_2^2 &= \|zQ - e^{i\varphi}Q\|_2^2 \\
&= 2\|Q\|_2^2 - 2\operatorname{Re}(\langle zQ, e^{i\varphi}Q\rangle) \\
&= 2\|Q\|_2^2 - 2\operatorname{Re}(e^{-i\varphi}\langle zQ, Q\rangle) \\
&= 2\|Q\|_2^2 - 2|\langle zQ, Q\rangle|\cos(\xi - \varphi)
\end{aligned}$$

where $\xi = \operatorname{Arg}(\langle zQ, Q\rangle)$. But

$$\operatorname{Re}(\langle zQ, Q\rangle) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(e^{it})|^2 \cos t \, dt,$$

and for every t, $|t| \leq \pi/2$,

$$(*) \qquad |Q(e^{it})| \geq |Q(e^{i(\pi - t)})|.$$

So $\operatorname{Re}(\langle zQ, Q\rangle) \geq 0$; that is $|\xi| \leq \pi/2$.

Moreover, if Q has at least one root inside the half-plane $\{\operatorname{Re} z < 0\}$, then inequality $(*)$ is strict for $|t| < \pi/2$, and one can therefore conclude that $\langle zQ, Q\rangle \neq 0$.

For $\theta > \pi/2$, we remark that if $|\varphi' - \xi| < |\varphi - \xi|$, then $\|P_{\varphi'}\|_2 < \|P_\varphi\|_2$. Since $|\xi| \leq \pi/2$, the result follows.

For $\theta = \pi/2$, it suffices to study the cases where $\langle zQ, Q\rangle$ vanishes. According to what we saw already, this can happen only if Q is of the form $Q(z) = (z - i)^p(z + i)^q$, with $p + q = n - 1$.

Moreover the quantity

$$\operatorname{Im}(\langle zQ, Q\rangle) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(e^{it})|^2 \sin t \, dt$$

cannot vanish if $p \neq q$. Indeed, if $p > q$, one has $|Q(e^{it})| < |Q(e^{-it})|$ for every t, $0 < t < \pi$, and thus

$$\operatorname{Im}(\langle zQ, Q\rangle) < 0.$$

Then, in order to prove Proposition 2.1, it suffices to note that if n is odd and $Q(z) = (z^2 + 1)^{(n-1)/2}$, then $\langle zQ, Q\rangle = 0$ and $\|(z - e^{i\varphi})Q\|_2 = \|(z - i)Q\|_2$ for every φ.

We have seen in this proof that if Q is a stable polynomial then $\operatorname{Re}(\langle zQ, Q\rangle) \geq 0$. This yields the following multiplicative estimates for the euclidean norm.

PROPOSITION 2.2. *Let Q be a stable polynomial and $\alpha \geq 0$, then*

$$\|(z - \alpha)Q\|_2 \leq \sqrt{1 + \alpha^2}\,\|Q\|_2 \leq \|(z + \alpha)Q\|_2.$$

The next proposition is the analogue of Proposition 1.4.

PROPOSITION 2.3. *In the set of the polynomials*

$$P_{p,q}(z) = (z - e^{i\theta})^p (z - e^{-i\theta})^q$$

with $p + q = n$, the polynomial $P_n = P_{\lfloor n/2\rfloor, n - \lfloor n/2\rfloor}$ is, up to the complex conjugation of the roots, the only polynomial minimal for $\|\cdot\|_2$.

Proof. Taking $p, q \in \mathbf{N}$, $p + q = n - 1$, $p > q$, we have to prove that $\|P_{p+1,q}\|_2 > \|P_{p,q+1}\|_2$.

For $\varphi \in \mathbf{R}$, $\theta \leq |\varphi| \leq \pi$, we consider the polynomial

$$R_\varphi(z) = (z - e^{i\varphi})P_{p,q}(z).$$

We have to show that $\|R_\theta\|_2 > \|R_{-\theta}\|_2$.

Setting $Q = P_{p,q}$, one writes, as above,

$$\begin{aligned}\|R_\varphi\|_2^2 &= 2\|Q\|_2^2 - 2\operatorname{Re}(e^{-i\varphi} \langle zQ, Q\rangle) \\ &= 2\|Q\|_2^2 - 2|\langle zQ, Q\rangle|\cos(\xi - \varphi),\end{aligned}$$

where $\xi = \operatorname{Arg}(\langle zQ, Q\rangle)$, and as we have already seen, $\operatorname{Im}(\langle zQ, Q\rangle) < 0$, since $p < q$.

Hence $\langle zQ, Q\rangle \neq 0$ and $\xi < 0$, which gives $\|R_\theta\|_2 > \|R_{-\theta}\|_2$ and proves our claim. The theorem follows as before.

3. Proof of the theorem for the norm $[\cdot]$

Once again, we will consider only the monic polynomials such that $a_0 = P(0) \neq 0$.

Schur's argument used for the first two norms no longer applies, and we cannot restrict ourselves to the polynomials such that all roots have modulus 1.

We first state the following proposition.

PROPOSITION 3.1. *Let P be a polynomial minimal for $f(P) = [P]/\sqrt{|a_0 a_n|}$ in the set of stable polynomials with degree n. Then P has at most one root in the open half-plane* $\{\operatorname{Re} z < 0\}$; *that is P is of the form*

$$P(z) = (z - z_1)Q(z),$$

where $\operatorname{Re} z_1 \leq 0$, *and all the roots of Q are purely imaginary.*

Moreover, if $\operatorname{Re} z_1 < 0$ *then $f(P)$ does not depend on* $\operatorname{Arg} z_1$. *Hence for every real φ, $\pi/2 \leq |\varphi| \leq \pi$, the polynomial*

$$\left(z - |z_1|e^{i\varphi}\right)Q(z)$$

is also minimal for f in the set of stable polynomials with degree n.

Proof. Suppose that the polynomial P, minimal for f in the set of stable polynomials with degree n, has a root $z_1 = \rho_1 e^{i\varphi_1}$ in the open half-plane $\{\operatorname{Re} z < 0\}$. We write $P(z) = (z - z_1)Q(z)$, and $Q(z) = \sum_{j=0}^{n-1} b_j z^j$. Then $\pi/2 < |\varphi_1| \leq \pi$ and Q is stable.

We also write, for every real φ, $\pi/2 \leq |\varphi| \leq \pi$,

$$P_\varphi(z) = \left(z - \rho_1 e^{i\varphi}\right)Q(z) \quad \left(\text{then } P = P_{\varphi_1}\right).$$

One has

$$f(P_\varphi) = \frac{[P_\varphi]}{\sqrt{\rho_1 |b_0 b_n|}}$$

and

$$\begin{aligned}[P_\varphi]^2 &= [(z-\rho_1 e^{i\varphi})Q]^2\\ &= [zQ-\rho_1 e^{i\varphi}Q]^2\\ &= [zQ]^2+\rho_1^2[Q]^2-2\rho_1\operatorname{Re}(e^{-i\varphi}[zQ,Q])\end{aligned}$$

where $[Q]^2$ is Bombieri's norm at the degree n, applied on Q (which has only degree $n-1$),

$$[Q]^2 = \sum_{j=0}^{n} \frac{|b_j|^2}{\binom{n}{j}} \quad \text{(with the convention } b_n = 0\text{)},$$

and [,] is the scalar product associated to Bombieri's norm at the degree n: if $R(z)=\sum_{j=0}^{n} c_j z^j$ and $S(z)=\sum_{j=0}^{n} d_j z^j$, then

$$[R,S] = \sum_{j=0}^{n} \frac{c_j \bar{d}_j}{\binom{n}{j}}.$$

So

$$[P_\varphi]^2 = [zQ]^2+\rho_1^2[Q]^2-2\rho_1|[zQ,Q]|\cos(\xi-\varphi),$$

where $\xi = \operatorname{Arg}([zQ,Q])$.

We can see that φ appears only in the quantity $\cos(\xi-\varphi)$. Let us show that $|\xi| \le \pi/2$, that is $\operatorname{Re}([zQ,Q]) \ge 0$.

For that purpose, we will use the following integral representation for Bombieri's norm, due to Boyd (see [5]).

If R is a complex polynomial with degree n, then

$$[R]^2 = \frac{n+1}{\pi}\int_0^{2\pi}\int_0^{+\infty} \frac{|R(re^{i\theta})|^2}{(1+r^2)^{n+2}} r\,dr\,d\theta.$$

For the associated scalar product, one gets

$$[R,S] = \frac{n+1}{\pi}\int_0^{2\pi}\int_0^{+\infty} \frac{R(re^{i\theta})\overline{S(re^{i\theta})}}{(1+r^2)^{n+2}} r\,dr\,d\theta.$$

So

$$[zQ, Q] = \frac{n+1}{\pi} \int_0^{+\infty} \frac{r^2}{(1+r^2)^{n+2}} \left(\int_0^{2\pi} |Q(re^{i\theta})|^2 e^{i\theta}\, d\theta \right) dr,$$

and

$$\mathrm{Re}[zQ, Q] = \frac{n+1}{\pi} \int_0^{+\infty} \frac{r^2}{(1+r^2)^{n+2}} \left(\int_0^{2\pi} |Q(re^{i\theta})|^2 \cos\theta\, d\theta \right) dr.$$

The stability of Q gives, for every r with $0 < r < +\infty$ and every θ with $|\theta| \le \pi/2$,

$$|Q(re^{i\theta})| \ge |Q(re^{i(\pi-\theta)})|,$$

which yields $\mathrm{Re}([zQ, Q]) \ge 0$, that is $|\xi| \le \pi/2$.

From this we will deduce that $[zQ, Q] = 0$. Indeed, suppose that $[zQ, Q] \ne 0$, and take $\varphi \in \mathbf{R}$, $\pi/2 \le |\varphi| \le \pi$, such that $|\xi - \varphi| < |\xi - \varphi_1|$ (this is possible since $|\xi| \le \pi/2$ and $\pi/2 < |\varphi_1| \le \pi$). We then have $[P_\varphi]^2 < [P_{\varphi_1}]^2$, so $f(P_\varphi) < f(P_{\varphi_1})$, which contradicts the minimality of $P = P_{\varphi_1}$.

Therefore $[zQ, Q] = 0$ and φ does not affect $f(P_\varphi)$.

Thus, in order to prove Proposition 3.1, it only remains to show that Q has all roots purely imaginary. To that aim, we note that if Q has at least one root in the open half-plane $\{\mathrm{Re}\, z < 0\}$, then one has, for every r with $0 < r < +\infty$ and every θ with $|\theta| < \pi/2$,

$$|Q(re^{i\theta})| > |Q(re^{i(\pi-\theta)})|,$$

so $\mathrm{Re}([zQ, Q]) > 0$, and therefore $[zQ, Q] \ne 0$, which completes the proof of Proposition 3.1.

Proposition 3.1 leads us to the study of polynomials having all roots purely imaginary. This is the aim of the following two propositions.

PROPOSITION 3.2. *Let $P(z) = \prod_{\nu=1}^{p}(z - \rho_\nu i)\prod_{\nu=p+1}^{n}(z + \rho_\nu i)$ be a polynomial with degree n having all roots purely imaginary. Writing $Q(z) = (z - i)^p (z + i)^{n-p}$, one has*

$$\frac{[P]}{\sqrt{\rho_1 \cdots \rho_n}} \ge [Q],$$

with equality if and only if $\rho_\nu = 1$ for every ν, $1 \le \nu \le n$.

Proof. In [2], it is shown that if R and S are polynomials with respective degrees p and q, then one has the inequality

$$(**) \qquad [RS] \geq \sqrt{\frac{p!\,q!}{(p+q)!}}\,[R][S].$$

So, with $R(z) = \prod_{\nu=1}^{p}(z - \rho_\nu i)$ and $S(z) = \prod_{\nu=p+1}^{n}(z + \rho_\nu i)$, we get

$$\frac{[P]}{\sqrt{\rho_1 \cdots \rho_n}} \geq \sqrt{\frac{p!\,(n-p)!}{n!}}\,\frac{[\prod_{\nu=1}^{p}(z-\rho_\nu i)]}{\sqrt{\rho_1 \cdots \rho_p}}\,\frac{[\prod_{\nu=p+1}^{p}(z+\rho_\nu i)]}{\sqrt{\rho_{p+1} \cdots \rho_n}}.$$

LEMMA 3.3. *Let P be a polynomial with degree d such that all roots $z_\nu = \rho_\nu e^{i\varphi}$, $1 \leq \nu \leq d$, have the same argument φ. Then*

$$\frac{[P]}{\sqrt{\rho_1 \cdots \rho_d}} \geq \left[(z - e^{i\varphi})^d\right],$$

with equality if and only if $\rho_\nu = 1$ for every ν, $1 \leq \nu \leq d$.

Proof. By rotating the roots of P (which does not affect $[P]/\sqrt{\rho_1 \ldots \rho_d}$) we may assume that $\varphi = 0$; that is, P has all roots real positive.

We write $P(z) = \sum_{j=0}^{d} a_j z^j = \prod_{\nu=1}^{d}(z - \rho_\nu)$. Since $(z-1)^d = \sum_{j=0}^{d}\binom{d}{j}(-1)^{d-j}z^j$, we have to prove that

$$\sum_{j=0}^{d} \frac{1}{\binom{d}{j}} \frac{|a_j|^2}{\rho_1 \cdots \rho_d} \geq \sum_{j=0}^{d} \binom{d}{j}.$$

It suffices to show that for every j, $0 \leq j \leq d$, one has

$$(***) \qquad \frac{|a_j|^2 + |a_{d-j}|^2}{\rho_1 \cdots \rho_d} \geq 2\binom{d}{j}^2.$$

With $j = 0$ or $j = d$, inequality $(***)$ becomes

$$\frac{1 + (\rho_1 \cdots \rho_d)^2}{\rho_1 \cdots \rho_d} \geq 2,$$

which is obviously true.

Let now j, $1 \leq j \leq d-1$. Then

$$a_{d-j} = (-1)^j \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \rho_{\nu_1} \cdots \rho_{\nu_j}$$

$$\begin{aligned} a_j &= (-1)^{d-j} \sum_{1 \leq \nu_1 < \cdots < \nu_{d-j} \leq d} \rho_{\nu_1} \cdots \rho_{\nu_{d-j}} \\ &= (-1)^{d-j} \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \frac{\rho_1 \cdots \rho_d}{\rho_{\nu_1} \cdots \rho_{\nu_j}} \\ &= (-1)^{d-j} (\rho_1 \cdots \rho_d) \sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \frac{1}{\rho_{\nu_1} \cdots \rho_{\nu_j}} \end{aligned}$$

Thus

$$\begin{aligned} \frac{|a_j|^2 + |a_{d-j}|^2}{\rho_1 \cdots \rho_d} &= (\rho_1 \cdots \rho_d) \left(\sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \frac{1}{\rho_{\nu_1} \cdots \rho_{\nu_j}} \right)^2 \\ &\quad + \frac{1}{\rho_1 \cdots \rho_d} \left(\sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \rho_{\nu_1} \cdots \rho_{\nu_j} \right)^2 \\ &= \left(\sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \frac{\sqrt{\rho_1 \cdots \rho_d}}{\rho_{\nu_1} \cdots \rho_{\nu_j}} \right)^2 \\ &\quad + \left(\sum_{1 \leq \nu_1 < \cdots < \nu_j \leq d} \frac{\rho_{\nu_1} \cdots \rho_{\nu_j}}{\sqrt{\rho_1 \cdots \rho_d}} \right)^2, \end{aligned}$$

which becomes, after a suitable change of notation,

$$\frac{|a_j|^2 + |a_{d-j}|^2}{\rho_1 \cdots \rho_d} = \left(\sum_{k=1}^{\binom{d}{j}} x_k \right)^2 + \left(\sum_{k=1}^{\binom{d}{j}} \frac{1}{x_k} \right)^2,$$

where the x_k's, $1 \leq k \leq \binom{d}{j}$, are real positive and satisfy $x_k = 1$, $1 \leq k \leq \binom{d}{j}$ if and only if $(\rho_1, \ldots, \rho_d) = (1, \ldots, 1)$.

The following lemma completes the proof of Lemma 3.3.

LEMMA 3.4. *Let $N \in \mathbf{N}^*$, we consider the function defined by*

$$g(x_1, \dots, x_N) = \left(\sum_{k=1}^{N} x_k\right)^2 + \left(\sum_{k=1}^{N} \frac{1}{x_k}\right)^2, \quad x_1, \dots, x_N > 0.$$

Then the only minimal point for g is in the point $(1, \dots, 1)$.

Proof. We expand the squares and use the fact that the function, $x \to x + 1/x$, $x > 0$, is minimal for $x = 1$ and only there.

Having proved Lemma 3.3 we return to the proof of Proposition 3.2. Applying Lemma 3.3 to the polynomials R and S, we get

$$\frac{[P]}{\sqrt{\rho_1 \cdots \rho_n}} \geq \sqrt{\frac{p!(n-p)!}{n!}} \left[(z-i)^p\right]\left[(z+i)^{n-p}\right],$$

with equality if and only if $\rho_\nu = 1$ for every ν, $1 \leq \nu \leq n$.

But the pairs (R, S) extremal for $(**)$ (that is for which equality holds in $(**)$) are exactly the pairs $(R(z) = (z - \alpha)^p,\ S(z) = (z + 1/\overline{\alpha})^q)$ with $\alpha \in \mathbf{C}$ (see [4] or [10] for a proof).

So the pair $(z - i)^p$, $(z + i)^{n-p}$ is extremal for $(**)$, which means

$$\sqrt{\frac{p!(n-p)!}{n!}} \left[(z-i)^p\right]\left[(z+i)^{n-p}\right] = \left[(z-i)^p(z+i)^{n-p}\right],$$

and Proposition 3.2 is proved.

Now we minimize $[\cdot]$ over the set of the polynomials

$$P_{p,q}(z) = (z-i)^p(z+i)^q \text{ with } p + q = n.$$

PROPOSITION 3.5. *In the set of the polynomials $P_{p,q}(z) = (z - i)^p(z + i)^q$ with $p + q = n$, the polynomial $P_n = P_{\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor}$ is, within the complex conjugation of the roots, the only polynomial minimal for* $[\cdot]$.

Proof. Since $(z - i)^p$ and $(z + i)^q$ are extremal for $(**)$, one has

$$\left[P_{p,q}\right] = 2^{(p+q)/2} \sqrt{\frac{p!\,q!}{(p+q)!}},$$

so we easily complete the proof with the inequality

$$p!(n-p)! \geq \left\lfloor \frac{n}{2} \right\rfloor! \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)!,$$

which is strict if $p \notin \{\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor\}$.

Using the previous three propositions, we can finally prove the theorem.

Let P be a monic polynomial minimal for f in the set of stable polynomials with degree n. According to Proposition 3.1, P has at most one root in the open half-plane $\{\text{Re } z < 0\}$.

We distinguish between two cases.

First case. P has no root in $\{\text{Re } z < 0\}$. Then, according to Proposition 3.2 and Proposition 3.5, we have either $P = P_{\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor} = P_n$ or $P = P_{n-\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$.

So:

If n is even, then $P(z) = (z^2 + 1)^{n/2}$.

If n is odd, then

$$P(z) = (z + i)(z^2 + 1)^{(n-1)/2} \quad \text{or} \quad P(z) = (z - i)(z^2 + 1)^{(n-1)/2}.$$

Second case. P has one root with negative real part, then

$$P(z) = (z - \rho e^{i\varphi})Q(z)$$

with $\pi/2 < |\varphi| \leq \pi$ and Q has all roots purely imaginary.

Moreover, according to Proposition 3.1, the polynomials $(z - \rho i)Q$ and $(z + \rho i)Q$ are also minimal for f. Then Proposition 3.2 implies that $\rho = 1$ and

$$Q(z) = P_{p,q}(z) = (z - i)^p (z + i)^q,$$

with $p + q = n - 1$.

Thus it remains to show that necessarily $p = q$ (and therefore n is odd).

The polynomials $(z - \rho i)Q = P_{p+1,q}$ and $(z + \rho i)Q = P_{p,q+1}$ are both minimal for f, so

$$[P_{p+1,q}] = [P_{p,q+1}]$$

But we know that

$$[P_{p+q,q}] = \sqrt{\frac{(p+1)!\,q!}{n!}}\, 2^{n/2}$$

and

$$[P_{p,q+1}] = \sqrt{\frac{p!\,(q+1)!}{n!}}\, 2^{n/2}.$$

Therefore, the condition $[P_{p+1,q}] = [P_{p,q+1}]$ implies that $p = q$.

To complete the proof of the theorem, it suffices to note that if n is odd, then every odd coefficient of the polynomial $(z^2+1)^{(n-1)/2}$ vanishes.

Therefore, writing $P_\varphi(z) = (z - e^{i\varphi})(z^2+1)^{(n-1)/2}$, $\pi/2 \le |\varphi| \le \pi$, we see that φ does not affect $f(P_\varphi)$ and so P_φ is minimal for f, for every φ, $\pi/2 \le |\varphi| \le \pi$.

4. A few optimal estimates for stable polynomials

The theorem we have just proved yields, with $\theta = \pi/2$, the following optimal estimates:

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a stable polynomial with degree n. Then

$$\frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}} \ge 2^{n/2} \tag{1}$$

$$\frac{[P]}{\sqrt{|a_0 a_n|}} \ge 2^{n/2} \binom{n}{\lfloor n/2 \rfloor}^{-1/2}. \tag{2}$$

Moreover, if n is even, then

$$\frac{\|P\|_2}{\sqrt{|a_0 a_n|}} \ge \binom{n}{n/2}^{1/2} \tag{3}$$

and if n is odd, then

$$\frac{\|P\|_2}{\sqrt{|a_0 a_n|}} \ge \sqrt{2} \binom{n-1}{\frac{n-1}{2}}^{1/2}. \tag{4}$$

Remark. The result of Erdös-Turan, with $c \sim 2.619$, yields the following estimate for stable polynomials:

$$\frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}} \ge e^{n/4c^2} \quad \text{and} \quad 4c^2 \sim 27.44.$$

Here we get

$$\frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}} \ge e^{(n/2)\log 2} \quad \text{and} \quad \frac{2}{\log 2} \sim 2.89.$$

5. Other norms

The result we get for the norms $\|\cdot\|_\infty$, $\|\cdot\|_2$ and $[\cdot]$ with $\theta = \pi/2$ is not valid for every usual norm.

Indeed, consider the L_1 norm

$$\|P\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(e^{it})|\,dt,$$

and write $P_\varphi(z) = (z - e^{i\varphi})(z^2 + 1)$. Then one has $\|P_\pi\|_1 \sim 1.552$ and $\|P_{\pi/2}\|_1 \sim 1.698$ so

$$\|P_{\pi/2}\|_1 = \|P_{-\pi/2}\|_1 > \|P_\pi\|_1.$$

Here $n = 3$, and both of these two polynomials have all roots of modulus 1. We see that the polynomial P_π, which has a root inside the left half-plane, is smaller (for the L_1 norm) than $P_{\pi/2}$, which is the polynomial $P_{n-\lfloor n/2\rfloor, \lfloor n/2\rfloor}$ of our theorem for $n = 3$ and $\theta = \pi/2$.

Hence our claim is not valid in this case, and one can check that P_π is minimal for $f(P) = \|P\|_1/\sqrt{|a_0 a_n|}$ in the set of stable polynomials of degree $n = 3$.

Acknowledgements. This work is part of the author's Ph.D., prepared under the supervision of Prof. Bernard Beauzamy.

The author wishes to thank Prof. Bruce Reznick for his careful reading and numerous suggestions which led to significant improvements of the manuscript.

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